**Engineering Mechanics**
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**Module - 03**
**Lecture - 03**
**Properties of Surfaces - III**

In the previous lecture, we have been talking about the first moment of a plane area and a centroid. Continuing on that in this lecture we define some more mathematical quantities and just work out some examples with them. The utility of defining such quantities would be clear later, when you do rotation dynamics and so on. But in this lecture we are just going to restrict to their definition and working them out. The quantities I will be talking about would be 1.

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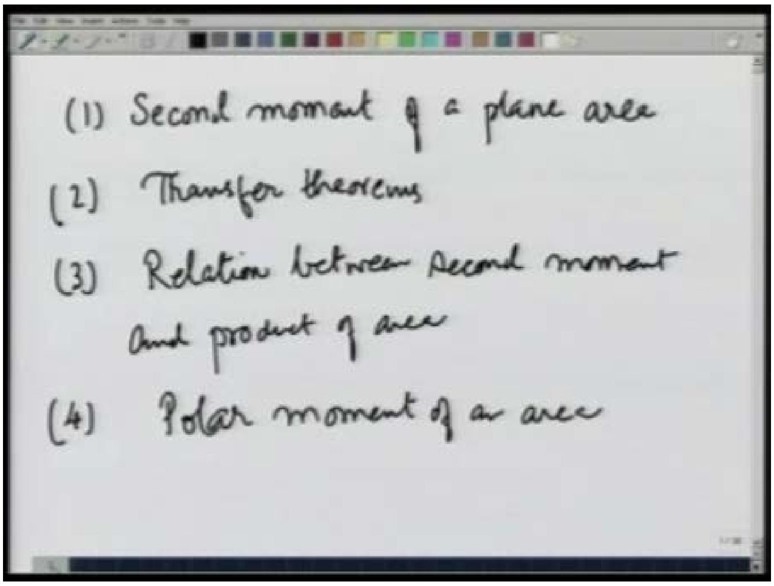


Second moment of a plane area 2 transfer theorems transfer, while transfer theorems we mean, how if we know the second moment of a plane area and one particular set of axis, how do we transfer them to another set of axis, 3. We will be talking about relation between second moment and product of area in particular.

We will be focusing on how second moments and products of area, which we will later in the lecture. Change when we go from one set of axis to another set, which allotted with respect to the other. And then we will be talking about polar moment of an area. So, to begin with let us define what is second moment and product of an area and product.

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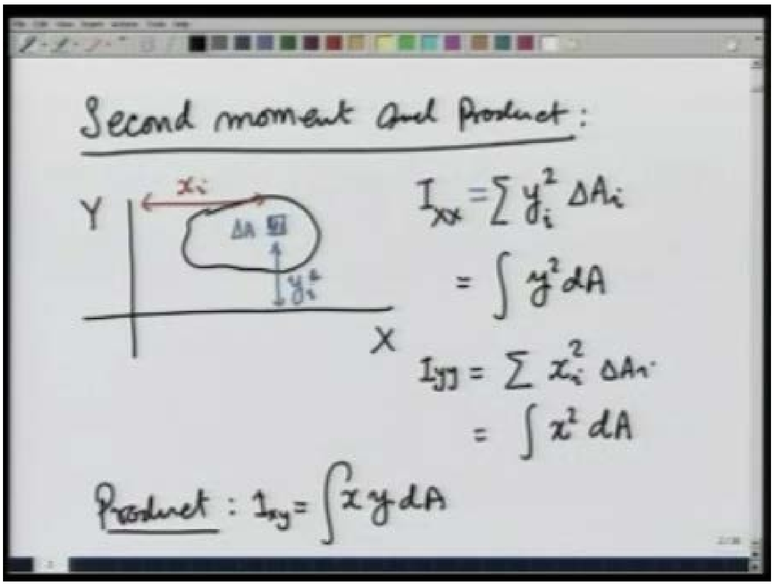


Suppose, I am given an area, a plane area in x y plane, like this is the x axis, this is the y axis. Then, the second moment I x x about the y axis is defined as I take a small area. Let me make it in blue delta a multiply this by the perpendicular distance from the x axis square. So, I x x is defined as take area delta A i multiply by its perpendicular distance from the x axis and add it up. This is the second moment of this area with respect to the x axis and this; obviously, goes to the integration y square d A.

Similarly, I y y that is the second moment about the y axis, is defined as the distance of the area from the y axis is chosen and then we write this as summation x i square delta Ai or limit of integration this becomes x square d A. These are just mathematical definitions and then the product of this area is defined as integral x y d A and I am going to call this I x y equals this. So, we have defined the second moment of a plane area and product of a plane area. Let us now work out some examples of how to calculate these.

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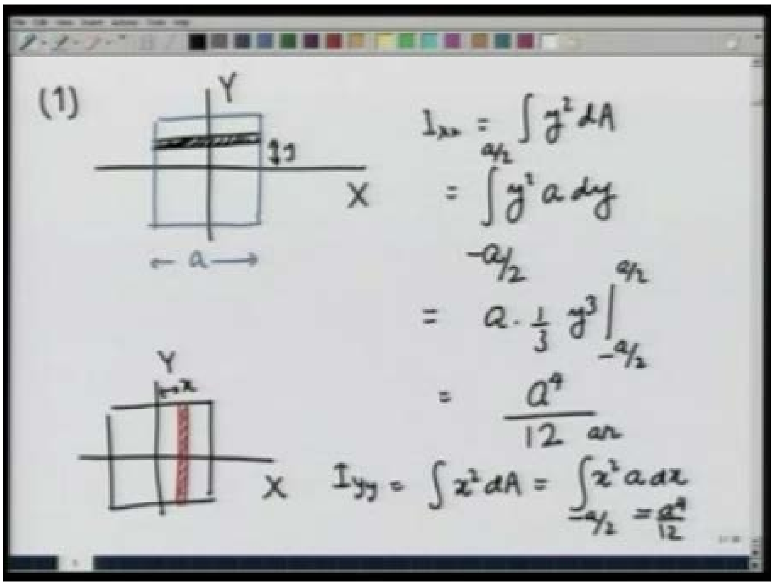


As a first example I take a square of side a, and calculate for it the moment of moment of area about the x axis, and about the y axis and its product of area I x x is equal to by definition y square d A. To calculate d A, I choose a strip at height y because for this entire strip the moment of the area is going to be the same. So, this becomes integral y square a d y and y changes from minus a by 2 to a by 2. And therefore, I x x is going to be a times one third y cube minus a by 2 to a by 2 or this comes out to be a raise to 4 over 12.

Similarly, to calculate I y y this is the square. I choose a strip like this and calculate I y y as integral x square d A, with this case would become integral, this is at distance x. So, integral x square a d x from minus a by 2 to a by 2, and this also in this case by symmetry would come out to be a raise to 4 over 12 third for the product of area.

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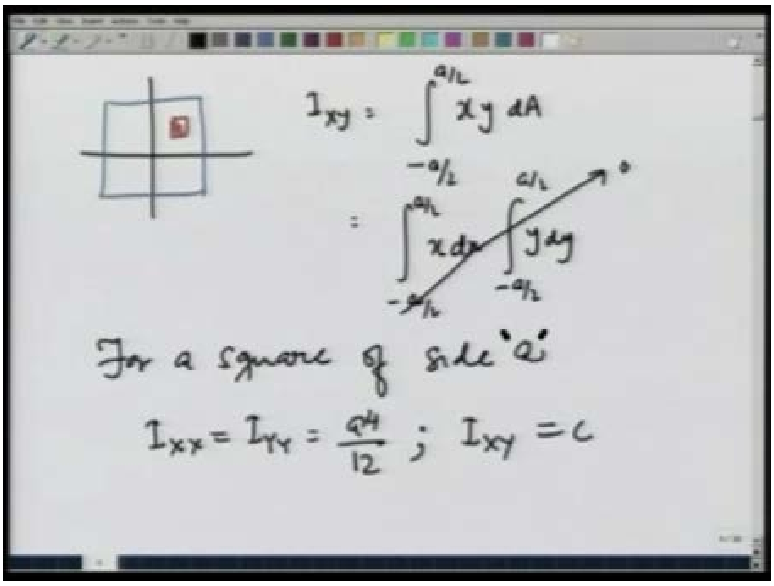


You see that I x y is equal to integral x y d A minus a by 2 to a by 2, and if I choose a small area d A here. This will be equal to integral minus a by 2 to a by 2 x d x integral minus a by 2 to a by 2 y d y and by and the symmetry of the wave of the function x and y this goes to 0. So, for a square of side a I x x equals I y y equals a raise to 4 over 12 and I x y equals 0.

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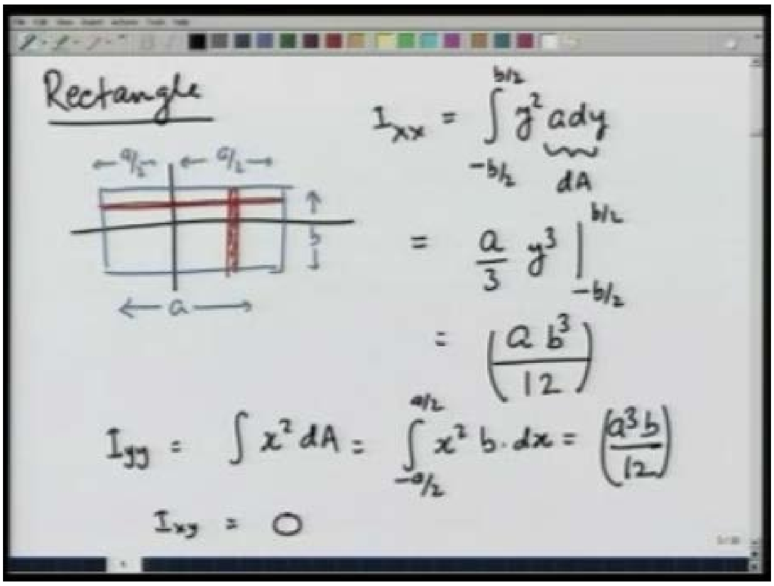


As a second example, let me take a rectangle of length a, and width b placed symmetrically about the origin. So, this side is precisely a by 2. So, is this is not made to scale please understand, this is a by 2 a by 2 divided on both sides, and I want to calculate I x x for that again I choose a strip here of width d y because for this entire strip y square is the same and calculate integral y square a d y is that small area d A and y changes from minus b by 2 to b by 2. And this comes out to be a over three y cubed minus b by 2 to b by 2 or a b cubed over 12.

Similarly, when I calculate I y y, for that I choose a strip parallel to y axis and calculate integral x square d A, which now becomes x square b d x x varying from minus a by 2 to a by 2, and this comes out to be a cubed b over 12, how about I x y? Again you will see by symmetry because the area is equally distributed on the negative side and the positive side of the y and x axis comes out to be 0. So, you have already also calculated the moment second moment and product of area for a rectangle. As the third example will make it slightly more complicated and I wish to calculate.

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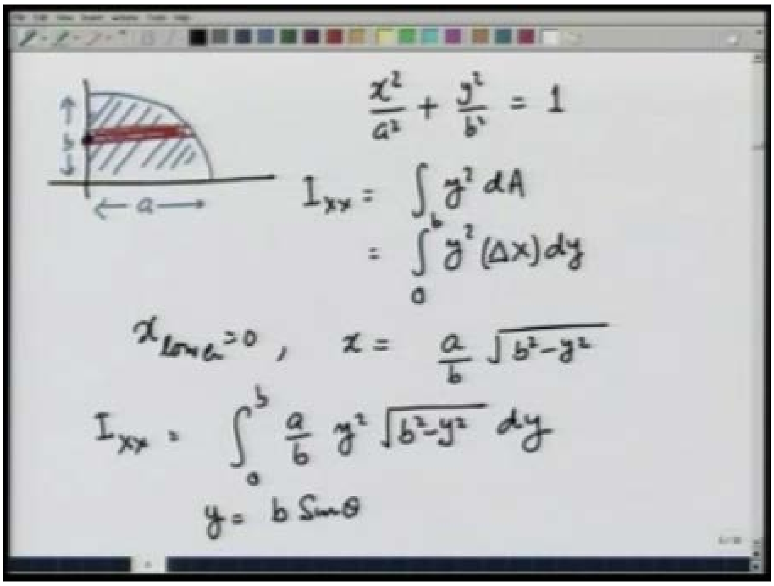


Second moment and product of area for an ellipse. The quarter of an ellipse here with semi major axis a, and semi minor axis b. The equation for the ellipse is x square over a square plus y square over b square is equal to 1. To calculate I x x, which is integral y square d A, I choose a strip parallel to the x axis of width d y and write this quantity as y square. The length of the strip, let us call it delta x d y.

My job is to calculate delta x and y varies from 0 to b. From the equation x lower is 0 that is this point and I also know from the equation of the ellipse that x is equal to a over b square root of b square minus y square. Therefore, I x x is going to be 0 to b a over b y square a square root of b square minus y square d y. This is what, to what we got to integrate to do this, we substitute y equals b sin of theta by doing.

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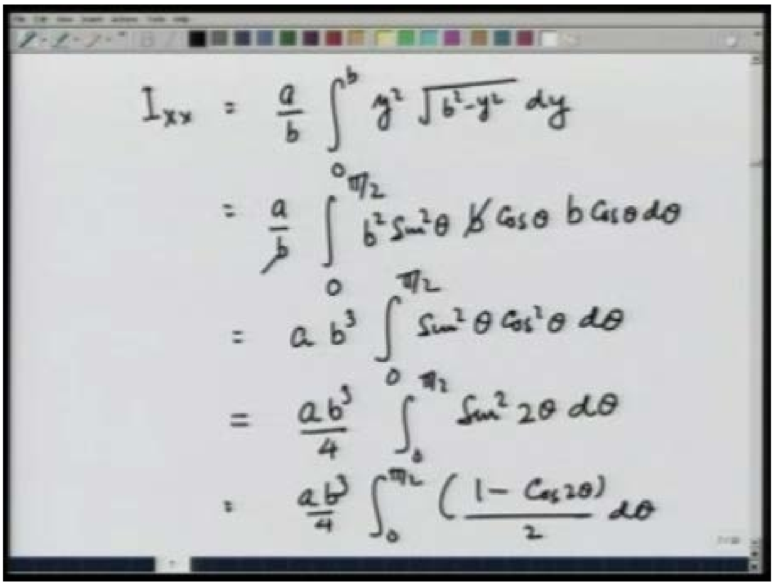


So, I get I x x, which is a over b integral y square the square root of b square minus y square d y 0 to b as a over b integral 0 to pi by 2 y square is b square sin square theta b square minus y square root, is going to be b cosine theta and d y is again b cosine theta d theta. And this gives me this b cancels, and I get a b cube integral sin square theta cosine square theta d theta 0 to pi by 2. This can be written as a b cubed divided by 4 0 to pi by 2 sin square 2 theta d theta, which is a b cube over 4 integral 0 to pi by 2 1 minus cosine 2 theta over 2 d theta. Doing this integral, when we do cosine 2 theta integral that gives me 0 and the first integral is going to be pi by 4 and therefore, I get for this shape where this is a.

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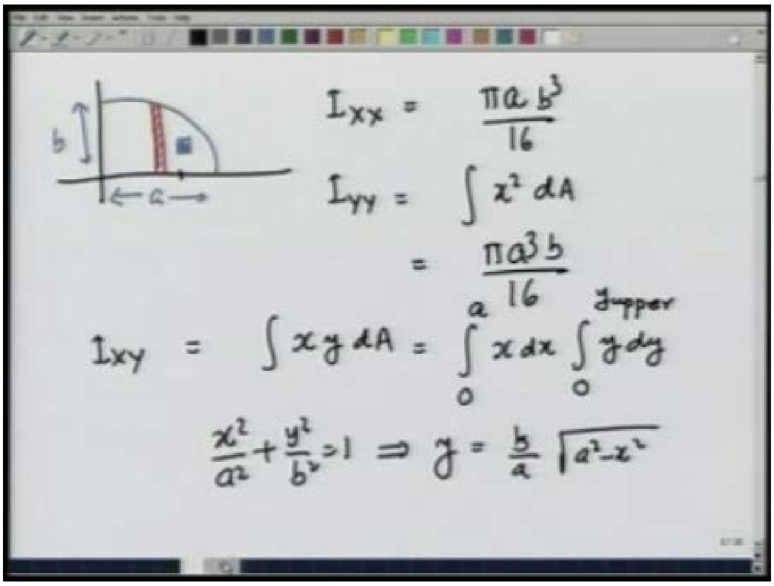


This is b I x x equals pi a b cube over 16. Similarly, I can calculate I y y, which is integral x square d A, where now I am going to choose a strip parallel to the y axis. The integral is very similar to what we just now did, and this will come out to be pi a cube b over 16, how about I x y? To calculate I x y I have to calculate the integral x y d A, where d A I take 2 be a small area at point x and y.

This can therefore, written as x d x y d y x square is from 0 to a. For any given x, the given x here y varies from 0 to a y upper, where y upper is given by the equation x square by a square plus y square over b square equals 1, and that gives for any given x, that y is going to be b over a the square root of a square minus x square.

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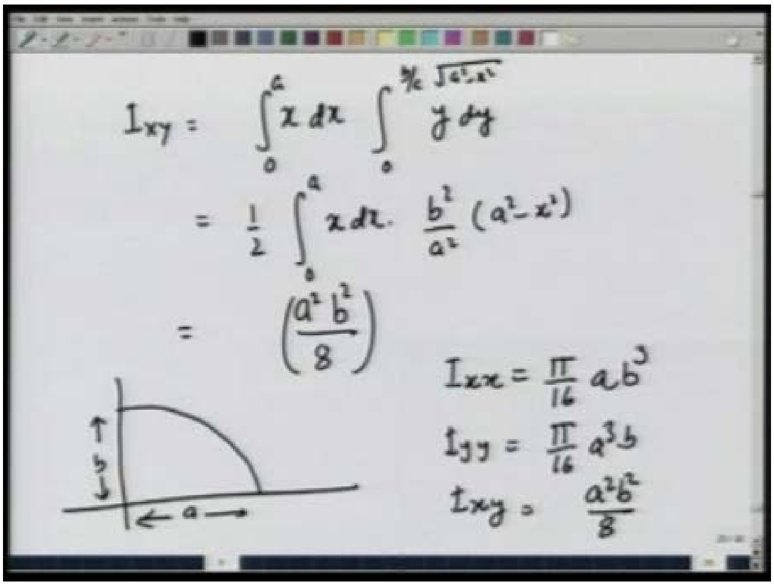

And therefore, I x y for this is going to be integral x d x 0 to a integral 0 to b over a is square root of a square minus x square y d y, which is nothing but one half 0 to a x d x b square over a square times a square minus x square. This is a standard integral. So, this gives me the answer a square b square divided by 8. Therefore, what we have determined is that for the quarter of an ellipse with semi major axis a and semi minor axis b, I x x is pi over 16 a b cubed, and I y y is pi over 16 a cube b, and I x y is a square b square over 8.

I will now leave it for you as an easy exercise to calculate I x x and I y y. That is the second moment of an area with respect to x and y axis for the entire ellipse, and also show that I x y for the entire ellipse is going to be 0, because of the symmetry between x and y axis. Using the second moment of area, we can also define, when we call the radius of gyration.

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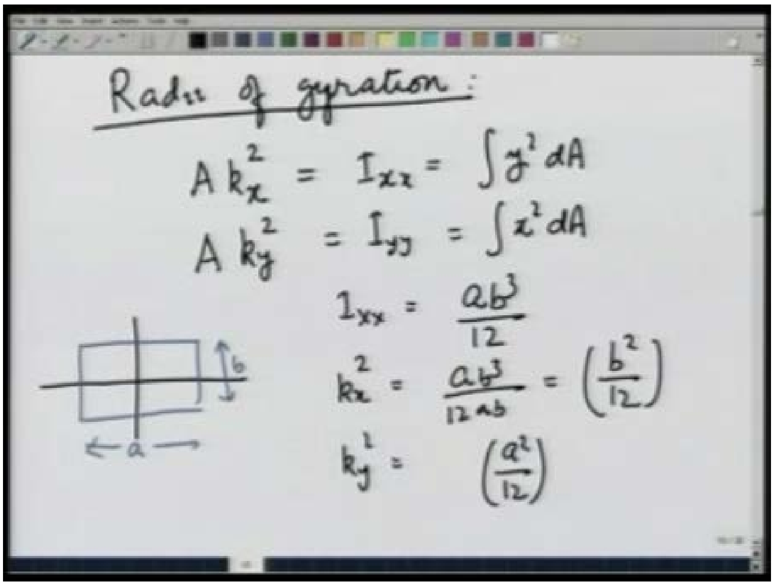


By gyration you can already see that these quantities are going to be useful, when we describe rotation. The radius of gyration of k x of an area A, about the x axis is defined through the relationship A k x square equals I x x equals integral y square d A. And radius of gyration for about the y axis is defined as A k y square equals I y y equals integral x square d A. Thus for example, when we look at a rectangle of length a, and width b, we have already calculated that I x x is equal to a b cubed over 12. And therefore, k x square is going to be a b cubed over 12 a b or equals b square over 12.

Similarly, k y square is going to be a square over 12, and from these area of gyration about the x axis and the y axis can be calculated. Having defined these quantities the second moment of inertia and the product of inertia. We now describe relationship between the second moments of an area about a set of axis passing through the centroid of the body, and another set x y axis, which are parallel to those passing through the centroid.

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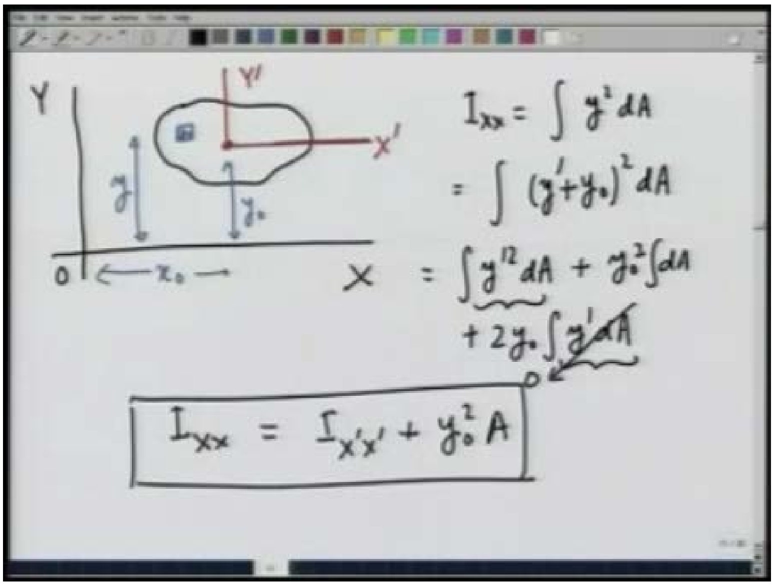


Thus suppose, I have an area the centroid with respect to a given set of axis x and y with origin at O such that the coordinates of a centroid are x 0 and y 0. Let us chose another set of parallel axis x prime and y prime passing through the centroid, X prime is parallel to x and y prime is parallel to y. And we want to now calculate show that I x x and I x prime x prime are related by a very simple theorem. So, are the other moments? Thus I x x is equal to y square d A.

So, suppose I have a small area d A here. This is y, which is also equal to y is equal to y prime, where y prime is the y coordinate of the same area with respect to x prime and y prime axis plus y 0 square d A. This is equal to y prime square d A plus y 0 square d A plus 2 y 0 integral y prime d A. Notice that y prime square d A is the I x prime, x prime. That is the second moment of inertia about the x axis passing through the centroid, and y prime d A is nothing but area times the y y coordinate of the centroid, in the centroid frame.

So, this is going to be 0, if I calculate the centroid with the origin of the centroid. The coordinates of centroid are going to come to 0. And therefore, I see that I x x is equal to y prime square d A. That is I x prime, x prime plus y 0 square times entire area. This way, if I know the second moment of inertia of a body about an axis passing through a centroid. I can easily calculate the second moment of inertia on the same body with respect to an axis, which is parallel to the first axis were displace by amount y 0.

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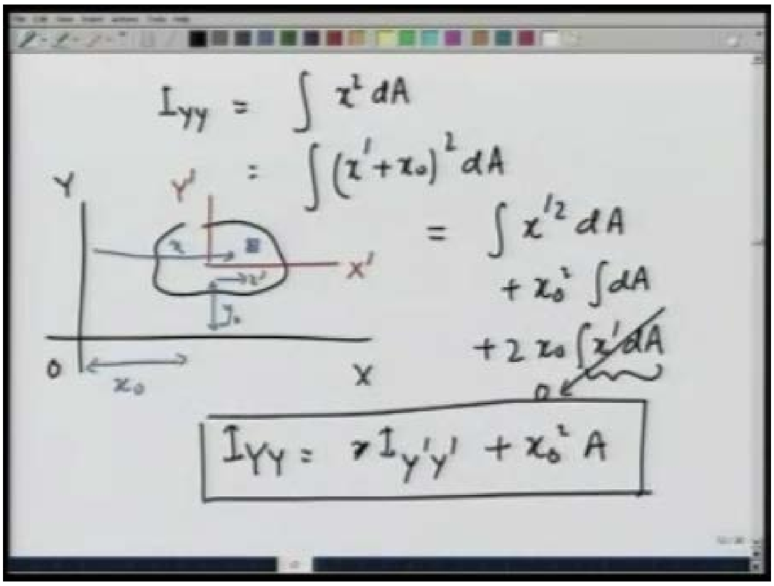


In the same manner, we can now calculate I y y which is equal to integral x square d A, which I can write as x prime plus x 0 square d A. Let me for convenience show in the picture, again what we are talking about? This is the body; this is the centroid set of axis x prime y prime. This is x y 0, this is x 0, this is y 0. If, I choose an area here, this is x and this is x prime, x is x prime plus x naught. So, this can be written as equal to integral x prime square d A plus x 0 square integral d A plus 2 x 0 x prime d A.

This again has the same logic as we applied earlier that this is the x coordinate of the centroid in the coordinate system, which has this origin at the centroid itself. So, this is 0. So, I get I y y is equal to I y prime y prime plus x naught square A. So, it is as if the entire area is concentrated at the centroid plus whatever the moment second moment of area is allowed the axis passing through the centroid.

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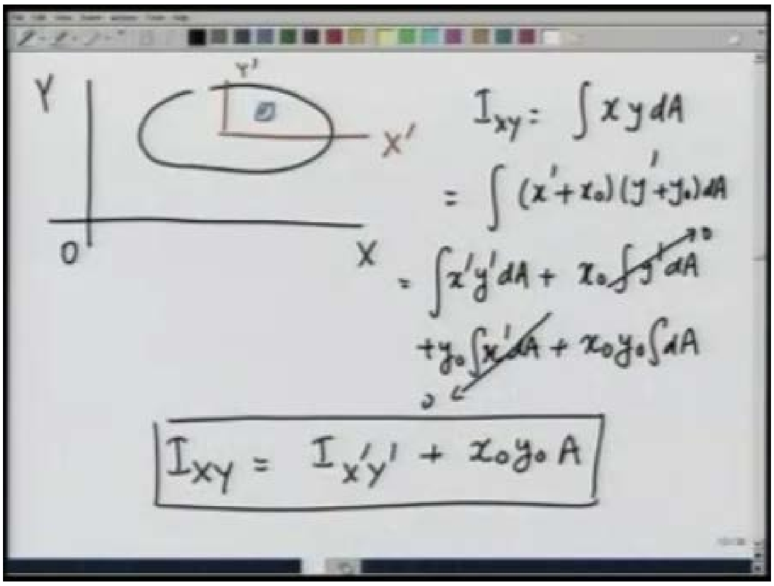


Next let us calculate the product of area in making these axis x prime y prime x y 0, and I take an area here. The product is I x y, which is equal to integral x y d A, and I substitute for x and y as x prime plus x 0 y prime plus y 0 d A, which comes out to be x prime y prime d A plus x 0 integral y prime d A plus y 0 integral x prime d A plus x 0 y 0 d A. Again by the arguments that we have used earlier these 2 terms drop to 0.

And therefore, I x y is equal to I x prime y prime plus x naught y naught times per area. So, what we have learnt is, if we know the second moment of inertia and the product of inertia about a set of axis passing through the centroid. I can calculate about any other set of axis, which are parallel to those passing through the centroid. Let us summarize these.

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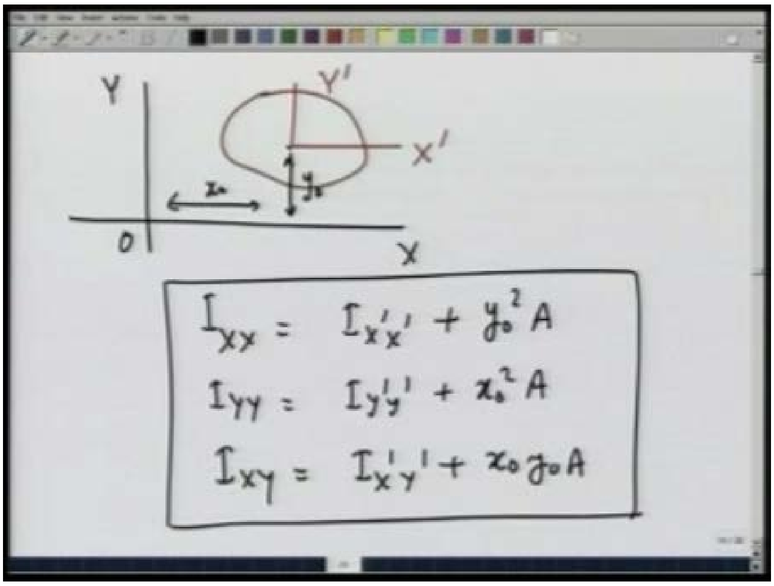


So, for an area whose second moment and product of area are known about the axis passing through the centroid. I have in general I x x equals I x prime x prime plus y 0 square times the area, where y 0 is the coordinate of the centroid. Y coordinate of the centroid I y y is equal to I y prime, y prime plus x 0 square times the full area, and I x y equals I x prime, y prime plus x 0 y 0 times entire area. These are known as transfer theorems, using these I can transfer the moment of area or the product of area from one coordinate system to another. As an example of the application of transfer theorem.

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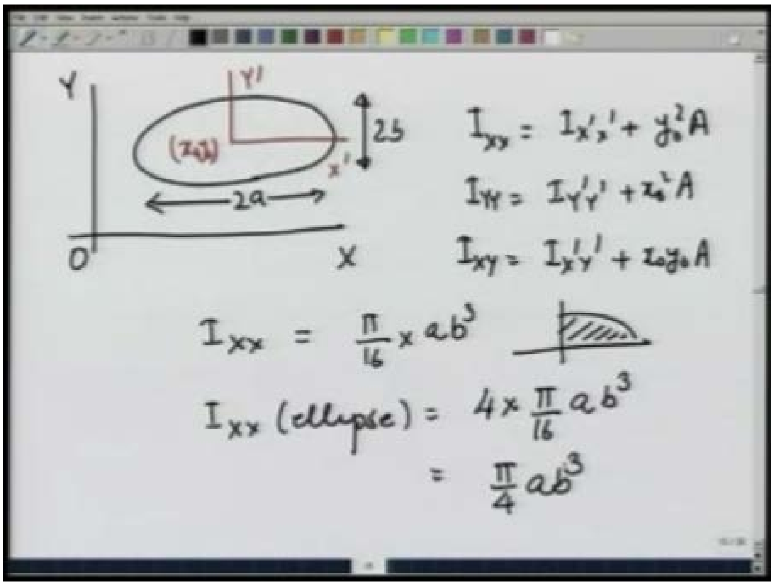

Let us take case of an ellipse with its length being 2 a, and this being 2 b with its centroid at point x 0 y 0. And calculate its moment of area second moment of area and product of area with respect to the x y axis shown here. So, by transfer theorems I have I x x equals I x prime x prime plus y naught square times the area of the ellipse I y y. Similarly is I y prime y prime plus x naught square times the area of ellipse and I x y is equal to I x prime y prime plus x 0 y 0 times the area of the ellipse, where I x x prime, x prime is the second moment of area with respect to the x prime axis parallel to the x axis passing through the centroid. I y prime y prime is the second moment of area with respect to the y prime axis parallel to the y axis and passing through the centroid.

Previously we have calculated I x x as pi over 16 times a b cube for quarter of an ellipse like this. So, for the full ellipse and this I left as an exercise for you there this is going to be 4 times as much. So, this is going to be pi over 16 a b cubed, which is pi over 4 a b cubed.

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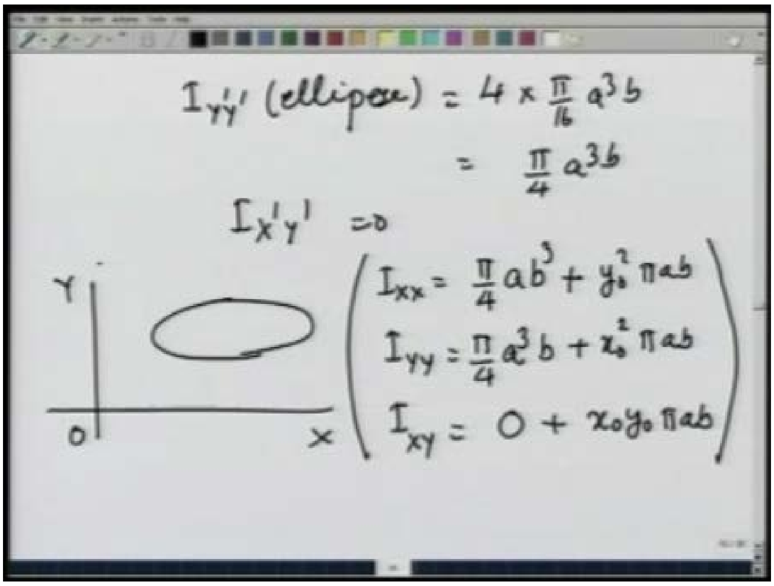


Similarly, I y y prime for the ellipse is going to be 4 times pi over 16 a cubed b, which is pi over 4 a cubed b and I x prime y prime is 0. Therefore, for this ellipse we will have I x x as pi by 4 a b cubed, which is the second moment of area about the x prime axis passing through the centroid plus y 0 square times pi a b, where pi a b is the area of the ellipse.

I y y is going to be pi over 4 a cubed b plus x 0 square pi a b, and I x y is going to be 0, which is the I x prime y prime by symmetry is the 0 for axis passing through the centroid plus x 0 y 0 pi a b. So, using transfer theorems we could calculate the second moment of area and the product of area, when it was given about the centroid, so far what we considered in the transfer theorem is the product.

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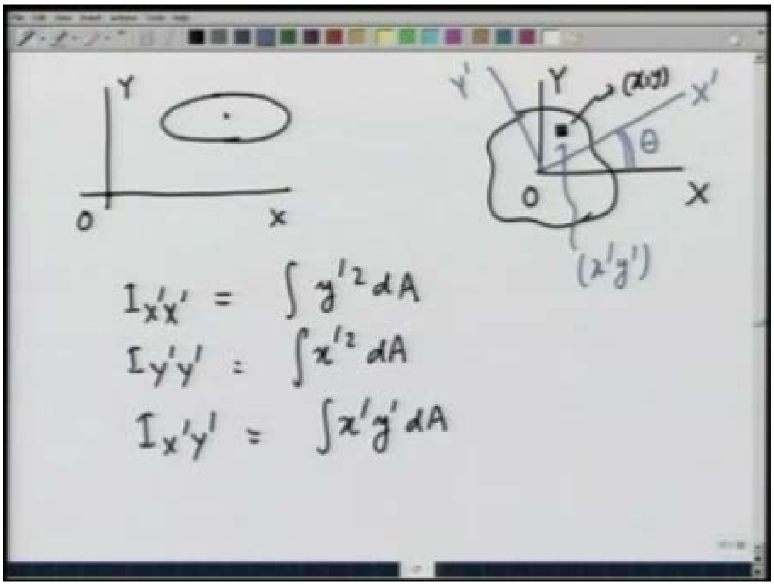


And second moment of area, when the centroid is displaced with respect to the origin of a given system. Now, we want to look at another transformation, where given an area, and its second moment of inertia and product of inertia about in set of axis x y. We wish to calculate it about another set x prime y prime, which is rotated with respect to the first strip by an angle theta. Let us see, what happens in this case?

So, if I want to calculate I x prime y x prime, x prime in the rotated set. This is going to be equal to integral y prime square d A, I y prime y prime is going to be equal to x prime square d A. And I x prime y prime in the second frame is going to be x prime y prime d A, where we chose a small area d A, whose coordinates in the origin system are x and y. In the new system x prime y prime, we can find out the relationship of I x x prime, with those similar quantities in the unrotated frame by a simple transformation laws of x and y coordinates. So, let us do that now.

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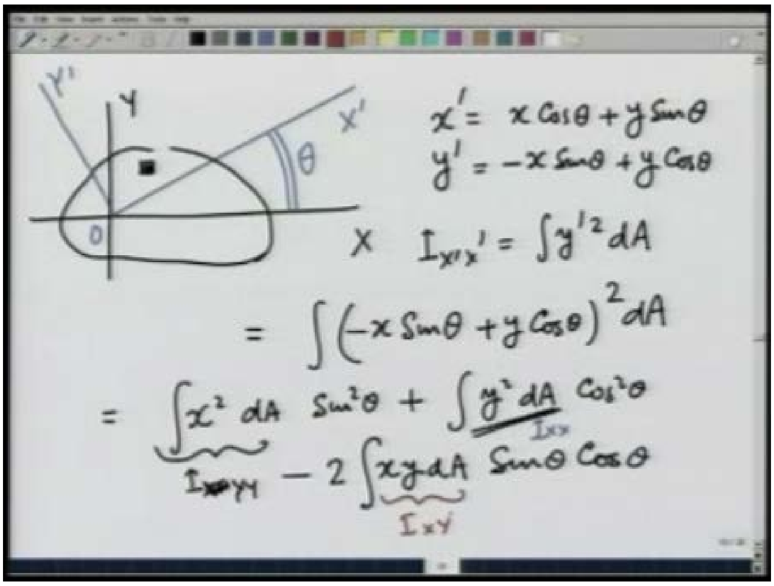


So, what we given is an area and we used to calculate its second moment of area and product of area with respect to a set of axis x prime y prime, when they are given in x and y. We know from our previous lectures that x prime for a given point is equal to x cosine of theta plus y sin of theta. Similarly, y prime is equal to minus x sin of theta plus y cosine of theta using these let us find what I x prime x prime is from the previous slide we know this is equal to y prime square d A, where d A is a small area.

Chosen Y prime square is going to be equal to integral minus x sin theta plus y cosine of theta square d A, which I can write as x square d A integral sin square theta plus integral y square d A cosine square theta minus 2 x y d A sin theta cosine of theta. But x square d A is nothing but I x x y square I y y. Sorry, I y y y square d A is nothing but I x x, and x y d A is nothing but I x y. Therefore, I can write this quantity as let us go to next page.

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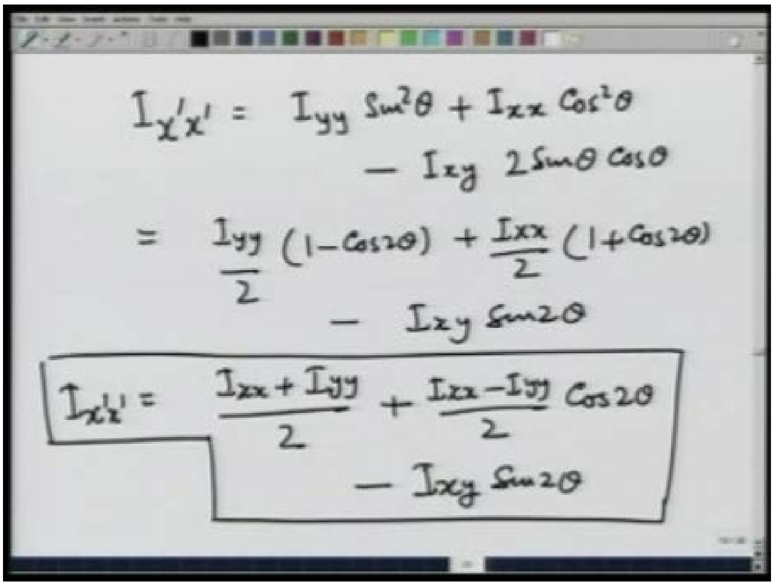


I x prime x prime as I y y sin square theta plus I x x cosine square theta minus I x y 2 sin theta cosine theta, which can be written as I y y divided by 21 minus cosine of 2 theta plus I x x divided by 2 1 plus cosine of 2 theta minus I x y sin of 2 theta, which is nothing but I x x plus I y y divided by 2 plus I x x minus I y y divided by 2 cosine of 2 theta minus I x y sin of 2 theta. Thus, if I know the second moment of area and product of area in one frame I can calculate it in the rotated frame. Let us do the same exercise for I y y prime.

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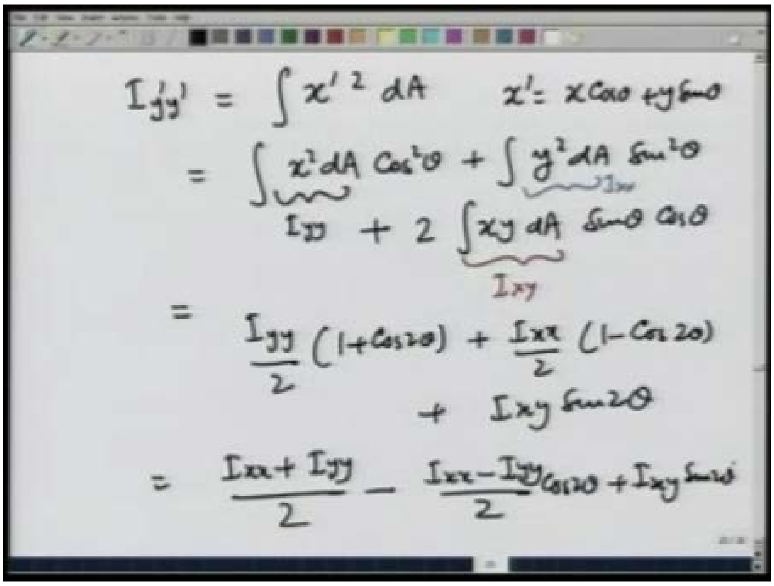

Y prime y prime, which is going to be equal to x prime square d A, but I know x prime is equal to x cosine theta plus y sin of theta. And therefore, I can write this as integral x square d A cosine square theta plus integral y square d A sin square theta plus 2 integral x y d A sin theta cosine of theta, which is this is nothing but I y y. This is nothing but I x x, and this is nothing but I x y. So, this whole thing can be written as I y y over 2 1 plus cosine 2 theta plus I x x over 2 1 minus cosine 2 theta plus I x y sin of 2 theta., which is nothing but I x x plus I y y divided by 2 minus I x x minus I y y divided by 2 cosine 2 theta plus I x y sin of 2 theta.

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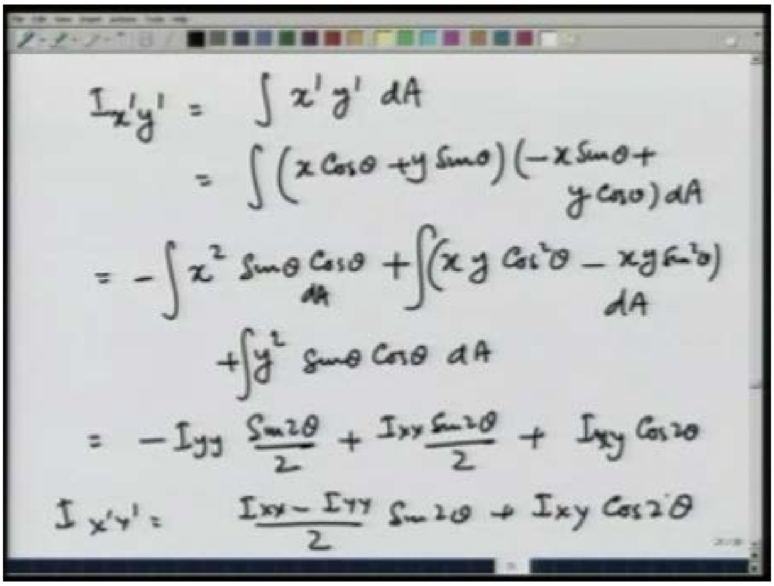


First we calculate I x prime y prime, which is nothing but x prime y prime d A, which is equal to integral x cosine theta plus y sin of theta times minus x sin theta plus y cosine theta d A. It comes out to be integral minus x square sin theta cosine theta plus x y cosine square theta minus x y sin square theta d A plus y square sin theta cosine theta d A. This is also d A x square d A is nothing but minus I y y.

This can be written as sin 2 theta divided by 2 plus y square d A is I x x sin 2 theta divided by 2 and x y d A is nothing but I x y cosine square theta minus sin square theta is cosine 2 theta. So, I x prime y prime is nothing but I x x minus I y y divided by 2 sin 2 theta plus I x y cosine of 2 theta. Let us summarize what we are looking for, is if we know for a body.

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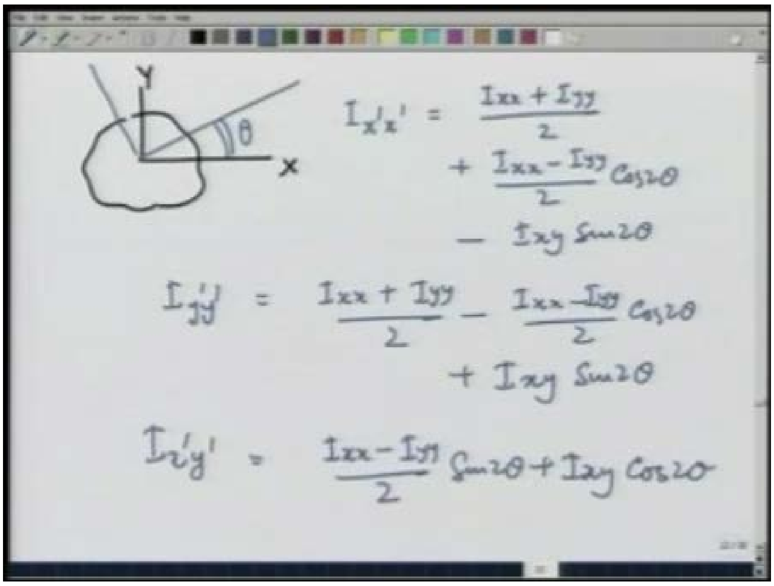


The products and second moments of inertia in one particular frame, how about its values in the rotated frame. In the rotated frame, let me now write it in blue I x prime x prime is nothing but I x x plus I y y divided by 2 plus I x x minus I y y divided by 2 cosine 2 theta minus I x y sin of 2 theta. Similarly, I y prime y prime is going to be I x x plus I y y divided by 2 minus I x x minus I y y divided by 2 cosine 2 theta plus I x y sin of 2 theta.

And I x prime y prime is going to be equal to I x x minus I y y divided by 2 sin of 2 theta plus I x y cosine of 2 theta, what these transformation laws give me? It is if I am given the second moment and product of area about a set of axis. I can calculate about any other set of axis, which is rotated with respect to the first set of axis. Let me just illustrate this thing by couple of examples, which are very interesting.

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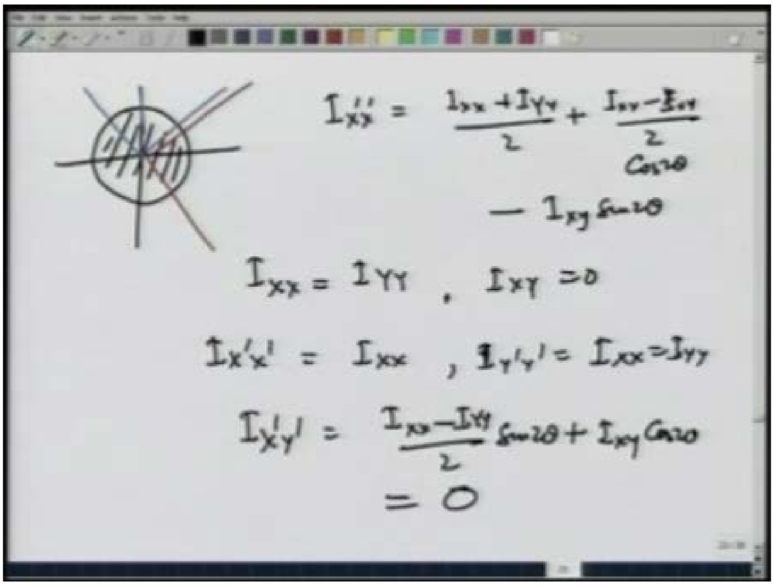

Suppose, I take a circle circular area for a circular area, no matter how I chose my rotated set of axis. Let us take third one, like this the circle always looks the same. And therefore, I x x and I y y should always come out to be the same no matter what cosine theta or sin theta is, and I x y should always come out to be 0. Let us see, if that happens. So, I x x we saw already is I x prime x prime is I x x plus I y y divided by 2 plus I x x minus I x y divided by 2 cosine of 2 theta minus I x y sin of 2 theta.

Now, for a circular area I x x is equal to I y y and I x y is 0. And therefore, I x prime x prime is going to be equal to I x x, and I y prime y prime is also going to be equal to I x x equals I y y. And I x prime y prime, which is equal to I x x minus I y y divided by 2 sin of 2 theta plus I x y cosine of 2 theta is also going to be 0. This is expected for a circle, what is very interesting that is the same thing comes out to be true for a square. Let us look at that case.

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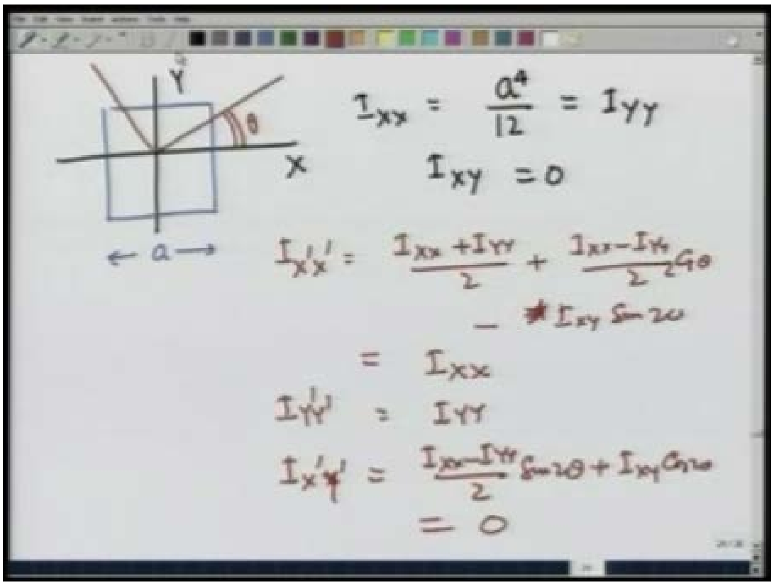


So, if I take a square of side a, we have already calculated that I x x for such a square is a raise to 4 over 12. So, is I y y where these are the x and y axis, and I x y is 0. The fact that I x x and I y y are equal, and I x y are 0 makes these quantities the same no matter, which other frame we look at. So, let me write it in red I x prime x prime, which is equal to I x x plus I y y divided by 2 plus I x x minus I y y divided by 2 cosine 2 theta plus I x y sin of 2 theta. This is not plus this is minus is going to be equal to I x x again.

Similarly, I y prime y prime is going to be equal to I y y and I x prime y prime is equal to I x x minus I y y divided by 2 sin 2 theta plus I x y cosine 2 theta. This is always going to come out to be 0, no matter how much you rotate the axis y. So, for a square about any set of axis I x prime x prime is always equal to a raise to 4 divided by 12 I y prime y prime is always equal to a raise to 4 divided by 12, and an I x prime y prime is always 0. Having given these two examples, I use these transformations to define something called the principal.

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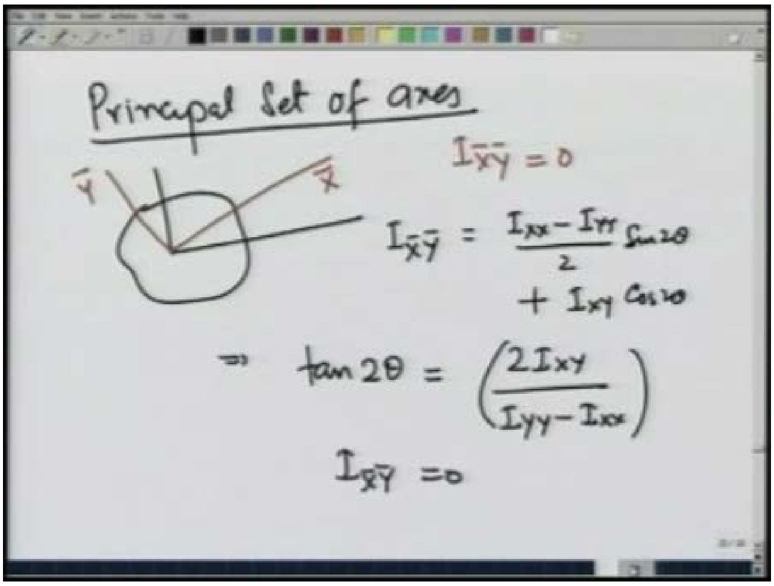


Set of axes. So, given an area I look for those set of axes, let me call them x and y. So, that I x y to 0 or rather given them a special name I x bar y bar. So, that I x bar y bar is 0, how do we accomplish that since we already known that I x bar y bar is going to be equal to I x x minus I y y divided by 2 sin of 2 theta plus I x y cosine of 2 theta. This implies that if I choose rotate the new set of axes thus a tangent 2 theta is equal to 2 I x y divided by I y y minus I x x.

I will get new I x bar y bar is equal to 0. Such a set of axis where the product of area vanishes is known as the principles of set of axes. And you can see from the construction that you can always find one set of axes because tangent 2 theta is varies from minus infinity to plus infinity, when always find a set of axes where the product of area would be 0. An interesting fact about the principle set of axis is that about.

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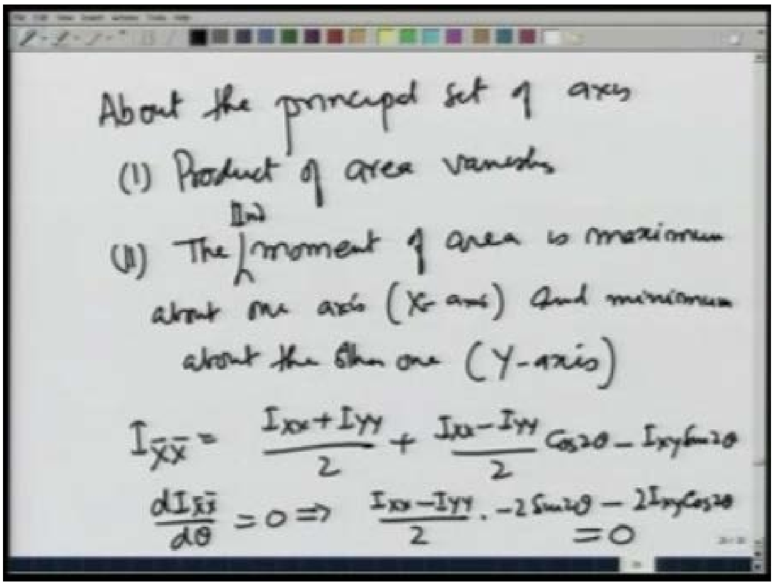


The principle set of axes the one product of area vanishes and 2 the moment or second moment of area is maximum about one axis. Say, the x axis and minimum about the other one, if it is maximum about the x axis, the other one is going to be y axis. Let us see, how does that come about?

So, let us look at I x bar x bar, which is I x x plus I y y divided by 2 plus I x x minus I y y divided by 2 cosine 2 theta minus I x y sin 2 theta. And asked for a new frame such that I x bar x bar is a maximum. So, for that I got to do I x bar x bar over d theta is equal to 0, and when I do that. Here, this implies that I x x minus I y y divided by 2 times minus 2 sin 2 theta minus 2 I x y cosine 2 theta is equal to 0, and that immediately gives me.

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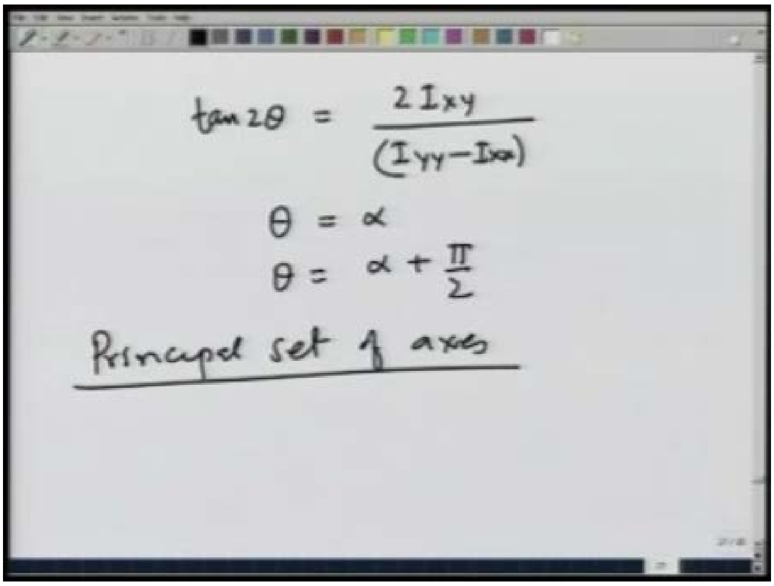


The tangent of 2 theta is equal to I x y times 2 divided by I y y minus I x x. So, when I accomplished by this rotation the fact that the product of area vanishes at the same time it maximizes or minimizes the moment of area, this equation has 2 solutions. Suppose, one of the solutions is theta equals alpha, then theta equals alpha plus pi by 2 is also a solution. So, by rotating it by angle alpha I maximize or minimize the moment of inertia about that particular axis. You can show that about the axis at alpha plus pi by 2. It will be the other way, if it maximizes at alpha at alpha plus pi by 2, it will minimize and the and vice versa.

So, we found a set of axis principle such that not only the product of inertia vanishes the second moment of area is also either maximum or minimum, if it is maximum about the x axis about the other y axis it becomes a minimum. If, it is minimum about the x axis it becomes maximum about the other axis the y axis. Have been made this point, this point let me now define something for you, which is known as the polar moment of area.

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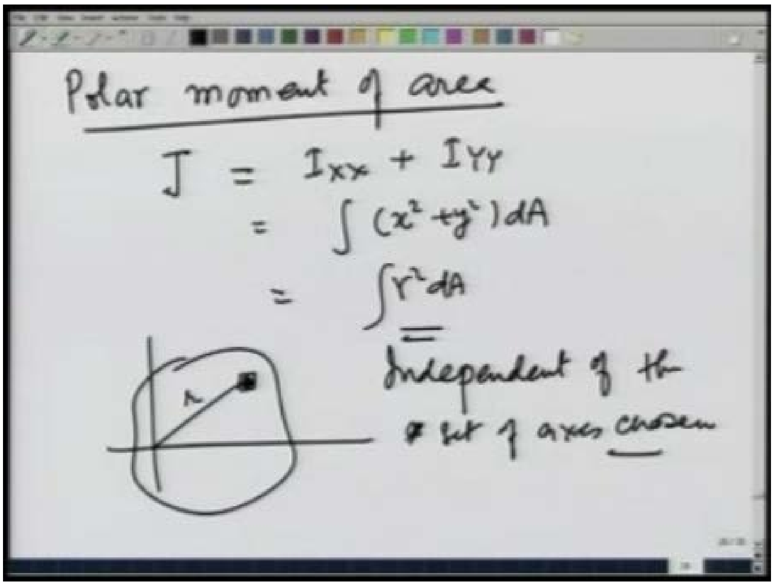


This is quite usually written as J, which is nothing but I x x plus I y y. And therefore, is equal to integral x square plus y square d A or r square d A. Given any area r square for a small area chosen is independent of which set of axes, we are talking about. So, this is independent of a set of the set of axes chosen.

And through this discussion you also see that for a square the any set of axes is the principle set of axes because as we have seen earlier the principle set of axes gives product of area 0, and second moment of area maximum or minimum for a square, any set of axes gives you product of area 0. So, therefore, any set of axes chosen for a square or a circle is the principle axes, what we have covered in this lecture. So, far is the second moment and product of an area a related quantity, which we will talk about in later lectures. We will discuss dynamics of rigid bodies would be the moment of inertia and product of inertia, and we will be using it, then in describing the rotational motion of a rigid body.