Mine Automation and Data Analytics

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Week - 8

Lecture 39: Continuous Random Variable Part I

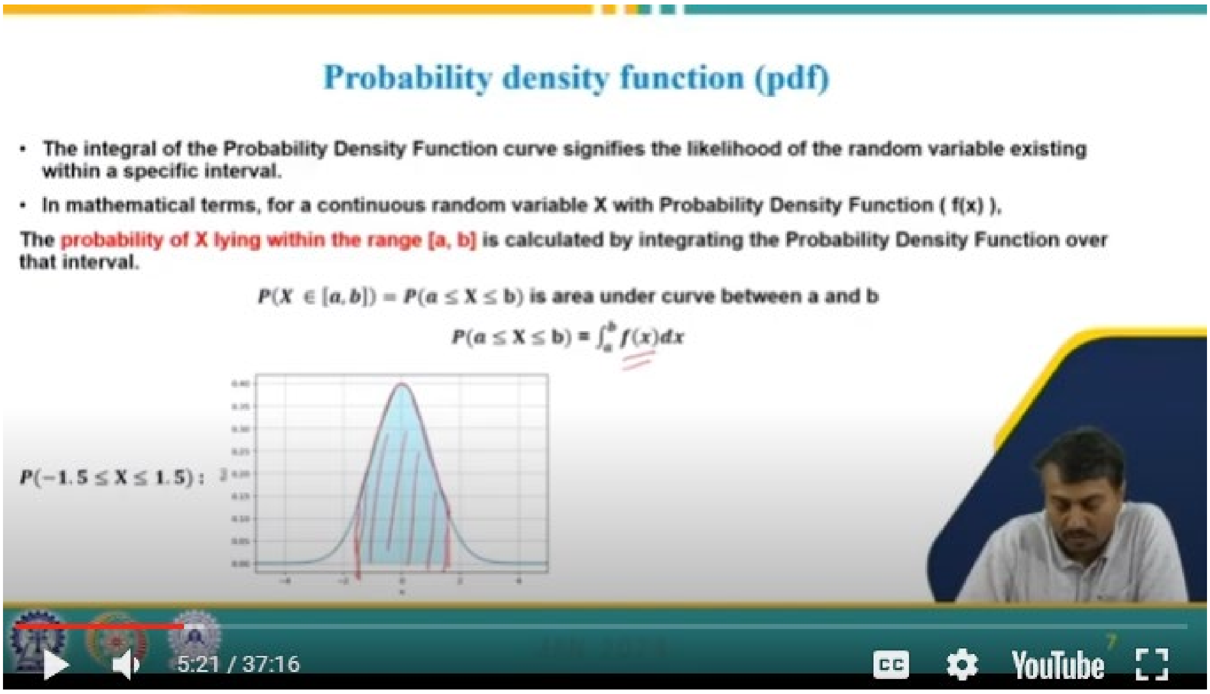
Welcome back to my course, mine automation and data analytics. In the last lesson, we discussed the discrete random variables. So, today we will introduce the continuous random variable. So, in this lesson, we will discuss the following: the continuous random variable and different distributions, for example, uniform distribution, exponential distribution, and normal distribution, with the concept of the probability distribution function. So, as you have seen in the discrete random variable, we have introduced the probability mass function. Here, we will deal with the probability distribution function.

So, let me revisit between the discrete and continuous random variables. A discrete random variable is characterized by its ability to assume a countable number of possible values at most. For example, 0, 1, 2, 3, 4, 5 like that, or suppose you are rolling a die, the number of times three or the number of times 6 is appearing. So, these are countable numbers, finite numbers.

Consequently, any random variable capable of adopting either a finite number or a countably infinite number of distinct values qualifies as a discrete random variable. So, it is worth noting that there are also random variables whose set of potential values is uncountably infinite. So, continuous random variables pertain to scenarios where outcomes of random events are numerical yet cannot be enumerated and are infinitely divisible. So, a discrete random variable is a variable that is characterized by having possible values that are distinct points along the natural number line. Discrete random variables are often associated with counting scenarios.

A continuous random variable is defined by its possible values spanning an interval along the natural number line. Continuous random variables typically involve measurement scenarios—probability density function. So, a probability density function is a mathematical function that describes the likelihood of a constant random variable falling within a particular range of values. That is very important.

In the probability density function, we are interested in measuring the likelihood of falling between that limit and the interval. So, in other words, it provides a way to represent the probability distribution of a continuous random variable, and it is denoted by f x like this. So, let me see. So, here, the f x is the function, and we want to measure the likelihood of lying in this range, and this can be found by integrating this particular area under the curve. So, this is the likelihood of the value occurring between minus 1.5 and 1.5. So, the integral of the probability density function, which is the integral of the probability density function curve, signifies the likelihood of the random variable existing within a specific interval. So, in mathematical terms, for a continuous random variable X, capital X with a probability density function of f x, the probability of X lying within the range, lying within the range. We are not interested in finding out the likelihood of this point right now.



No. We are interested in finding out the likelihood, or the probability of X lying in the range from a to b, which is calculated by integrating the probability density function over that interval. The probability of X represents this in between a and b, and that can also be represented by the probability of X greater than equal to a and less than equal to b is the area under the curve between a and b, and this is represented by the integral of a to b f x dx, fundamental concept.

Now, what are the properties? Critical, the total area under this curve, for example, is minus 3 to 3, so this total area under the curve is 1, ensuring the probability encompassing all potential outcomes. So, this range indicates all potential outcomes outside this 0, or this can be abbreviated as f x dx, which is equal to 1, and the total summation over this area is 1, which is another property. So therefore, the area beneath the probability distribution curve of a continuous random variable between any two points falls within the range of 0 to 1 because you have seen in the earlier slide that if something in this range, left side or right side, or outside this is 0 and in maximum likelihood is 1. So therefore, the area beneath the probability distribution curve of a continuous random variable between any two points falls within the range of 0 to 1, and this is represented by the probability of X of xi, which is always greater than equal to 0 and less than equal to 1.

$$\mathbf{0} \leq \mathbf{P}(\mathbf{X} = \mathbf{x}\boldsymbol{i}) \leq \mathbf{1}$$

The area beneath the probability density function graph between points a and b remains constant irrespective of whether endpoints a and b are inclusive or exclusive. The probability of X greater than equal to a less than equal to b is the same as the probability of X more significant than a less than b because it is a definite integral that calculates area over the range a to b f x dx—cumulative distribution function. So, for a continuous random variable, f, a is equal to the probability of X less than equal to a. So, earlier, we are interested in finding out the likelihood of that particular function between this range. Now, when we are interested in finding out the point probability encompassing less than equal to a means that includes the whole area less than equal to a that is from minus infinity to a.

$$\boldsymbol{P}(\boldsymbol{a} \leq \mathbf{X} \leq \mathbf{b}) = \boldsymbol{P}(\boldsymbol{a} < \mathbf{X} < \mathbf{b}) = \int_{a}^{b} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx}$$

So, minus infinity to a f x dx summing over that area is the probability. Still, we must remember the total area under the probability curve, or the curve is 1. Given that the likelihood of continuous random variable X taking any single value is 0. So, we consequently have a probability of X less than equal to a, which is equal to the probability of X less than a, and that is represented by minus infinity to a f x dx. Now, what is the expectation? The expectation of continuous random variable is integration X f x dx. The variance is X minus exponential X whole square f x dx, and we have already seen the expectation and variance of a discrete random variable.

$$\textbf{Expectation E(X)} = \int \boldsymbol{x}\, \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{dx}$$

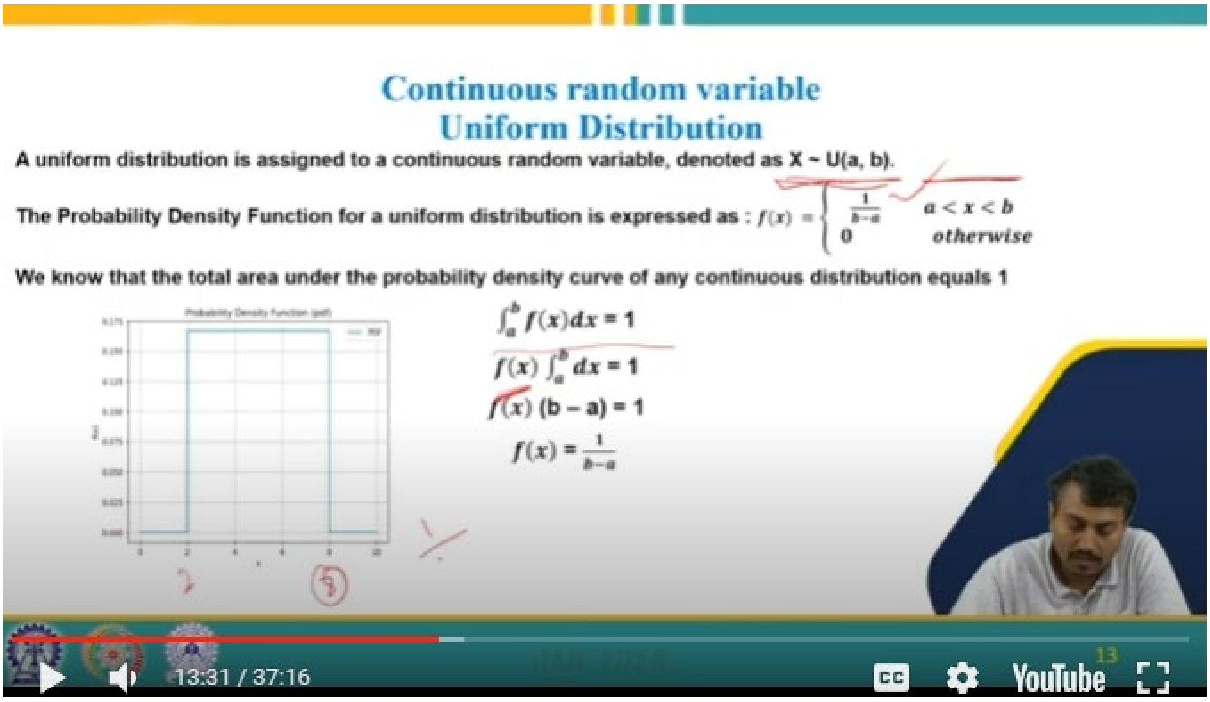
$$\textbf{Variance Var(X)} = \int \left(\boldsymbol{x} - \boldsymbol{E}(\boldsymbol{X})\right)^{\mathbf{2}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx}$$

The summation of i equals 1 to infinity xi probability of xi variance is exponential of X minus mu whole square. So, these are the distribution for the continuous random variable, uniform distribution, standard uniform distribution, exponential distribution, normal distribution, standard normal distribution, t distribution, and chi-square distribution. Continuous random variable, now let us see the uniform distribution. A uniform distribution is assigned to a constant random variable denoted by X of you a to b, you a b, and the probability density function for a uniform distribution is expressed as one by b a for the range a to b, X range a to b it is a constant value, and otherwise the value is 0. We know that the total area under any continuous distribution's probability density curve equals 1.

So, the probability of the integral a to b f x dx is equal to 1. The whole area under this curve from a to b is 1. And now we know that f x is a constant. So, we keep outside the f x. So, the integral of a to b dx equals 1, then b to a b minus a multiplied by f x equals 1.

So, f x equals one by b minus a, which is the uniform distribution here from 2 to 8, and 1 by eight minus 2 equals 1 by six is around 0.166. So, it is around that value of 0.166. So, that is, this is constant over this range.

We have already seen a continuous random variable for 2 to 8, one by 6. So, this value is between 2 and 8 and is one by 6. For standard uniform distribution, a continuous random variable follows the standard uniform distribution with a minimum value of 0 and a maximum value of 1 represented by X of you 0 1. So, what is the probability density function of the uniform distribution? This is 1 for X in the range of 0 to 1; the left and right sides are 0. So, similarly, f x dx 0 to 1 equals 1. So, f x again it is a constant. So, 0 to 1 dx equals 1, and 1 minus 0 f x equals 1. So, f x is equal to 1. So, what does it look like? So, from 0 to 1, it is a constant value of 1. So, these are the characteristics of the standard uniform distribution.



So, the cumulative distribution function of the uniform distribution denoted by f x is given by x less than equal to a is 0, x greater than equal to b is one always because there is no other value because it is the highest value and in between this value what is the value we have to calculate. So, for a continuous random variable, f a is equal to the probability of X less than equal to a, and that is basically minus infinity to a f x dx. So, the f x is equal to the probability of X less than equal to x, and that is minus infinity to x f x dx. So, if I divide that into two parts, minus infinity to one part, another is a to x another part, and we know that the value for X is less than equal to a; this is the left side value.

So, this is 0. So, this is rightly represented, and this is x to a or a to x f x dx, and f x is one by b minus a. Then it is x minus a because that is integral of a to x dx x minus a. So, this is the cumulative distribution function in the range a to b. So, for a to b, from this a to b, this is the value. So, it is taking 3, so it is three minus two divided by 3, this is one by 3, 1 by 3, 0.33. Then you are taking it as 4, so four minus two divided by five minus 2, 3 is equal to 2 divided by three is equal to 0.66, 0.66. Now, check with 4.Five minus two by 3, so it is 2.5 by three, equals 0.8 and above. So, it is basically 0.8 and above. So, this is the cumulative distribution function for the uniform distribution.

So, we have seen the cumulative distribution function for uniform distribution 2 to 5, and we have calculated the different values in this interval. This cumulative distribution function for the standard uniform distribution is denoted. So, for that less than equal to the value of 0 and above one, it is one, and between this value of 0 to 1 is the x. So, again, we subdivide this problem of f x of x probability less than equal to x minus infinity to x. So, minus infinity to 0 f x dx one part, another is 0 to x f x dx another part. So, for the less than equal to less than 0 values, it is 0, 0 so it is 0, and for f x, it is a constant coming outside. So, 0 to x is x minus 0, f x is equal to 1, so this is x. So, any value you take is 0.5 means 0.5, 0.25 means 0.25 and 0.75 means 0.75. So, this is the distribution pattern. The expectation of continuous random variable x of you a b is b plus a divided by 2.

How we estimate expectation is a to b integral x f x dx. So, x dx because f x is a constant one minus one by b minus a, this will go out. So, x dx of a to b, x square divided by 2, so b square minus a square divided by 2. So, this will give the expectation value of the continuous random variable b plus a divided by 2. Variance is the expectation of x square and expectation x f whole square the minus with the difference between these two.

So, what is the expectation of x square? So, x square f x dx and f x is a constant value that goes out one by b minus a, and x square dx is nothing but x cube divided by 3. So, b cube minus a cube divided by one by b minus a equals a square plus b square plus a b divided by 3. And we have already estimated the expectation of x. We have to square it whole and then find out this value minus that value is equal to the variance of a continuous random variable of x you a b. So, this is estimated and calculated. So, this is the value we have calculated in the last slide, and this value we have calculated is a plus b divided by 2, and then it is the whole square we require.

So, this is coming as b minus a divided by the whole square divided by 12. Now, we want to do some examples: x is a uniform random variable over 0 to 1. Now, find the following probabilities: the probability of x less than 0.5. So, the probability of x less than x is nothing but the likelihood of 0 to 0.

1. $\mathbf{P(X < 0.5) = 0.5}$

$$P(X \le x) = P((X \le x) = P((X < 0.5) = \int_0^{0.5} f(x)dx$$

$$= \int_0^{0.5} 1.\ dx$$

$$= 0.5$$

2. $\mathbf{P(X \ge 0.8) = 1 - 0.8 = 0.2}$

$$\mathbf{P}\,((\boldsymbol{X} \geq \mathbf{0.8}\,)=\mathbf{1}-\mathbf{P}\,((\boldsymbol{X}<\mathbf{0.8})= \mathbf{1}-\int_{0}^{0.8} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{d} \boldsymbol{x}$$

$$= \mathbf{1} - \int_{0}^{0.8} \mathbf{1}.\ \boldsymbol{dx}$$

$$= \mathbf{1 - 0.8}$$

$$= \mathbf{0.2}$$

Five and f x are constant values of one, so it will go out. So, 0 to 0.5 dx is nothing but 0.5 because we know the pdf value of the standard uniform distribution is 1.

For x greater than 0.8, x greater than 0.8 is the probability of x greater than 0.8 equals one minus the probability of x less than 0.8 and 1 minus f x dx 0 to 0.8. So, the f x is a constant one that goes out. So, 0 to 0.8 dx is nothing but 0.8. So, one minus 0.8 is equal to 0.2. A probability between 0.2 and 0.9 that can be represented as the probability of x less than equal to x minus the probability of x less than equal to y, and that is the probability of less than equal to 0.9 and 0.2 difference between these two, and for every case f x is the constant. So, 0.9 minus 0 minus bracket 0.2 minus 0. So, that gives us 0.7.

1. **P(0.2 ≤ X < 0.9) = 0.9 – 0.2 = 0.7**

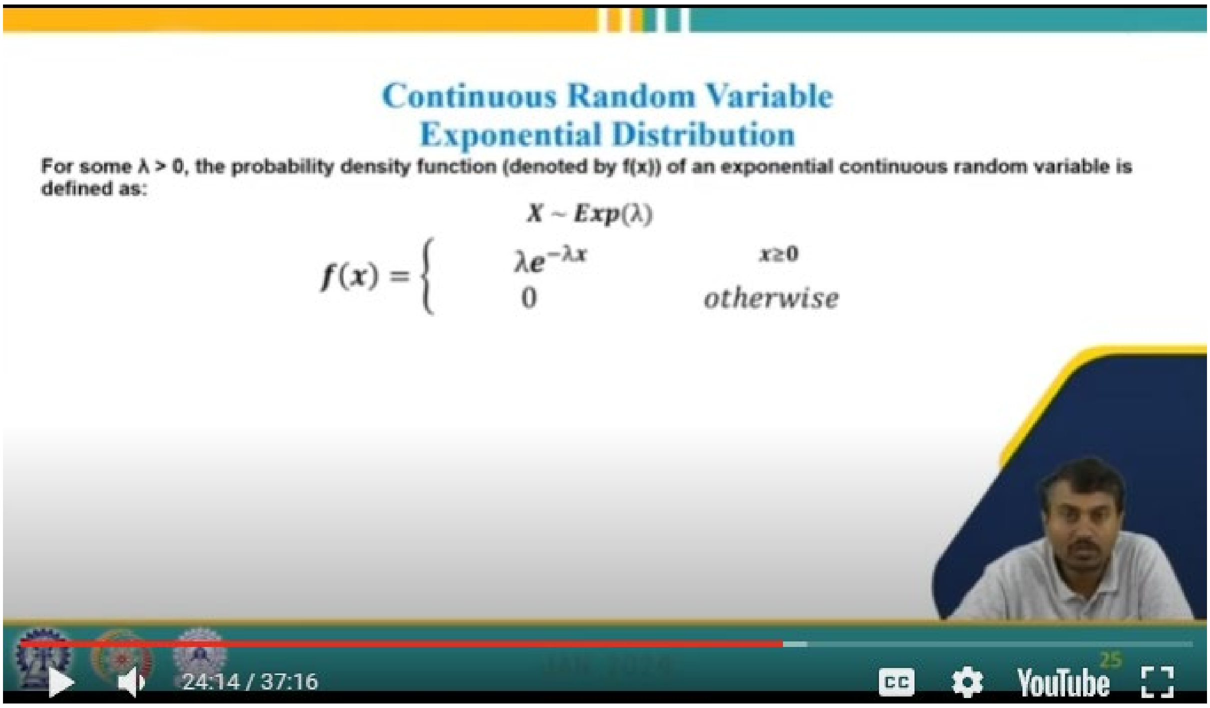
$$\boldsymbol{P}(\boldsymbol{y} \leq \boldsymbol{X} \leq \mathbf{x})=\mathbf{P}\,((\boldsymbol{X} \leq \mathbf{x})-\mathbf{P}\,((\boldsymbol{X} \leq \mathbf{y})=\mathbf{P}\,((\boldsymbol{X} \leq \mathbf{0.9})-\mathbf{P}\,((\boldsymbol{X} \leq \mathbf{0.2})$$

$$= \int_{0}^{0.9} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{dx} - \int_{0}^{0.2} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{dx}$$

$$= \mathbf{0.9 - 0.2}$$

$$= \mathbf{0.7}$$

Exponential distribution, so for some lambda probability density function denoted by f x of an exponential continuous random variable is defined by x of exponential lambda for x more significant than is equal to 0 it is lambda of e to the power minus lambda x otherwise, it is 0. So, this is the shape of the exponential distribution for different lambda values.



This is for 0.5 blue, 1.5, and for the green, it is 3. For the cumulative distribution function, the exponential distribution is denoted as f a. We know the cumulative distribution function of any continuous random variable f a is equal to the probability of x less than equal to a, minus infinity to a that is f x dx. So, that can be subdivided into minus infinity to 0 f x dx and 0 to an f x dx, and in the last slide, we have seen that more significant than equal to 0 is the value; otherwise, it is 0. So, we know f x equals 0 for less than 0, an exponential distribution. So, we can estimate 0 to a lambda e to the power lambda x dx.

So, f a is calculated as one minus e to the power minus lambda a., this is the exponential distribution of the cdf with different values of lambda, and here you will notice that with the increase of the lambda value, it goes up quickly, the peak values and for 1.5 value later and for three it goes very quickly to the peak values. Expectation for the exponential distribution: the expectation of x to the power n is 0 to infinity x to the power n f x dx, and that is 0 to infinity x to the power n lambda e to the power minus lambda x dx. So, lambda goes out 0 to infinity x to the power n e to the power minus lambda x dx.

So, here we want to apply the integration by parts formula integration of you dv is equal to uv minus the integral of v du. So, the exponential of x to the power n is equal to n by lambda expectation of x to the power n minus 1, and for n is equal to 1, the expectation of x 1 is equal to 1 by lambda expectation of x 1 minus 1 is equal to 1 by lambda. The expectation of exponential distribution exponential lambda is equal to 1 by lambda. Then variance is n is equal to 1. We have found that the e of x is equal to 1 by lambda for n equal to 2, and the expectation of x square is equal to 2 by lambda square. So, variance is the expectation of x square minus the expectation of x the whole square.

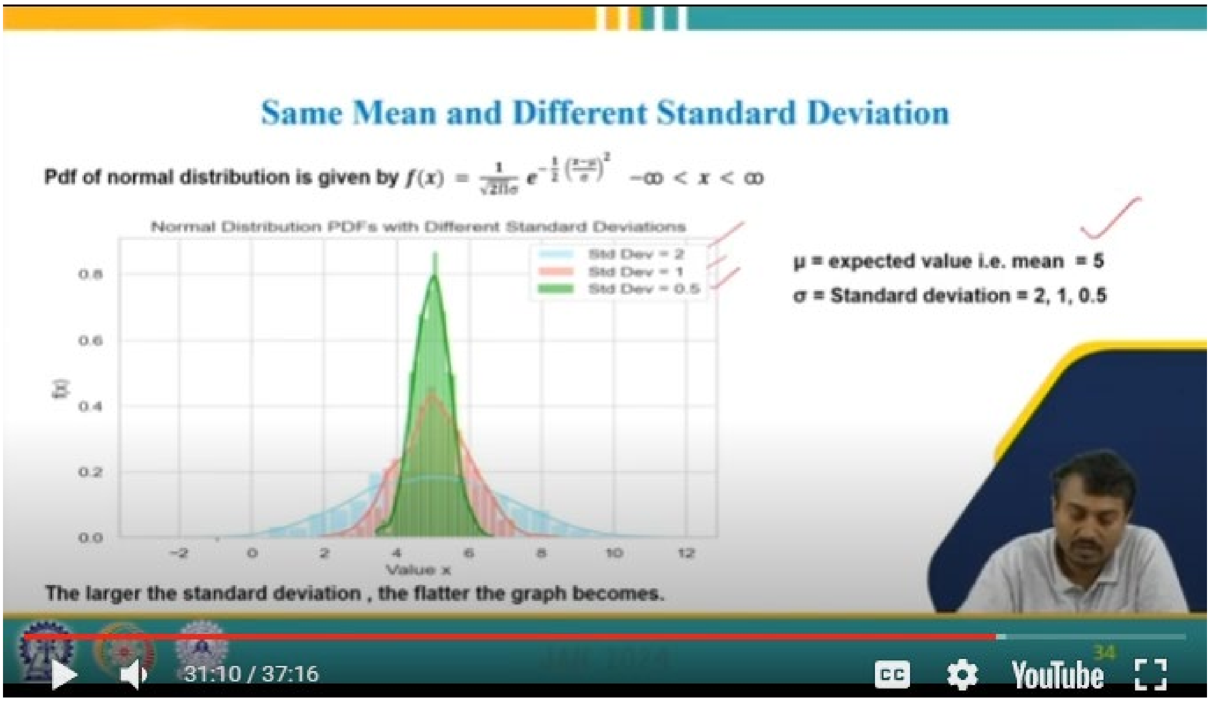
So, two minus lambda square minus one by lambda whole square equals 1 by lambda square. So, the variance of the exponential distribution is defined by one by lambda square. The normal distribution alternatively referred to as Gaussian distribution or Bell's shaped curve is a continuous probability distribution exhibiting symmetry around its mean which coincides with its median and mode. So, defined by its mean mu and standard deviation sigma the normal distribution takes on a Bell's shaped form. Consider paramount in statistics for both theoretical understanding and practical application the normal distribution hold significant importance.

So, the continuous random variable x follows a normal distribution. It is denoted by x of n mu sigma square where mu is the mean and sigma square is the variance. So, mu is the expected value of the mean and sigma square is the variance. And pd above the normal distribution given by fx is equal to 1 by root over two pi sigma and e to the power half x minus mu divided by sigma whole square for the range minus infinity to infinity. x varies from minus infinity to plus infinity. The shape of the probability distribution function, this is the Bell's shaped curve, and this is, basically, this is 5.

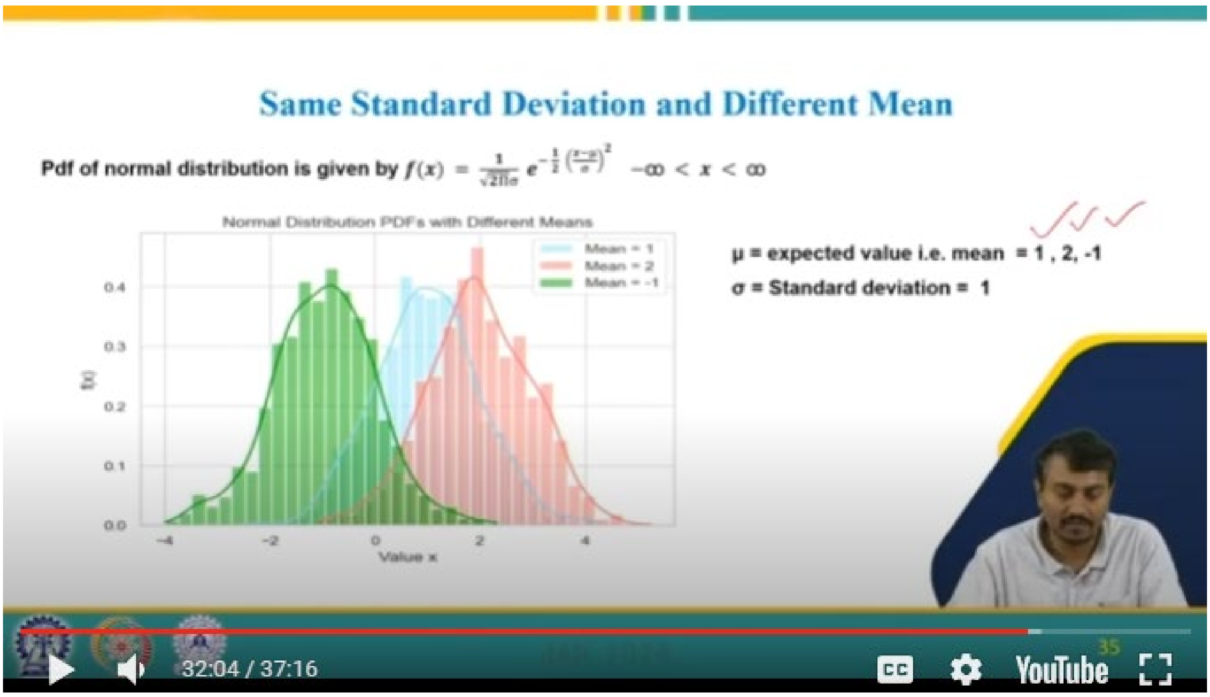
$$f(x) = \frac{1}{\sqrt{2\Pi\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

So, around 5, this particular curve is symmetrically distributed. This side and this side are symmetrical and here this is the height of the curve. So, for the expected value mean of 5 and the standard deviation of 2, this is the shape of the curve. For that same value, the height of the curve is around 0.

Two or approximated as 0.199. For mu is equal to 5, and sigma is equal to 2. Similarly, for sigma is equal to 1 and mu is equal to 5 height is 0.399 nearly 0.4. So, based on standard deviation, the height changes. So, here we have deliberated that for different standard deviations starting from 0.5, 1, and 2 with the same mean of 5, you see this was the distribution for standard deviation 2, this is the distribution for standard deviation 1, and this is the distribution for standard deviation 0.5. So, this shoot up and distribution spread is reduced for the less standard deviation or the lower standard deviation.



So, the larger the standard deviation, flatter the curve. So, here we see that the standard deviation remains the same that is 1, but the mean is different. So, the mean is taken as 1, 2, and minus 1, and for that, this basically shifts in the x-axis. Symmetric around mean, so the pdf of normal distribution given by fx is equal to 1 by root over two pi of sigma, then e to the power minus half of x minus mu by sigma whole square for the x range values of minus infinity to plus infinity. The probability density function of a standard random variable x exhibits symmetry around its mean. So, this means that the x is equally likely to occur on either side of the mean, that is, mu.



So, the probability of x being less than equal to mu is equal to that of x being greater than equal to mu, which is 0.5. So, the probability of x less than equal to mu is this side. The probability of mean x greater than equal to mu is this side, both are 50 percent, and that is symmetry, and that is the beauty of normal distribution. So, here in this particular curve, we are showing you this is the mean, the red color line dotted line this is the mean, and the curve is spread around the mean, and there are different domains, there is a blue domain that represents mu plus one standard deviation, then the green represents mu plus two standard deviations, and the orange indicates mu plus three standard deviations. So, it is a bell-shaped curve, and it is centered around the mean. We have identified the different zones in the horizontal axis between the value of mu plus three sigma and mu minus three sigma, the right side mu plus three sigma mu minus three sigma.

Approximation rule for the normal distribution. So, the interval between mu plus sigma and mu minus sigma, that is, mu plus minus sigma one plus minus sigma, encompasses 68 percent of the distribution, and the interval between mu plus minus two sigmas encompasses 95 percent of the distribution. The interval mu plus minus three sigma encompasses approximately 99 percent of the distribution. So, let us see the example x of normal distribution 10, 16. So, here, mu is equal to 10, and sigma is equal to 4. So, what is the probability of x between 6 and 14? So, 6 and 14 means ten minus 4 equals 6, 10 plus 4 equals 14.So, this is mu plus one sigma. So, this is nothing but the first zone. So, this is 68 percent, which is 0.68. For the same, the probability of less than six and the likelihood of less than 16 minus 4 equals 6.

So, mu minus sigma, so this particular zone. So, this was 68.68. So, both sides are total, and this side was 0.32. This is symmetry. So, 0.32 divided by two is equal to 0.16. The probability of x being more significant than 14, similarly greater than 14, 6 plus 4, is 0.16, similar to the 6. The probability of x is more important than 22; 22 is nothing but 10 plus 3 sigma, three into 4, 12 plus 10, 12.

So, more significant than three is 0. So, these are the references. So, we have covered the continuous random variable in this lecture, followed by the PDF probability distribution function and its properties. Then, we have covered the cumulative distribution function, which is CDF and uniform distribution, their expectation and variance, exponential distribution, its expectation and variance, and normal distribution approximation rule. Thank you.