Research Methodology
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Lecture - 40
Propagation Errors Part 02

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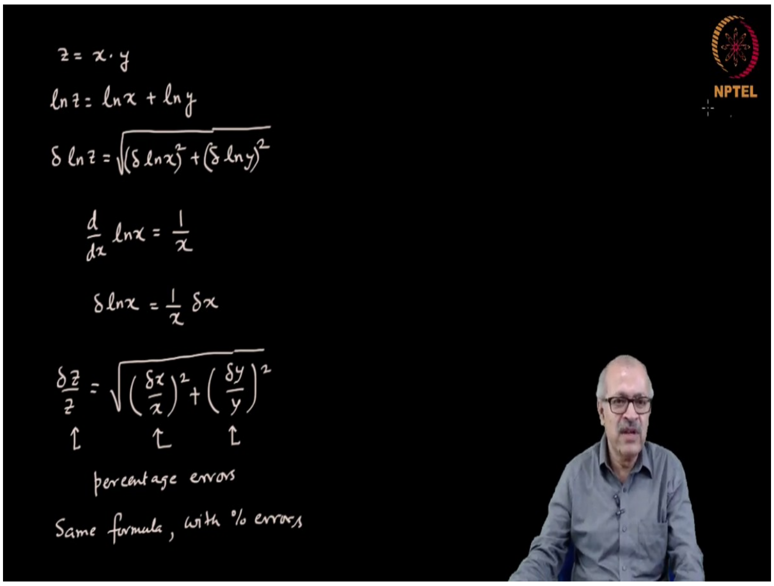

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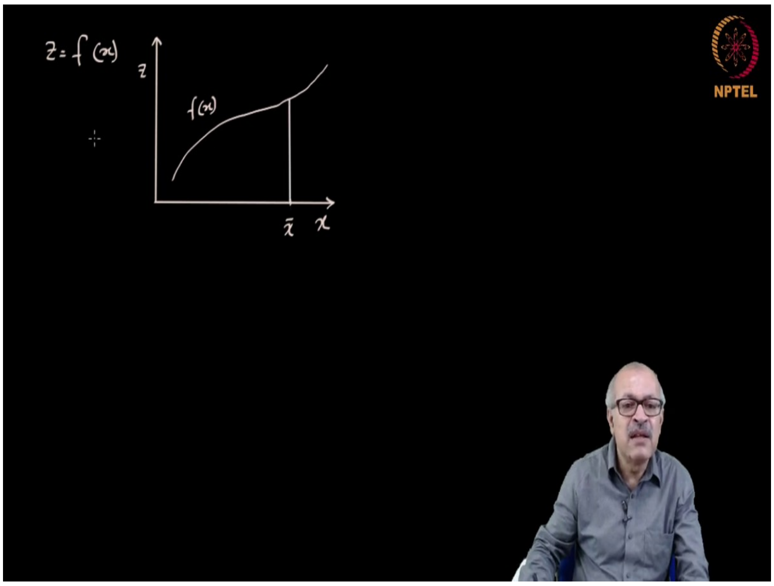

Let us go ahead. I said that we encounter another kind of situation, where what we are trying to measure is actually a function of x. z is a function of x. We have measured x and we are trying to find z, some function, whatever that function can be.

Now, in such situations where we are talking about a function of z, the functional form has to be given, and let us assume the functional form to be something like that, whatever it is.

So, here I have x, here I have z, and this is my function of x. In that case I have made a measurement of x and we are trying to find out what will be the mean value of z, and what will be the error in z. Let us try to work that out.

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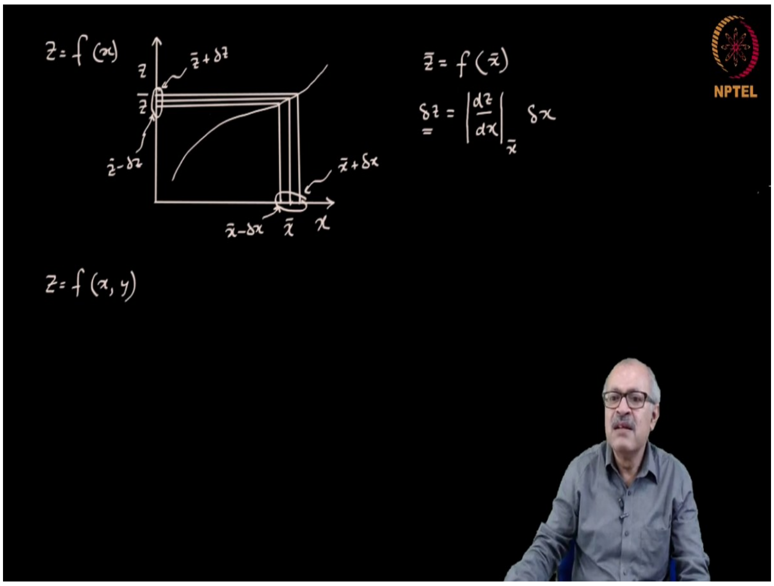

Suppose I have measured the x bar value here. So, x bar, the mean value measured, is here. In that case, we can draw a line going up. So, if x is this and z is a function, then following that function this will be the measured value of z.

What about the error? The error will be a range something like this. If we have that range then we can go up like this, and come to the left like that. So, this much. This will be the error. This is x bar, this point is x bar plus delta x and this point is x bar minus delta x. These then propagate like this. As a result, this becomes z bar plus delta z, this becomes z bar minus delta z.

So, we have this error propagating to this error. It is easy to infer that the z bar will be the same function (I have said that functional form has to be known). If the functional form is known and x bar is known, we can know z bar. But the delta z will propagate depending on the slope here.

If the slope is smaller than 1, then actually the error will reduce, while if it bigger than 1 the error will increase. So, it depends on the derivative of the function at that point. So, this is actually the derivative of the function dz/dx. But the dz/dx is different at different parts. So, it has to be calculated at x bar.

Not only that. Depending on whether it is a positive slope or a negative slope, there will be no difference. So, it is actually the mod of that, times your delta x, that will be the value of the delta z. Is that clear from the construction that I have shown in the graph?

So, your delta z is actually given as a the error in z, error in x times the slope of the function calculated at the mean value of x. So, this is how we calculate the value of the z. So, we can again express z as z bar plus minus delta z, the error bar.

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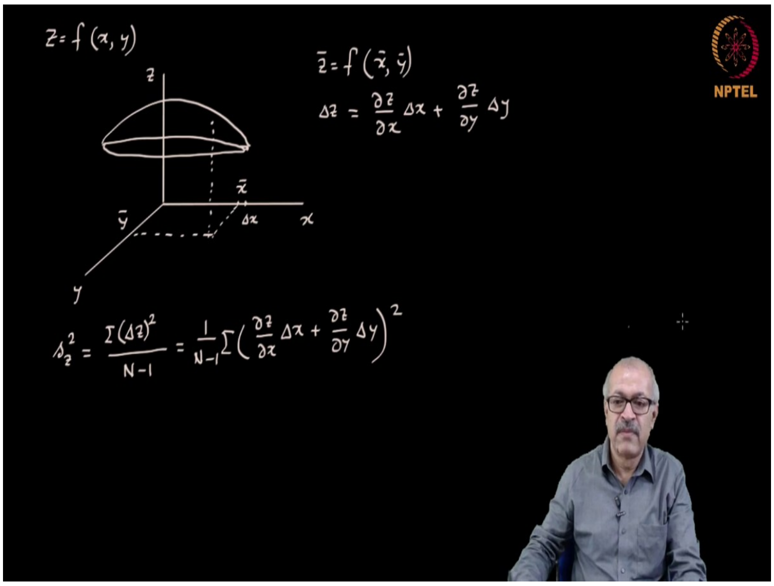

What if z is the function of two variables x and y? What then? Well it is not difficult to see that. Then we will have to depict the situation as: this is my x, this is my y, and then this is my z and the function will then take the form of some kind of a surface. So, it will be something like this.

So, if you have a surface; that means, z is a function of x and y. It will be some kind of a surface. Then I have the measured values of x and y, which can be somewhere here. So, I have a measured value of x and y somewhere here. This is my measured value of x, this is my measured value of y, and we are trying to find out this.

So obviously, we can find out the measured value of z as the function of x bar and y bar; that is not a problem. Now, let us consider: I have got an x bar and let us consider a perturbation, some error in x, and let us call it, say, delta x. It is not the standard error I am talking about. I am just talking about a perturbation, a change from the value of x bar.

Take it this way: if I had measured the x bar again I would have got a different result; a third time it would be a different result. So, there could be different results. This is another result. The other result is x bar plus delta x. Similarly there can be another result if you measured it again.

That means, whenever you measure it, you are measuring with a number of readings and you have measured again with 25 readings, and you got another value of the x bar which is the earlier x bar plus delta x, and the earlier y bar plus delta y (or minus delta y, whatever it is).

In that case the delta z, what we had obtained in terms of the slope, in this case we have to obtain it in terms of the slope in the x direction and the slope in the y direction, which means in terms of the partial derivatives. So, this will be equal to the delta z = del z / del x times this perturbation in x plus del z del y times this perturbation in y.

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

So, these are the deviations of the data points, this capital delta x is the deviation from the value of x bar that you have obtained, and we are trying to find out, due to these deviations in x and y, what would be the deviation in z? That can be found from this formula.

But now we are trying to find out what will be the standard error, standard division in z. So, we proceed by stating the standard deviation. Let us work it out in terms of the variance, the variance would be s z square ok.

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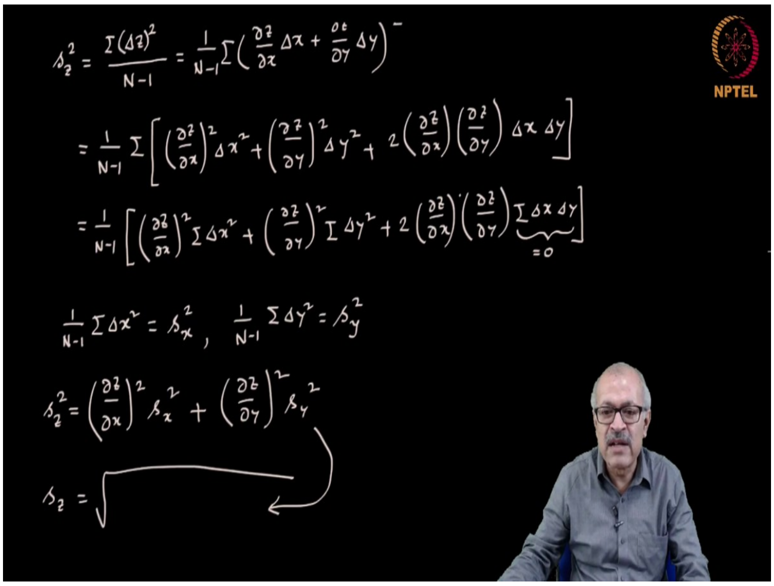

How do you define the variance? Variance is the difference of a data point from the mean value that you have obtained, which in this case, is delta z. So, its summation, it is delta z, this is the difference between the value optioned minus the mean value, its square summed over all data minus N minus 1. That is the definition of the variance.

Now, this is something that we know. So, we can substitute it here. So, this will become 1 by N minus 1 summation of now its square del z del x delta x plus del z del y delta y square.

$$
\begin{aligned}
s_z^2 &= \frac{\sum(\Delta z)^2}{N-1} \\
&= \frac{1}{N-1}\sum\left(\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y\right)^2 \\
&= \frac{1}{N-1}\sum\left[\left(\frac{\partial z}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \Delta y^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\Delta x \Delta y\right] \\
&= \left(\frac{\partial z}{\partial x}\right)^2 \frac{1}{N-1}\sum(\Delta x)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \frac{1}{N-1}\sum(\Delta y)^2 \\
&\quad + \frac{2}{N-1}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sum(\Delta x \Delta y)
\end{aligned}
$$

Then we can expand that as 1 by N minus 1, this will be: let me put the whole thing in a bracket, del z del x square delta x square plus del z del y square delta y square plus 2 del z del x del z del y delta x delta y, this whole thing right.

Now, this is 1 by N minus 1 we keep it, but now we would like to put the summation inside, so that it becomes del z del x square the summation over delta x square plus del z del y square summation over delta y square plus twice del z del x del z del y summation over delta x delta y. Well we can also put the 1 by N minus 1 in all these quantities, but we will do that slowly.

Now, let us see what this is this. Let me write it separately, this was 1 by N minus 1 delta x square. So, sigma delta x square. What is that? 1 by N minus 1 and this we are keeping separate. So, what is this? This is actually the variance of x 1 by N minus 1 and the summation over delta y square is nothing but this the variance of y.

$$\frac{1}{N-1}\sum(\Delta x)^2 = s_x^2 \text{ and } \frac{1}{N-1}\sum(\Delta y)^2 = s_y^2,$$

So, the variance of x will be multiplied by this variance of y. But what is this? Notice, that it is a summation over delta x delta y. delta x and delta y are the errors the perturbations. So, these could be positive or negative and as a result of which this product could also be positive or negative and if you take a large number of such data points then positives and negatives will cancel off and as a result of this this part equals 0 for a large number of readings.

And therefore, the whole thing will be 0, which means s z square then becomes this term del z del x square s x square plus del z del y square s y square. Then, that the variances in this case are related in this way and therefore, the standard deviation as standard deviation as the square root of this whole thing. This is how the error would propagate from the error in x and the error in y to the error in z and the measured value of z will be stated like this and the standard error in z will be stated like this.

Square root of del z del x square, standard error of x square plus del z del y

$$s_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 s_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 s_y^2$$

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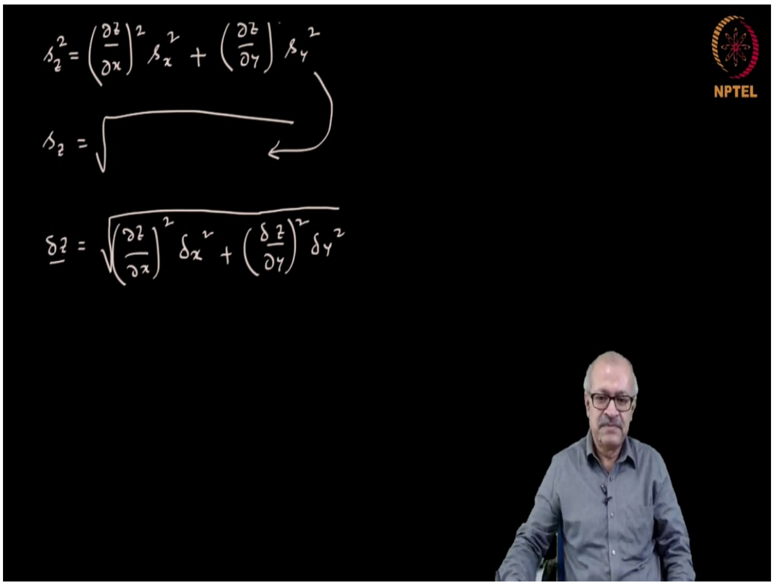

I should write the delta z will be square root of del z del x square delta x square plus del z del y square delta y square. This is how the error in x and y would propagate into the error in z.

So, having the measured value of x and the value of y, which means the mean value of x and mean value of y, measured standard errors in x, standard error in y measured, this is how it will propagate into the standard error in z.

And this way the errors propagate. Often we do make measurements of things which are different from things that we need and they are somehow related. Depending on that relationship, we can find out ways of stating the result of measurement of the thing that we ultimately want to measure or state. So, this is how we ultimately make the measurements and state the results in a paper.

Thank you.