Engineering Mechanics
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Module – 01
Lecture - 02
Equilibrium I

In the previous lecture, we had been looking at vectors in particular how to represent them algebraically and then we looked at the product of vectors.

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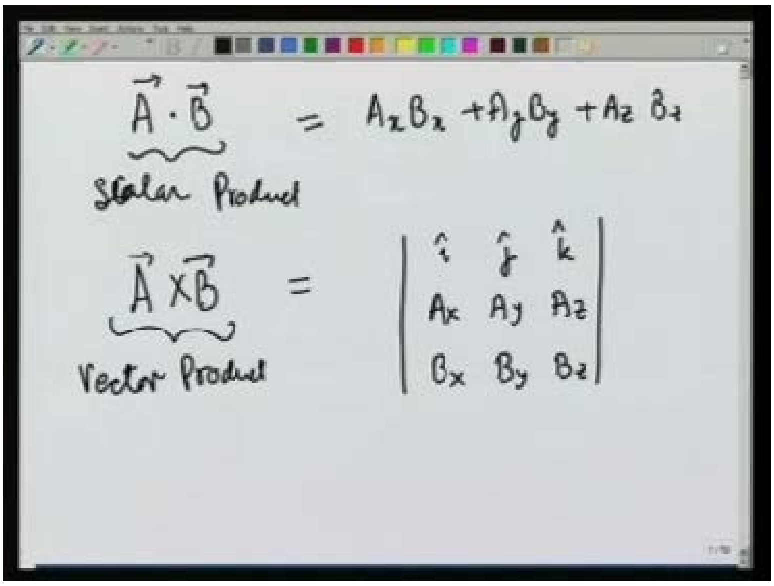

And defined a scalar product which we wrote as Ax Bx plus Ay By plus Az Bz and a vector product which gives me a vector A cross B an in a short way I wrote this as determinant of Ax Ay Az Bx By and Bz. And I had asked you to check that this gives the correct answer we define this to products in this manner because, this satisfy certain transformation properties of vector components under rotation.

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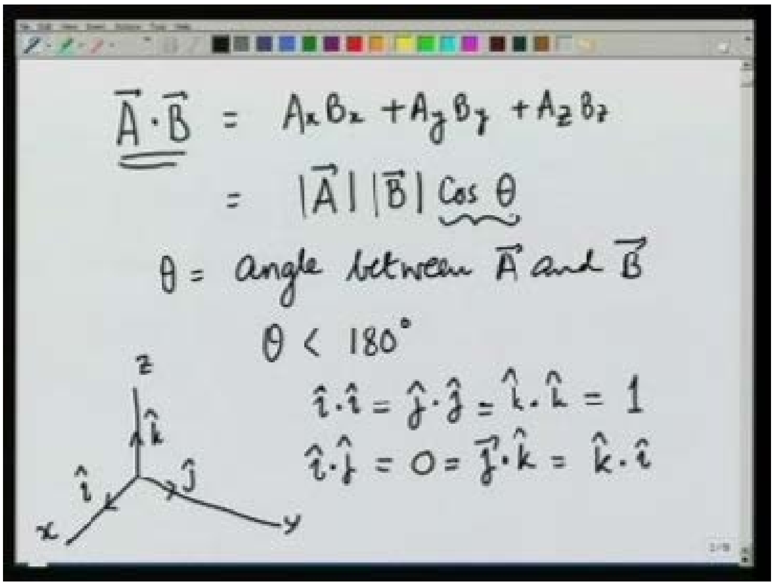

Let us look, at these products now slightly more carefully. So, let us look at the dot product A dot B which is Ax Bx plus Ay By plus Az Bz this can also be written as magnitude of A times, magnitude of B times cosine of theta where theta is the angle between A and B and theta is a smaller of the angle. So, theta is less than one eighty degrees. In particular if I look at the unit vectors along x direction, y direction and z direction we find that i dot i same as j dot j they are all equal to one because, their magnitude is one and the angle between j and j or k and k is zero.

Similarly, i dot j is zero. So, is j dot k and. So, is k dot i notice that the dot product or the scalar product can be both negative and positive because cosine theta can be negative or positive.

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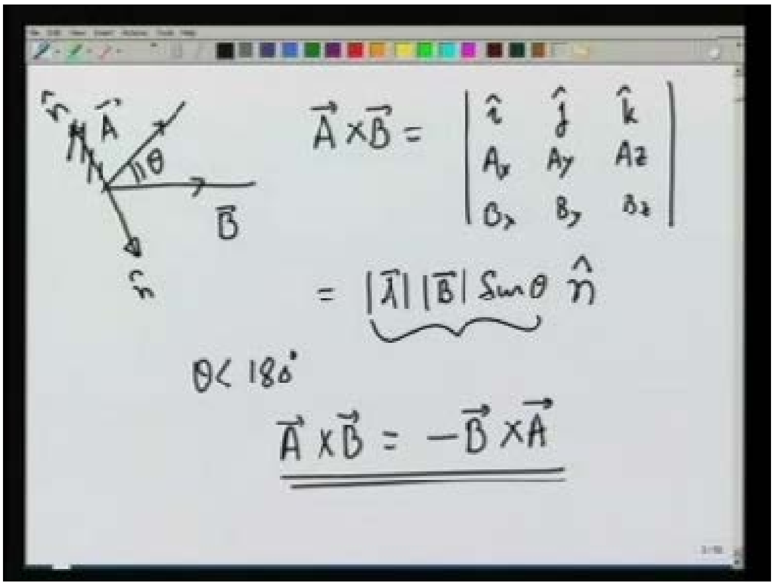

Similarly, if I look at two vectors A and B then A cross B which I can write as the determinant i j k Ax Ay Az Bx By Bz is also equal to magnitude of A times, magnitude of B times sin of theta where theta is the smaller of the angle between A and B. So, less than 180 degrees times a unit vector in direction n where n is perpendicular to both A and B so, it is perpendicular to the plane form by A and B.

So, it will be somewhere like this may be coming out of this plane of the board how about whether n is this way or this way this is answered by right hand rule you turn A towards B through the smaller of the angle and the thumb gives you the direction of the vector product. So, in this case n would be going down you can see that A cross B is going to be equal to minus B cross A because the magnitude will remain the same. But the direction of n would change because now you will be turning B towards A this is also clear from the components if you look at them carefully.

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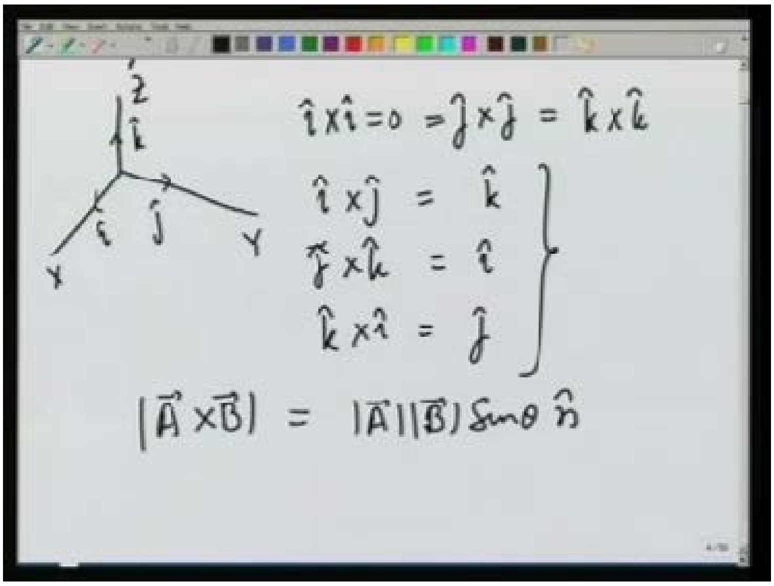

So, this is how the cross product was in particular if I look at the unit vectors i j and k in x y and z direction then i cross i is 0 and. So, is j cross j because the angle between them is 0 and. So, is k cross k on the other hand i cross j is equal to k j cross k is equal to i and k cross i is equal to j these are the relationships between unit vectors. Let us now look at 1 or 2 examples where you see these relationships carefully first I want show you that A cross B through an example is really equal to A B sin theta times n where n as I told you is in the sense of rotating A towards B through this small were the angles.

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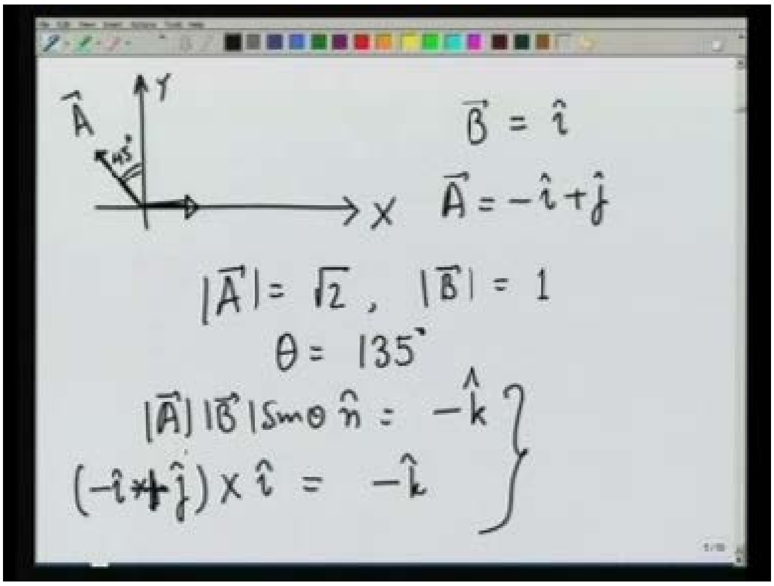

So, let us for simplicity take so, let us for simplicity take this as x axis this as y axis consider B to be a vector unit vector in x direction. So, B is equal to i and consider A vector to be in this direction which is 45 degrees from the y axis or the x axis. So, A is equal to minus i plus j you can see that the magnitude of A is square root of two magnitude of B is one and the angle theta between them is 135 degrees. So therefore, A B sin theta n is going to be equal to sin of 135 is one over root two is going to be equal to 1 times n turn A towards B through this smaller angle and that gives you a direction in the opposite to z axis.

So, this is going to be minus. Let us see if you get the same answer if I explicitly do component wise a vector product in that case A cross B would be minus i cross sorry plus j cross i i cross i is 0 and j cross i is minus k. So, you see both give you the same answer. Now, I want to demonstrate to you that A cross B gives you a vector that is perpendicular to both A and B.

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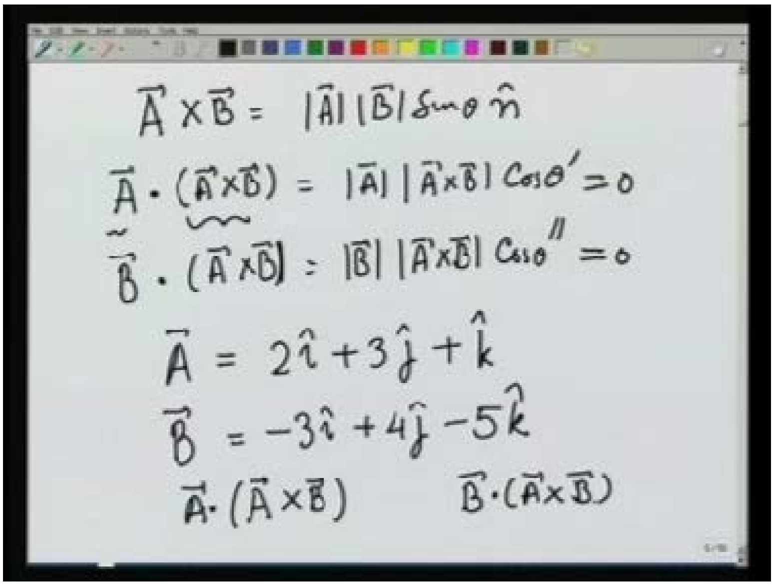

Since A cross B is equal to modulus A modulus B sin theta and where n is perpendicular to both A and B. That means, that A dot A cross B which is equal to modulus of A modulus of A cross B cosine of theta that’s called theta prime. Because now, this is the angle between A and A cross B should be 0 because A cross B is in direction perpendicular to A. So, should be B dot A cross B which is equal to modulus or magnitude of B magnitude of A cross B cosine of this a theta double prime again since A cross B suppose, to be perpendicular to A and B both this should also be 0.

Let us take an example, let us take A to be a vector let us say two i plus three j plus k and B to be a vector say minus three i plus 4 j minus 5 k. We want to take that cross product or vector product and show that the resulting vector is really perpendicular to both A and B. So, what we will do is we will calculate A cross B and then take dot product with A and take dot product with B.

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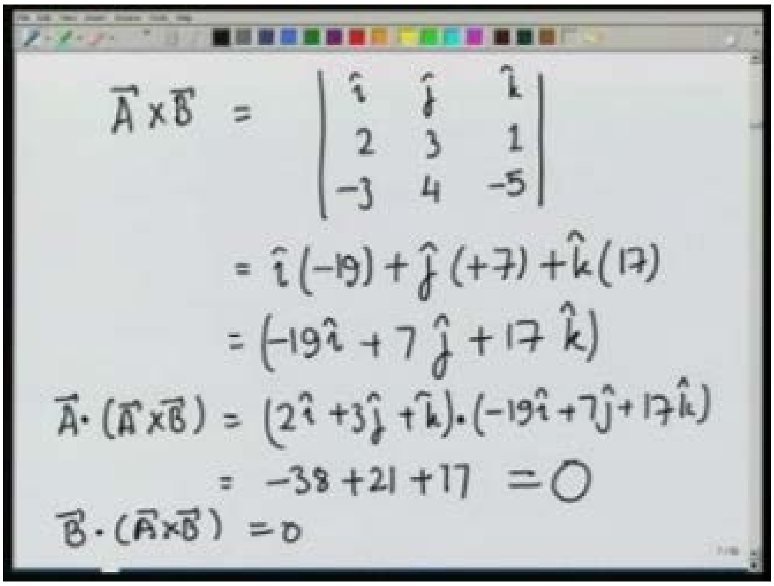

So, let us do that A cross B is going to be equal to i j k plus vectors were 3 2 3 1. So, 2 3 1 and minus 3 4 and minus 5 minus 3 4 and minus 5. So, cross producted i minus 15 minus 4. So, minus 19 plus j minus 3 plus 10. So plus 7 plus k 8 plus 9 17. So, this gives me a vector minus 19i plus 7j plus 17k this is A cross B. Let us take A dot A cross B which is equal to 2i plus 3j plus k dotted with minus 19i plus 7j plus 17. Now, I can use the distributive property and show that i dot i is one.

So, this is going to be minus 38 i dot j is 0 i` dot k is 0. So, no contribution from here plus 3 would multiply only the j component plus 21 and k would be plus 17 which is equal to 0. Since, A and A cross B both magnitudes are non zero this can only happen if the angle between A and A cross B is 90 degrees I, leave it for you to show that B dot A cross B is also 0 thus we see that, A cross B is perpendicular to vectors A and B both.

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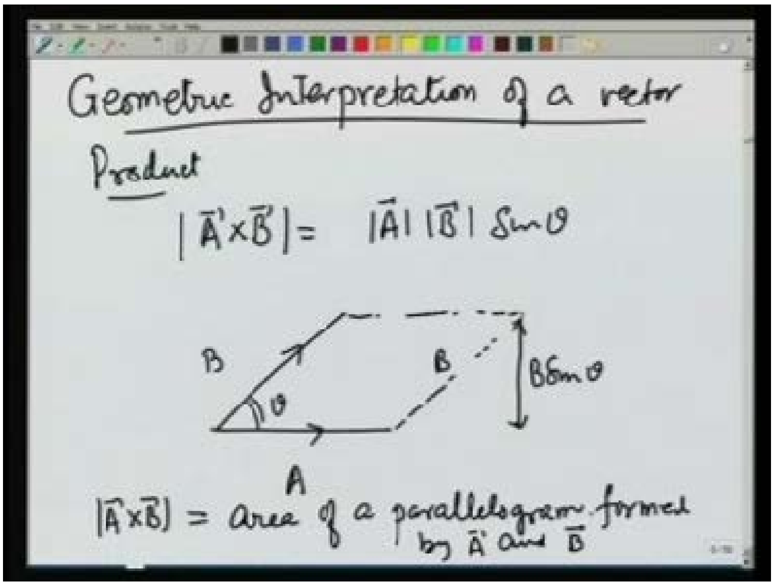

Let us now look at, geometric interpretation of a dot product or cross product of a vector product A cross B magnitude is nothing but, magnitude of A is magnitude of B times sin of theta where theta is the angle between them. So, if there is a vector A and there is a vector B and angle between them is theta you can see that A cross B is nothing but, A times B sin theta this is B magnitude. So, this is B sin theta. So, this is the base times height of this parallelogram. And therefore, magnitude of A cross B is the area of parallelogram formed by A and B.

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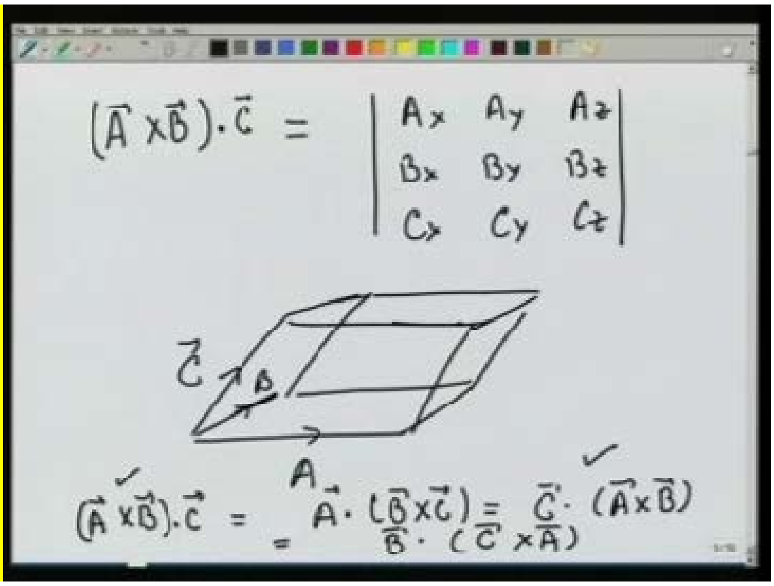

Next we look at A cross B dotted with C this I, leave for you to show that is Ax, Ay, Az, Bx, By, Bz and Cx, Cy and Cz determinant its values equal to this and this has a geometric interpretation of if, I form a parallel pipette by vectors A, B and C it is the volume of this parallel pipette. So, that is the geometric interpretation. Since I, can write A cross B dot C as this determinant form is also shows since determinant does not change if, I change its rows twice this is also equal to A dot B cross C. This is also equal to C dot A cross B which is same as this and there is one more term which is also equal to B dot C cross A all these are equal. This is cyclically the dot product and cross product go like this.

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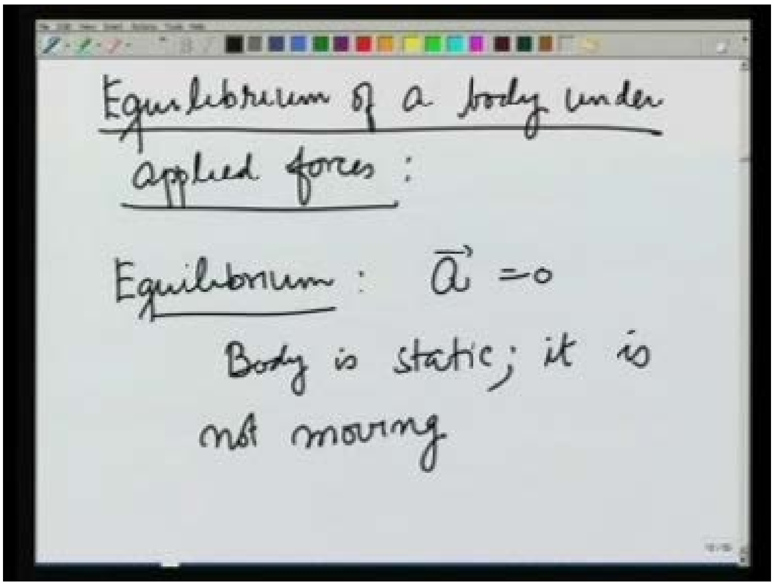

So, this is a general review of vectors that we have done and now we are ready to look at Equilibrium of a body under applied forces. As I said, in my previous lecture by equilibrium in general we would mean that the acceleration is 0, but here we would also mean that the body is static it is not moving. So, let us look at the conditions at a necessary and sufficient to make sure of that. Let us go step by step and look at a point suppose, I have a point mass and I want it remain stationary, but there are several forces applied on it.

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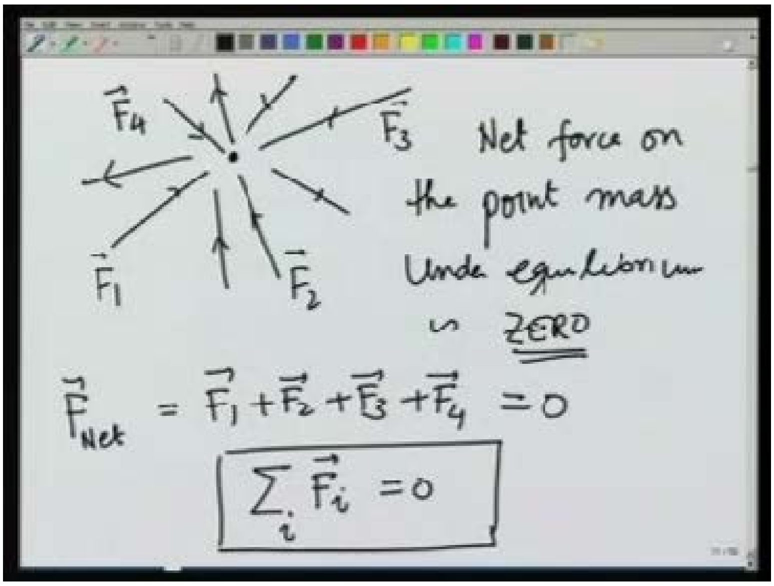

So, this could be F1 this could be F2 this could be F3 and this could be F4 if, the body does not move and its acceleration is 0; that means, the net force on the point mass under equilibrium is 0. And that means, in this particular case at F1 plus F2 plus F3 plus F4 the vector sum of all these forces gives me the net force and that must be 0. In general if, there are more than 4 forces there may be n forces I should have 1 condition summation over i all forces is equal to 0 this is one condition for equilibrium.

So, there could be more than 1 force, more than 4 forces in all sort of directions this direction, this direction, sum of all that must be 0. In order for the point mass to be in equilibrium I have been talking about point mass because in that case we are all just up to…it cannot do anything else just move. On the other hand in engineering problems we do not always have a point mass.

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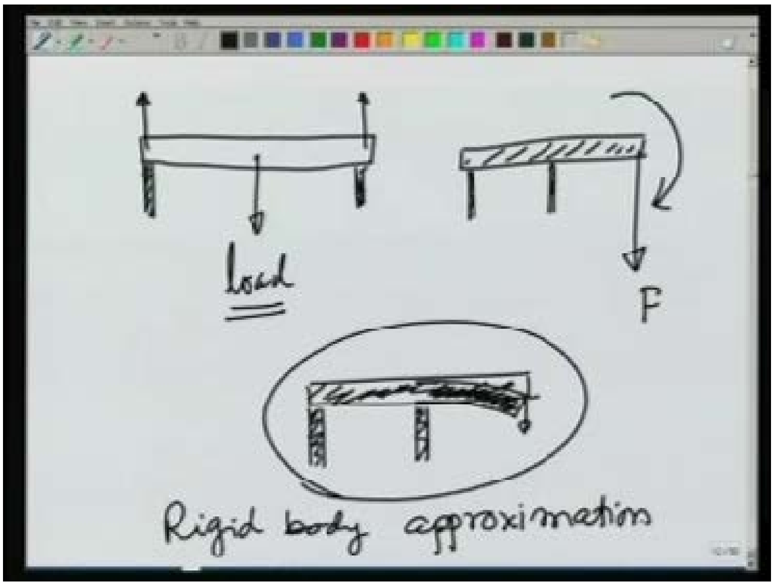

We have extended bodies for example, I could have a beam that is supported on 2 pillars and there is a load here and the 3 forces that is one force. Force applied by the support in this direction, force applied by the other support in this direction and the load not only these three forces should be 0 there should be some other conditions, what are those conditions? We want to look at them. When we apply force on an extended body experience tells us, that the extended body not only can move you can have some more effect on the extended body.

For example if, I put the support here one support here and apply the force this way you know that this body is going to rotate like this. Other effect that the force can have is again looking at the same beam being supported by two supports, two pillars. And if, I apply a force here it could bend like this. So, there all sorts of things that a force can do besides has giving it a motion it can make the body rotate or make it bend.

In this course, we are going to assume that this does not happen. The body is so rigid its internal forces so good that they adjust and do not let the body to deform this is called as rigid body approximation. The body so rigid that, it does not deform at all. So, only two effects then the force can have is number 1 move it number 2 rotate it.

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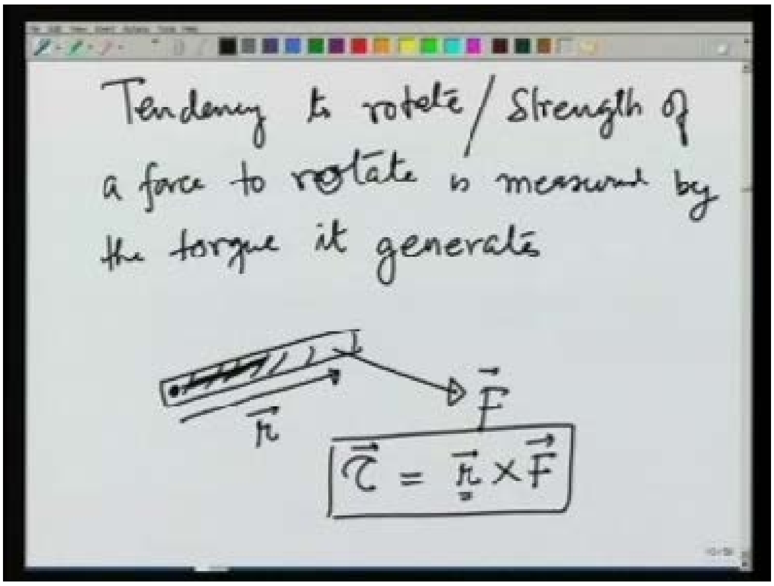

So, we are going to consider these two things in this course for equilibrium the tendency to rotate or strength of a force to rotate is measured by quantity call the torque; by torque it generates. For example suppose, I have a body which is held at 1 point and if, I apply a force in this direction at distance at displacement r from the point at which it is held. Then the torque is given by r cross F or the vector product of the displacement times point at which the force is applied. Experience tells us further the force on the point at which the body is held more it rotates more it tends to rotate.

So, therefore, the distance is important, the vector product is important because, if the force is applied in the direction of this point, the force is along the same direction it will not rotate. So, vector product in that case would be 0. So, this is how the torque is defined.

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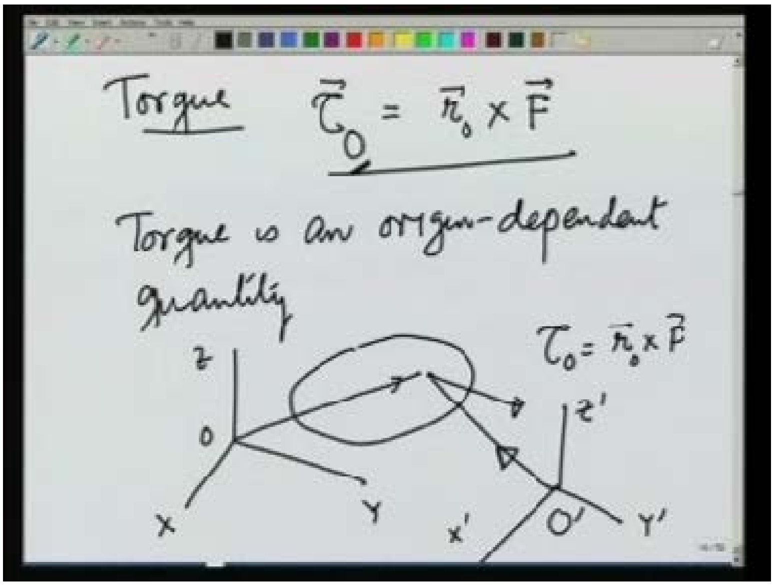

So, let us write now, torque tau from the origin o is going to be given as r measured from o cross F ro measures the point at which the forces being applied compare to the origin r. The point at which the body is being held naturally when you define torque like this; it is an origin dependent quantity. For example, let us take x, y and z axis plus take a body here if, I apply a force at this point in this origin the torque would be r cross F it is a ro on the other hand. If, I have a different frame of reference centered at o prime the torque you can see is going to be different. How do we relate the direction of torque, to direction of rotation.

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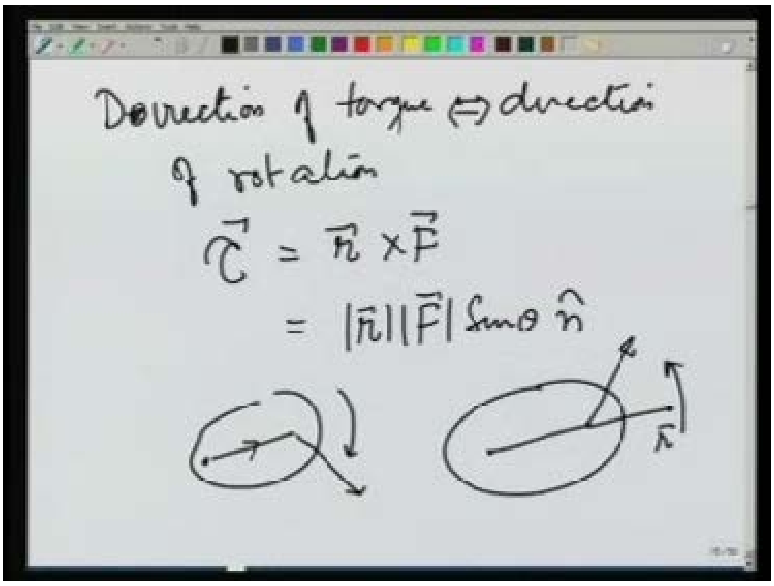

So, if torque is given by r cross F it is going to be equal to magnitude of r, magnitude of F sin of theta n where n is the vector that is given by this thumb, when r is turned towards F through the small angle. The tendency of rotation is also the same if, n is the direction of torque the fingers right hand fingers give me the direction of rotation. As you can see if, there is a body here held here is a force is this way then r cross F is going to go into the board.

So, body you can see as a tendency to rotate this way on the other hand is a force is this direction then r cross F is coming out of the plane of this, this board and the body has a tendency to rotate in this manner. So, this is how we relate the torque direction and the direction of tendency of rotation. So, if we want that in addition to not moving the body also does not rotate for equilibrium then, the torque on the body should also be 0.

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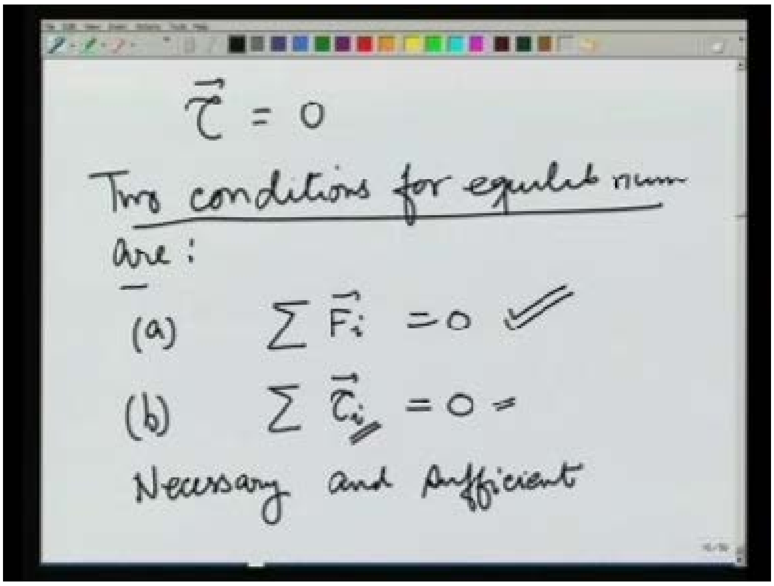

So, two conditions for are the sum of all the forces Fi is 0. Then, the body would not move and B sum of all the torques applied on the body should also be 0. You may ask, I have just now said the torque depends on the origin and I have not put any origin here. Once you satisfy this condition at sum of all the force is 0 the torque, the net torque is going to be independent of the origin as we show later.

So, therefore, this condition is also independent of the origin. Once, I make sure that summation of Fi 0 and these two conditions are necessary and also sufficient for equilibrium of a body.

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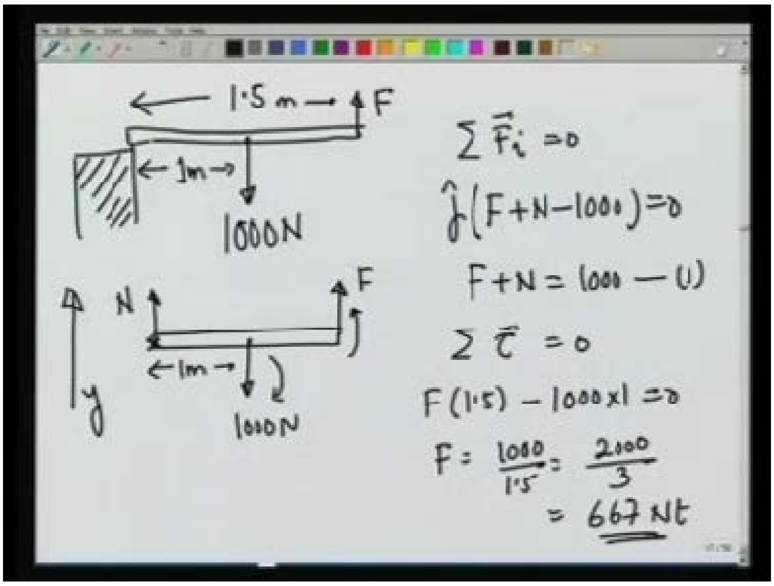

Let us take an example a simple 1 to start with suppose: I have a beam which is being supported, on a block of say cement, the beam is light weight. So, I can ignore its mass suppose it is holding a weight of 1000 Newton's, but force F should I apply in order to hold this weight. Let us say this distance is 1.5 meters and the load is being held at a distance of 1 meter. If I look at the beam the forces that are being applied on the beam are the force by me force of 1000 Newton at a distance of 1 meter from the support and the support could also be applying a force.

So, first condition that summation Fi is equal to 0 gives me if, I assume this to be the y direction that j unit vector F plus N minus 1000 is equal to 0 or F plus N is equal to 1000 that is condition 1. On the other hand torque should also be 0 about any point, I take. So, let us take this point about which, I am going to take the torque. And then you see that 1000 Newton gives me a torque clockwise the F gives me a torque counter clockwise and there should both balance each other.

So, if I write summation tau is equal to 0 since all the torques are in the same direction up or down, I can straight away write this as F times 1.5 minus 1000 times 1 should be equal to 0 or the force is 1000 over 1.5 which is 2000 over 3 or 67 Newton's approximately. So, you see this is how I am going to apply the condition for equilibrium for the force and the or torques.

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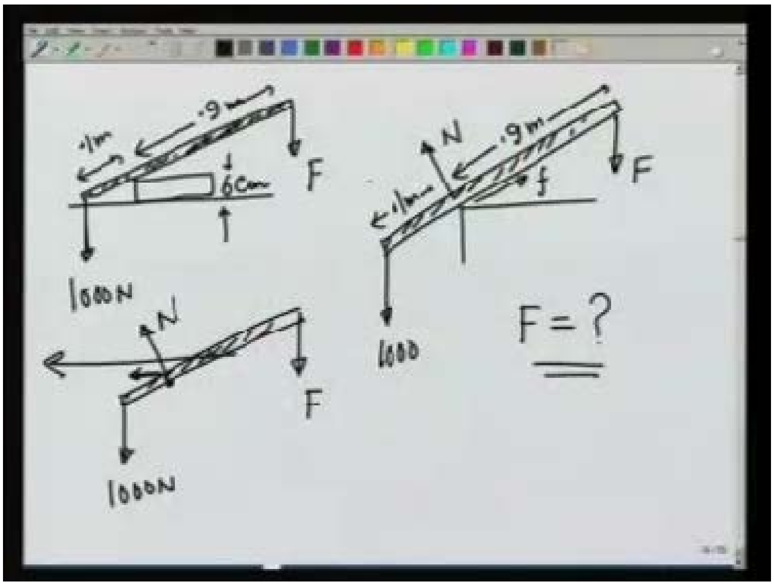

For the second example let us take, the case of lever that we use everyday. Let us suppose, we have a very hard rod which is put on a brick on the ground. Let us say the height of the brake is 6 centimeters. Let this length be point 1 meter, let this length be 0.9 meters and I am trying to lift a load which is pushing the rod down of 1000 Newton's by applying a vertical force here what would happen?

Suppose, initial I ignore the friction between the brick and the rod then you can see that on the rod the forces that, we are applying are number 1 the force downwards number 2 the force downwards because, of this load 1000 Newton's. And at this point the brake is going to apply a force perpendicular to the rod in this direction you can see right away if, there is no friction this N is going to have the horizontal component that is not balance. So, the rod is going to have a tendency to move this way in order that is the rod does not move, there should also be a frictional force between the rod and the brick.

So, we are going to assume that, there is a sufficient frictional force that, prevents the slipping to the left. So, let there be a frictional force in this direction f, let there be a normal reaction n. Let there be the force that, we are applying F vertically down and let there be a force 1000 Newton's here this distance is 0.1 meter this distance is 0.9 meters and the friction is sufficient to prevent slipping. We want to know the value of F how much force should I apply down in order to lift this weight of 1000 Newton's.

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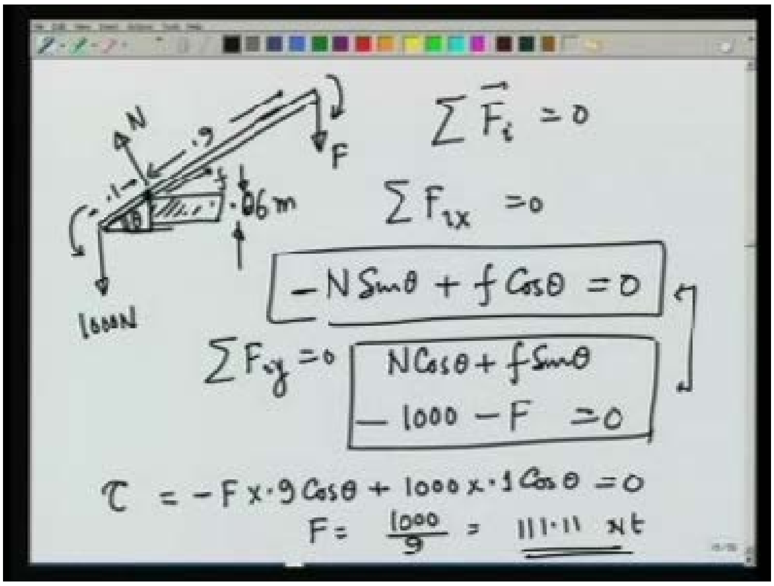

Again we are going to apply the 2 conditions that we talked about this is a brick of height 0.6 0.06 meters 6 centimeters. Let this angle be theta this is 0.1 this is 0.9, this is N, this way is frictional force F, this is F and this is 1000 Newton's. So, when I first write the force condition summation Fi is equal to 0. Let us, write its x and y components. So, summation Fix is equal to 0 gives me N sin theta which is in a negative direction plus f cosine of theta is equal to 0.

Similarly, when I write summation Fiy the force component y direction is 0 this comes basically by writing force in terms of i and j and making each component of the vector 0 this gives me N cosine of say theta plus f sin of theta minus 1000 minus F is equal to zero. So, these are the two conditions for the force balance. How about the torque? For the torque let us choose a convenient point because it does not matter which point I choose.

Let us choose the support right here, as a point about which I will take the torque then you can see that this F has a tendency to rotate the rod clockwise and this has a tendency to rotate counter clockwise and therefore, two are working in opposite directions. The torque is all going to be this we take this plane to be xy plane torque is going to be in k or minus k direction. And therefore, I can just write the scale of numbers right away it is going to be equal to clockwise would give me a minus k.

So, F times 0.9 cosine of theta minus sign plus 1000 times 0.1 cosine of theta is equal to 0. And therefore, F comes out to be 1000 over 9 which is roughly 111.11 Newton's and that is your answer. If, you wish to find what N and F are those can be find from these equations you will require sin theta and cosine theta for that and that you can obtain from this triangle.

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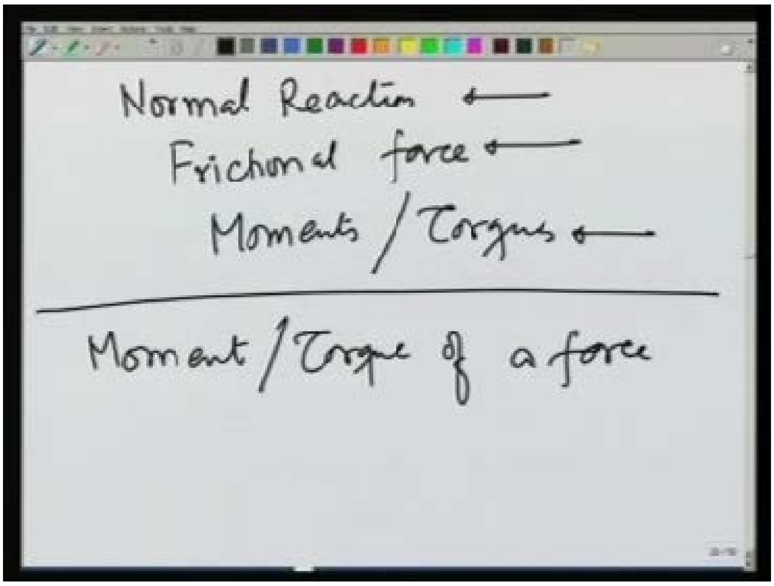

So, in these 2 examples you saw that, I have used terms like Normal Reaction, Frictional force, moments, torques etcetera. So, now in the specialized and look at each of these things more carefully for example, if I take a support in which direction does it apply the Normal Reaction. In which direction are the Frictional Forces what are different types of Frictional Forces? Will study moments in slight the more detail to understand how it can affect the motion? And what special moments are.

So, will take these topics 1 by 1 I want to start by discussing moment of a force moment or torque of a force as, indicated earlier.

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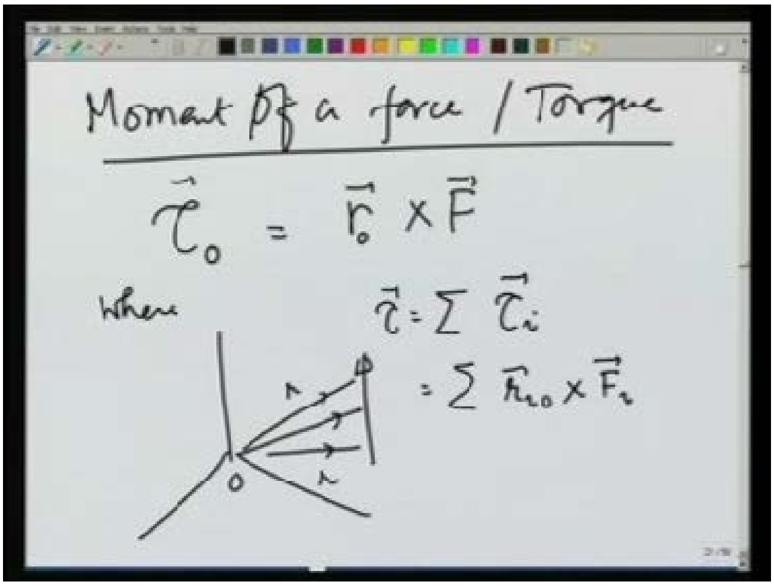

The moment of a force or torque is given by r cross F, where r is a vector from the origin to any point on the line of action of the force. If there are many forces then, the total torque is going to be equal to sum of all these torques which is going to be summation rio cross Fi. Let us see that, through an example.

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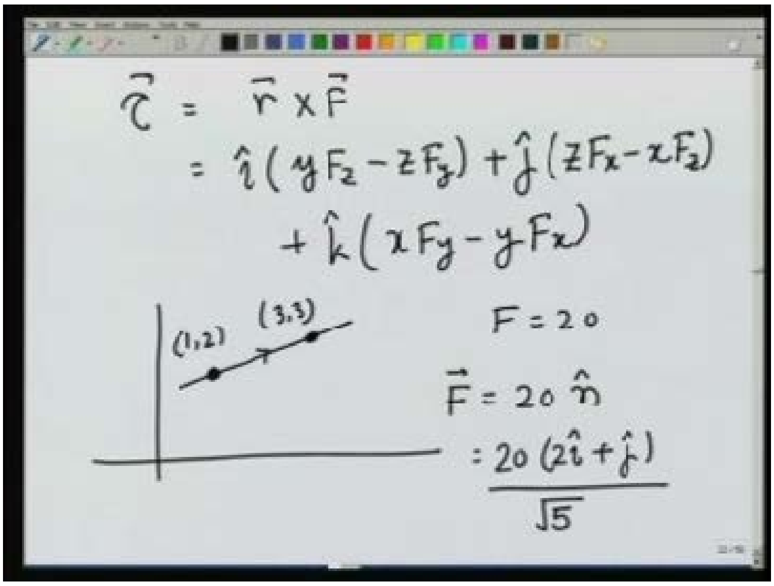

So, if I am calculating the moment tau which is r I am dropping o, now it is understood that it is depending on the origin cross F is going to be equal to x component x direction is going to be equal to y Fz minus zFy plus component in the y direction is going to be zFx minus x Fz plus the component in the z direction is going to be xFy minus yFx. So, let us take a simple 2 dimensional example, let there be a force of 20 Newton's acting along the line which is passing through say 1 2 and 3 3. So, that if I would write the forces of vector, it would be twenty times a unit vector in this direction which I can write as 20 times a vector. In this direction would be 2i plus j divided by square root of the magnitude of this vector 2i plus j which will be square root of 5. So, this is the force that is being applied.

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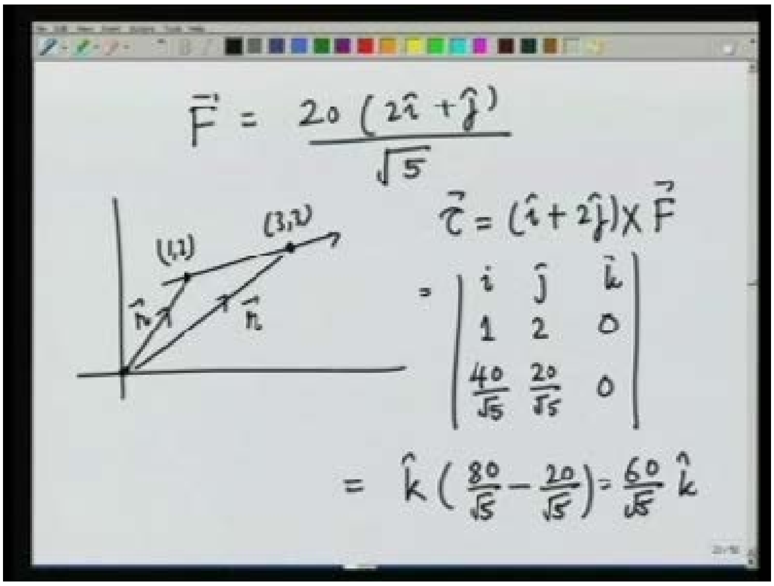

So, that we are applying is F equals 20 2i plus j over square root of 5, it is being applied along the line which is passing through 1 2 and 3 3. Let us calculate the torque about the origin by first taking r from 0 to 1 2. So, that this vector is going to be i plus 2j and we are crossing it with F which is going to be nothing but, i j k the determinant way i is 1 2 0 for the force it is 40 over root 5 j component is 20 over root 5 and 0. This gives i component 0, j component 0 and k component is going to be 80 over root 5 minus 20 over root 5 which is 60 over root 5 k that is the torque. What will torque be if, I take r instead to be a vector from 0 the origin to 3 3. Let us calculate that.

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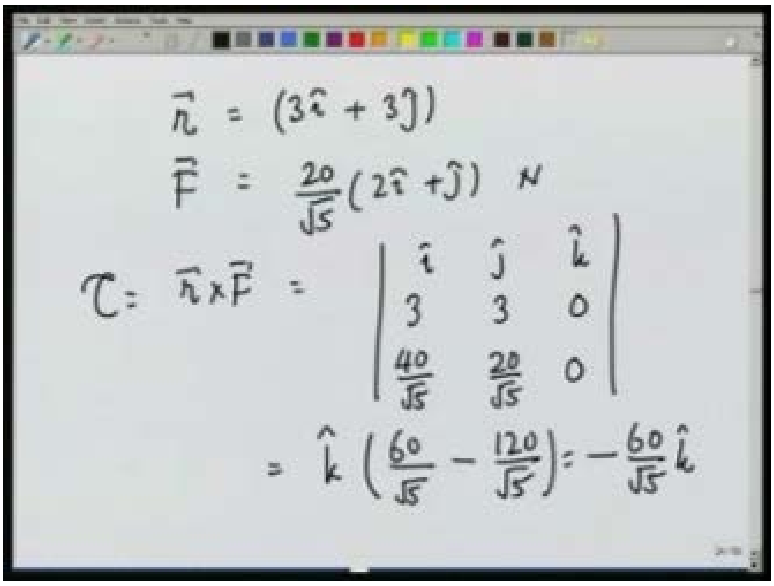

So, that vector is going to be 3i plus 3j force of course, is 20 over root 5 2i plus j Newton's. So, if I calculate torque which is r cross F, it is going to be equal to i j k this determinant 3 3 0 40 over root 5 20 over root 5 and 0. Again it gives only non zero component direction k is going to be equal to 60 over root 5 minus 120 over root 5 which is minus 60 over root 5 k. The direction seems to be opposite, I must, made a mistake in the previous part.

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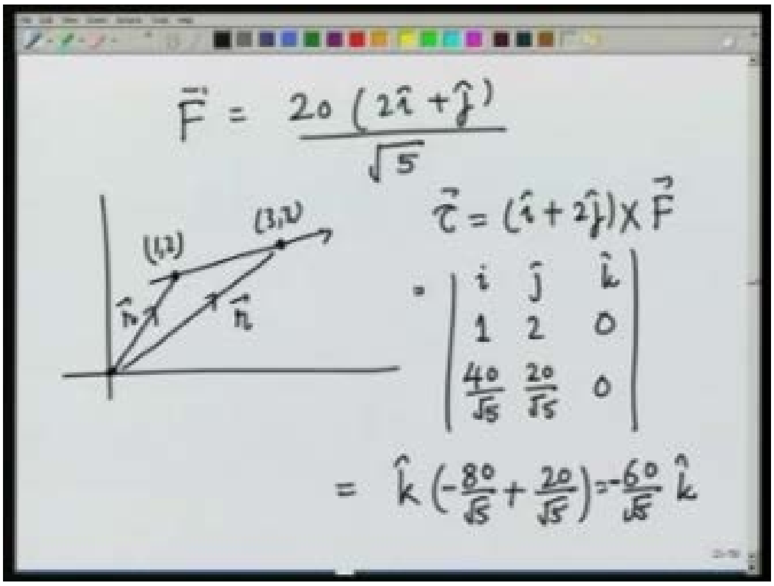

Let us check that and yes indeed there is a mistake, this should be minus, this should be plus. So, in the previous part also the answer was minus 60 over root 5 k and this part also the answer we get is minus 60 over root 5 k.

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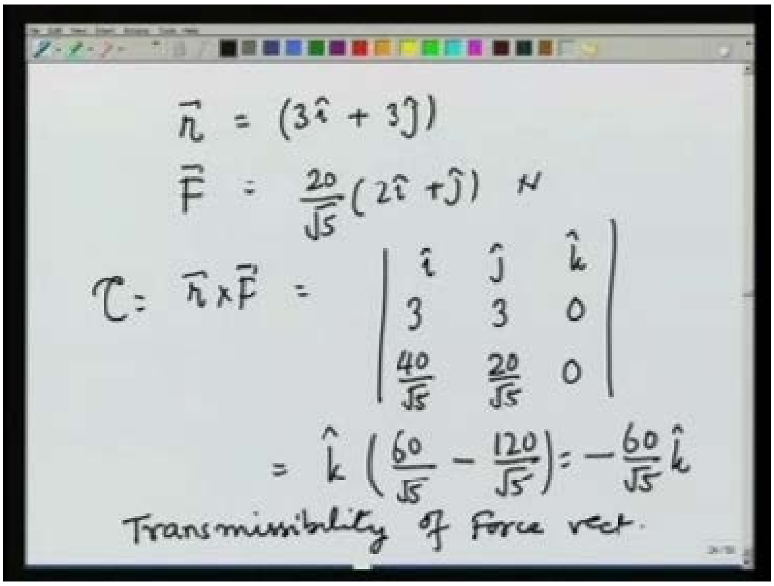

And therefore, you see that no matter where, I take the vector r to be along the line of application of the force the torque comes out to be the same. This is also an example of transmissibility of force vector that, we talked about in the previous lecture.

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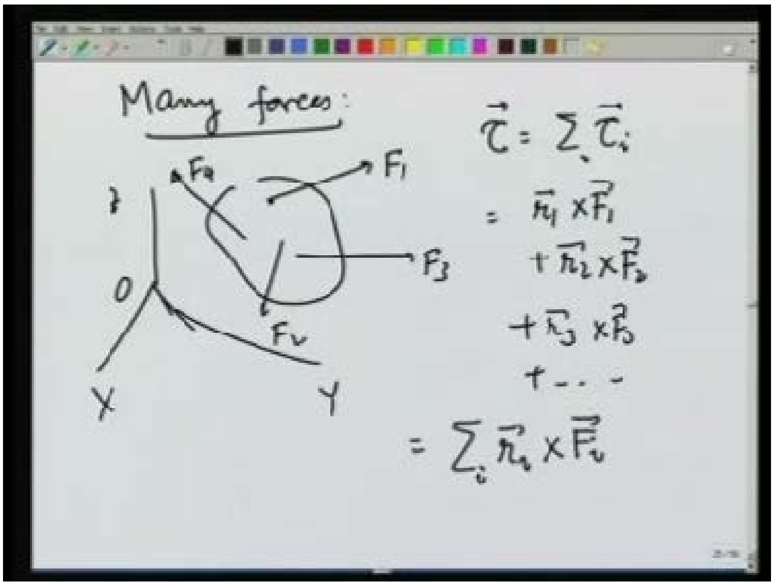

As, I said earlier if, there are many forces on a body origin x, y, z there many forces working F1, F2, F3, F4 and so, on. Then the total torque would be equal to summation of individual torques which will be equal to r1 cross F1 plus r2 cross F2 plus r3 cross F3 and so, on which is equal to summation ri cross Fi, where ri is a vector from the origin to any point on the line of action of the force Fi.

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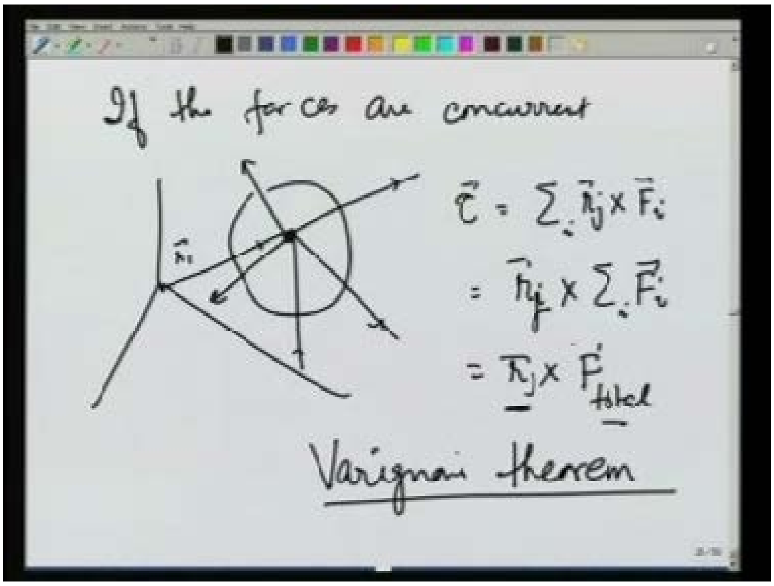

If, the forces are concurrent that means they all meet at some point this may be 1 force, this may be the other force, this may be third force, fourth force, fifty force and so, on. Then the torque, I can take this point where they meet to be the point where, I am going to take the torque displacement. And therefore, this is going to be rj. Let us call this rj cross Fi this equal to a rj can be taken out because, you same for all forces cross Fi which is nothing but, rj cross F total.

So, if some all the forces meet at a certain point, the torque due to all the forces equal to the torque of the total force cross with the direction up to that point. This is known as Varignon's theorem and through only if, when all the forces are meeting at a certain point.

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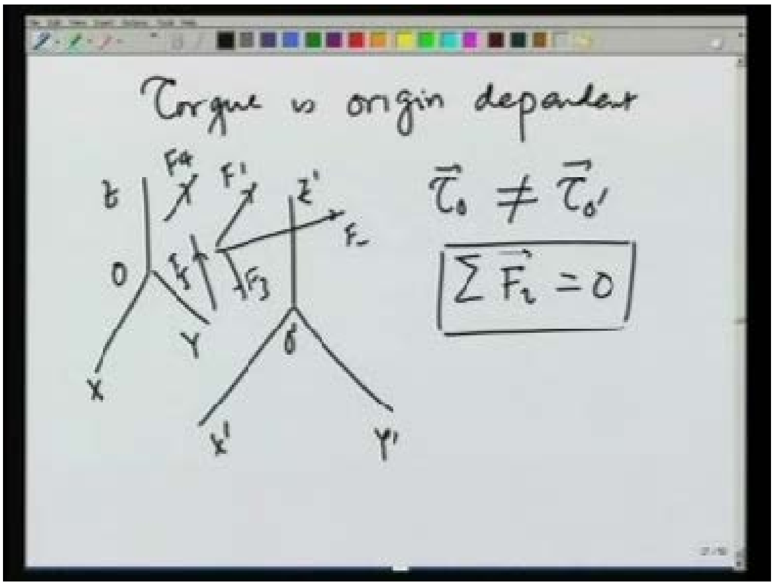

In general since, the torque is origin dependent therefore, if I take 2 particular frames o and o prime x, y, z, x prime, y prime, z prime, tau o is not going to be equal to tau o prime. For a given set of forces F1, F2, F3, F4 may not even cross then F5 and so on. However, there is a very special circumstance under which the 2 torques are equal and that is when summation of Fi is equal to 0 and I am going to show that to you.

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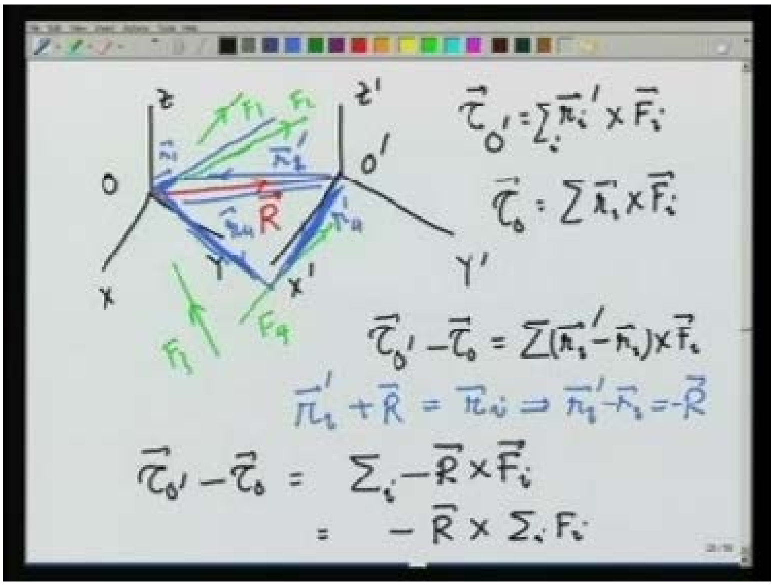

Let us take 2 Different frames o, o prime x, y, z, x prime, y prime, z prime. Let a distance vector from displacement vector from o to o prime be R, I am going to draw in green the forces that this be F1, that this be F2, that this be F3, that this be F4 and so, on. Then the torque about o prime is going to be equal to ri prime cross Fi summed over i. And torque about o is going to be summation ri cross Fi where ri and ri prime I take to be the vectors let me draw them in blue touching at the same point.

So, let us say if, I extend this force like this, this could be r1, this could be r2 and up to this point, this is r1 or for F4, this is r4 and this is r4 prime, this is sorry this should be r one prime. So, if I now, take the difference between tau o prime minus tau o this is going to be equal to summation ri prime minus ri cross Fi and from the figure it is clear, this is r4 prime this is r and this is r. So, ri prime plus R is equal to ri like r4, this is r4 is equal to R plus r4 prime similarly r1 is equal to R plus r1 prime.

So, ri prime in plus R in general ri and therefore, ri prime minus ri is equal to R with the negative sign. And therefore, tau o prime minus tau o is equal to summation i minus R cross Fi are being the same vector, I can write this as minus R cross summation i Fi. That is the difference between the torques taken about point o prime and point o.

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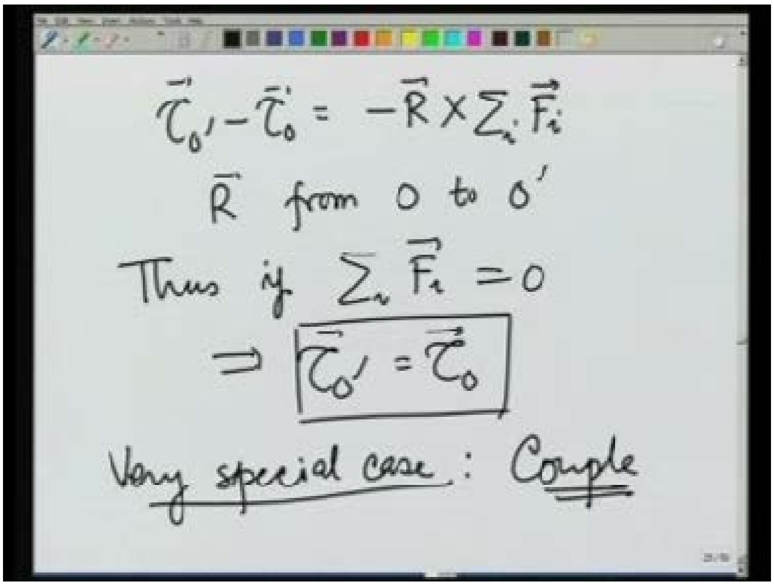

So, let us write this again tau o prime minus tau o is equal to minus R crossed summation i Fi R is a vector from o to o prime. Thus if, the total force summation Fi is equal to 0 this employees tau o prime is going to be equal to tau o the torque becomes independent of the origin, because I took o and o prime arbitrarily this is what I promised you earlier as show that if, the total force on a system is 0. Then the torque is independent of the origin no matter about which point, I take the torque it will always come out to be the same. A very special case of this, is something called a couple which is used in engineering mechanics quite a lot. And what is the couple?

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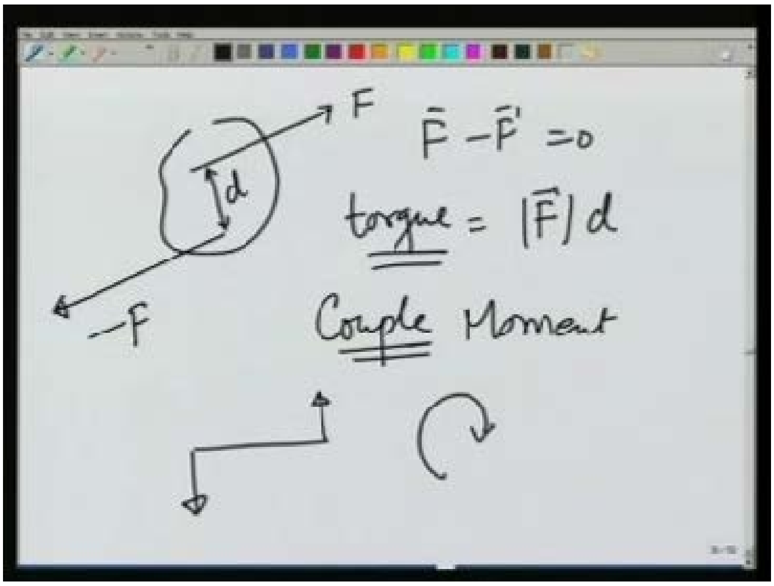

A couple is suppose, I take a body apply a force in 1 direction apply a force in the opposite direction minus F. And they displaced parallely from each other by a distance d the net force F minus F is equal to 0 and the torque may not be 0, because of forces are displaced from each other is going to be magnitude of F times d where d is the perpendicular distance between the 2 and this is known as a couple or couple moment. It is indicated either by this symbol where these 2 arrow show the forces applied or a symbol like this.

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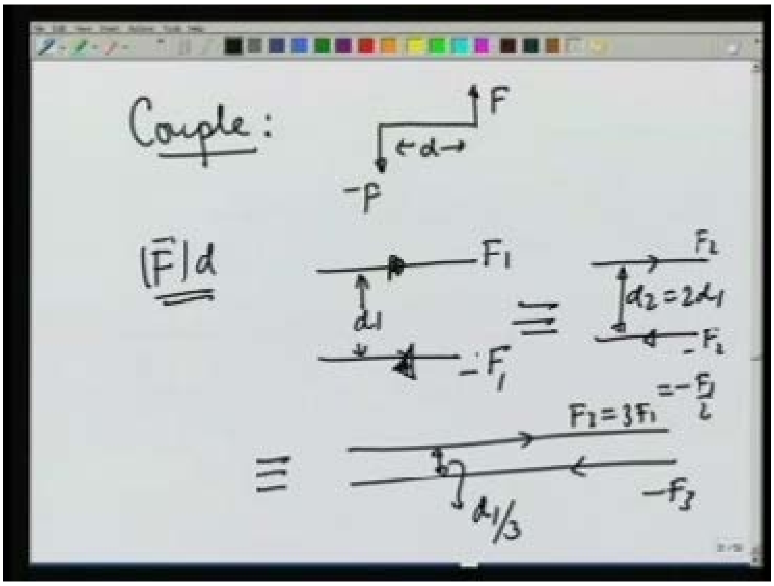

So, it is a pure moment without any force. Thus, a couple is a force the 2 forces F minus F displaced by parallel distance between them d as moment is given by F d. Since, the net force is 0 a couple the same couple can be generated by different kinds of forces. For example, I could have a force F1 go in the other way, minus F1 displaced by distance d1 from each other this would be equivalent to a force F2, F2.

Let us say, is minus F1 divided by 2, but the distance d2 which is twice d1 which will, also be equivalent to a force F3 minus F3 plus say which is equal to 3 times F1. But displace from each other by distance of d1 over 3 they all 3 of them represent the same couple, because a net force is 0. So, there is no effect of the force.

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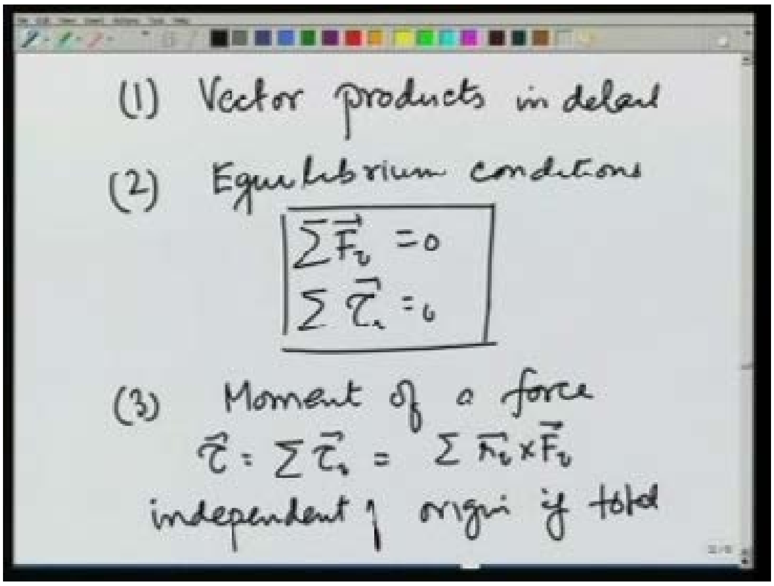

So, let me just recap, what all did we do in this lecture 1, we looked at Vector products in detail, then we looked at equilibrium conditions and found that summation Fi is equal to 0 and summation torque i is equal to 0 are both necessary and sufficient condition to ensure equilibrium. Then we started looking at different aspects of equilibrium separately, and third, we looked at moment of a force and found that, tau the total torque taui summed over which is equal to ri cross Fi is origin dependent, but becomes independent of origin if, total force is 0.

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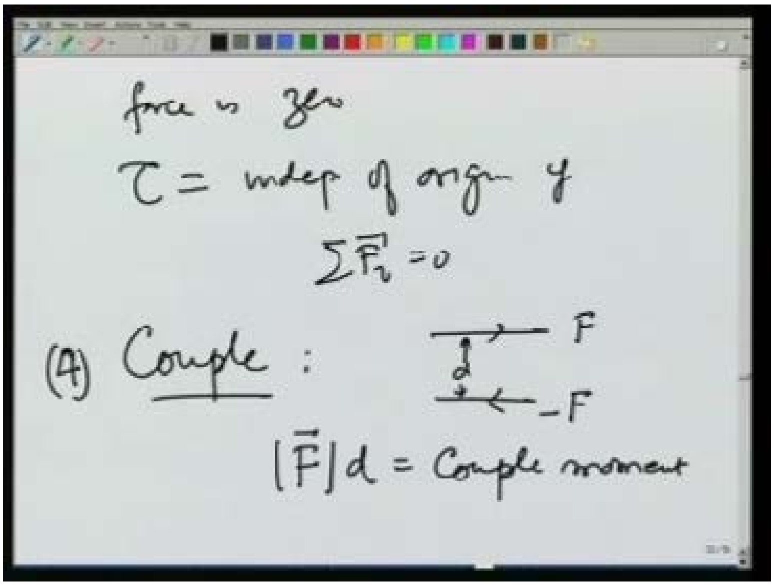

Let me, rewrite this again tau is independent of origin if, summation Fi is equal to 0 and then, we looked at a very particular special kind of torque. Under, these circumstance called a couple which is 2 forces equal and opposite forces separated by parallel distance d in this case a force magnitude times d is a couple moment and the direction you can find by right hand rule on the cross product rule yourself. In the next lecture, we look at how to define the moment about an axis or a couple about an axis and they go on to discuss, other things about equilibrium.