Numerical Methods and Computation

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Lecture No # 10

Solution of Nonlinear Equations (Continued)

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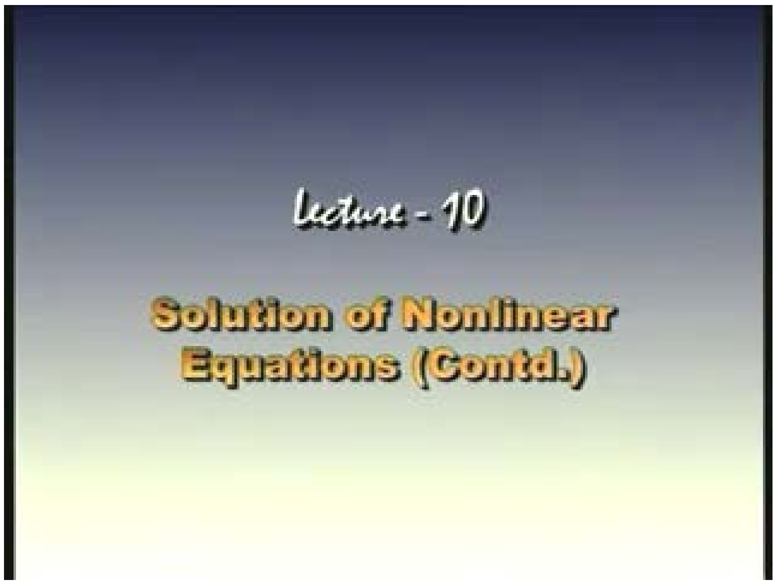

In our last lecture we derived the Bairstow method for extracting a quadratic factor from a given polynomial. This quadratic factor may give us two real roots or it may give a complex pair of roots. We have shown that the approximations would be obtained from evaluating the values of h_1 and h_2, the increments for the initial approximation. This is the formula that we have derived last time for finding these values of h_1 and h_2, so that the next approximation can be obtained as pk plus one is pk plus h_1 and qk plus one is equal to qk plus h_2. Now we have also seen that there are six quantities to be evaluated at the point pk qk. All these quantities are to be evaluated at pk qk and to derive these six quantities we had used the synthetic division procedure and the synthetic division procedure that we have written was this particular procedure in which we have written all the coefficients of the given polynomial. If there are any missing coefficients, we will insert a zero over there.

Then we have applied the synthetic division by using the factors minus p minus q and the first row of the synthetic division of the first level would give us b_0, b_1, b_2, b_n minus two, b_n minus one and b_n. The second level of the synthetic division will give me c_0, c_1, c_2, c_n minus two, c_n minus one. Based on the formulas that we have derived earlier we have found that these are the formulas that can be written for these six quantities. We can also write down what are these quantities, h_1, and h2. From this we can pick up the value of h_1 from here, I can insert these quantities and write a formula for that also. Alternatively I can obtain these six quantities independently, substitute in this formula to get my next approximation. For example, if I substitute these quantities in this what I would get will be simply this.

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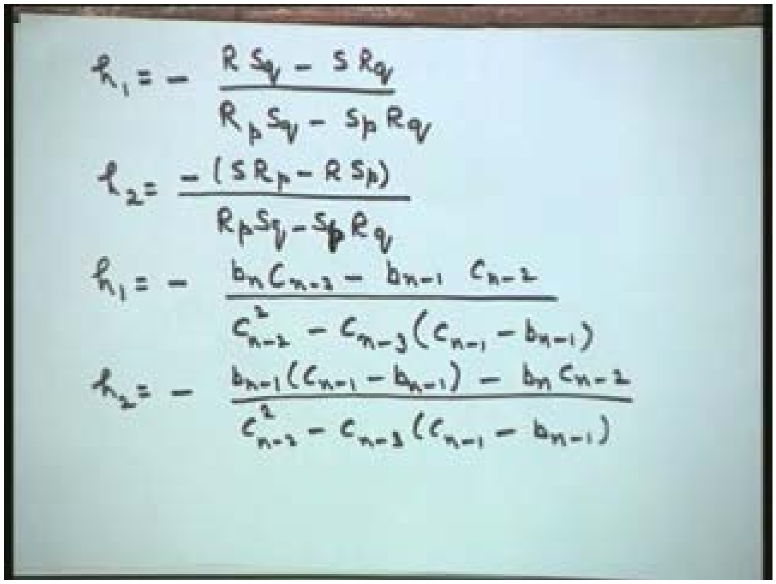

Let me write down what is your $h_{1;}$ from this formula I will just keep it here. This is minus of R partial derivative with respect to q minus s partial derivative with respect to q divided by the denominator that we have obtained earlier sp Rq. Now this is the value of h_1 and this is the value of h_2. We have the same denominator and the numerator is s into the partial derivative of this R, the partial derivative with respect to p divided by Rp sq minus sp Rq. So what we would now like to write down is to write these quantities that we have derived over here in terms of this b_n minus one. Therefore if I do that I would get here h is equal to minus of (because R is equal to bn minus one and so on) $b_n$cn minus three , b_n minus one cn minus two; denominator will be c_n minus two square minus cn minus three cn minus one minus b_n minus one. The denominator is Rp sq minus sp; h_2 obtained from here is minus of b_n minus one cn minus one minus b_n minus one minus b_n cn minus two divided by the same denominator cn minus two square cn minus three cn minus one b_n minus one. The denominators are both are the same the numerator is here bn minus one cn minus one minus bn minus one minus bn cn minus two. Now as I mentioned to you earlier, we can use either one of these formats, either we find out the six quantities from the synthetic division directly and substitute it here or alternatively I can use this particular formula.

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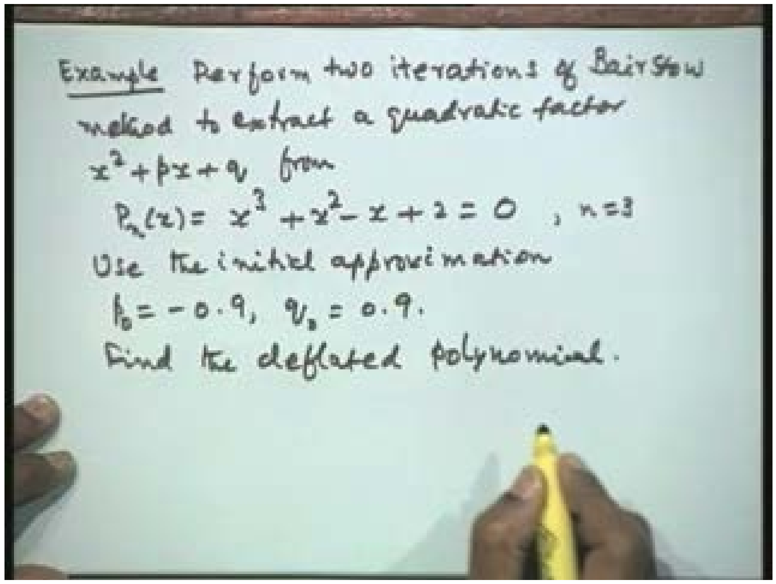

Now let us take an example to illustrate this particular way of obtaining the quadratic factor. We will perform two iterations of Bairstow method to extract a quadratic factor x squared plus px plus q from the given polynomial. Pnx is x cubed plus x squared minus x plus two is equal to zero; n of course is equal to three. Use the initial approximations. We are giving here the initial approximations for carrying out this procedure; p_0 is – 0.9, q_0 is 0.9; then find the deflated polynomial. So after you find the root to the required accuracy, we will perform the first level of synthetic division and that will give us the deflated polynomial; that is b_0x to the power of n minus two plus b_1x to the power of minus one and so on. So we would perform the synthetic division using the coefficient that we have over here.

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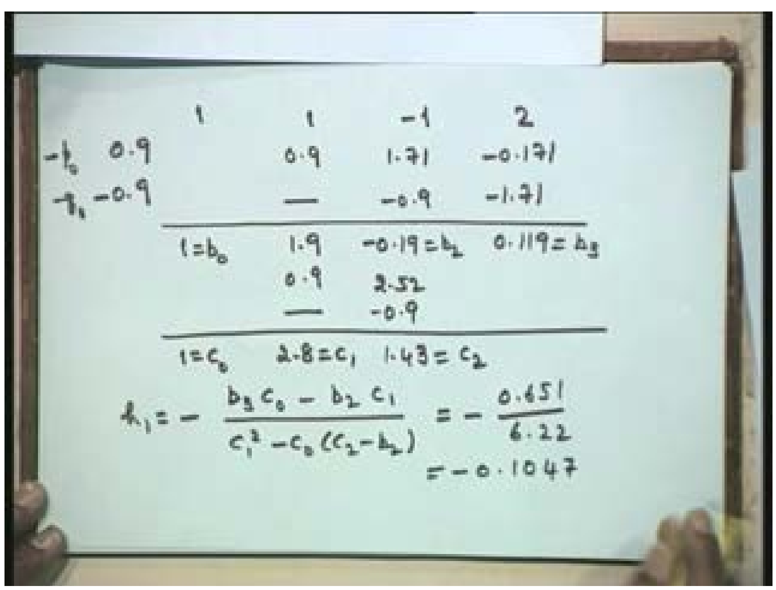

Let us take these coefficients. So the coefficients are 1, then we have 1 here, - 1 here and 2 over here and we have to use the opposite signs of this minus p_0 minus q_0 minus p, minus q. So I have to use here 0.9, 0.9. So I perform the synthetic division with respect to this. Therefore I get here 1 that is equal to b_0. This is your minus p_0 and minus q_0 and this will be – 0.9; because we are performing the extraction of the factor x square plus px plus q, so we will be performing the synthetic division with minus p_0, minus $q_{0.}$ So this is p_0 into b minus p_0 into b that is 0.9, so there is nothing here, therefore 1.9; 1.9 into 0.9, so we will have here 1.71. Then – 0.9 into 1, so I will have – 0.9, this is the product of these two products. We add these two; we will get 0.19 which is our b_2. Then I multiply 0.9 with the current – 0.9 that is – 0.171 and I multiply – 0.9 into 1.9 that is – 1.71; we add it, we will get 0.119 that is equal to $b_{3.}$ So the first step of synthetic division procedure is complete. Now we proceed to the next step. So I will again get 1 that is equal to c_0; 0.9 into 1, which is 0.9; add it, 2.8 that is $c_{1.}$ Then 2.8 into 0.9 that is 2.52 minus 0.9 into 1 add up, 1.43 that is equal to c_2. Therefore these are the quantities that we have for evaluating our h_1 and h_2.

So let us evaluate h_1 here itself. What I would do here is, I will put here n is equal to three and write down this particular formula for h_1. So if I put n is equal to three, what I would get here is minus b_3 c_0 minus b_2 c_1 divided by c_1 square minus c_0 c_2 minus b_2.

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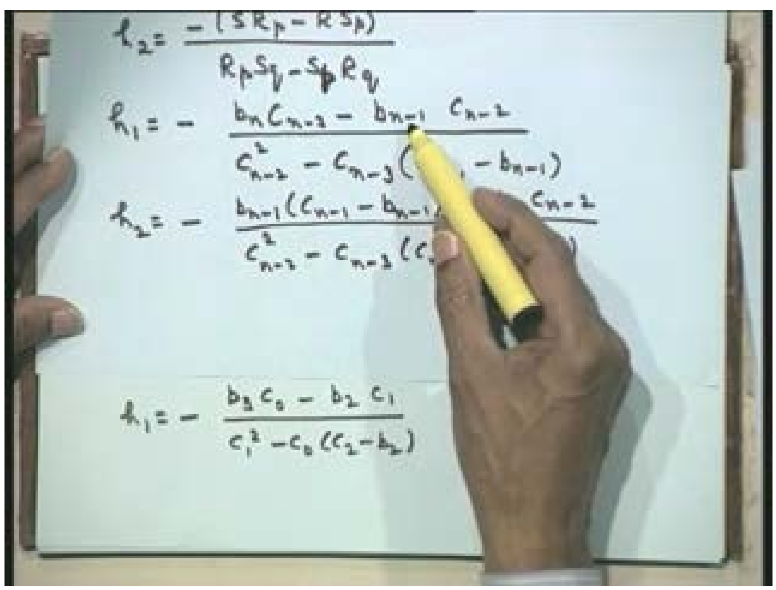

So I will just put it over here on this slide, so that it will be easier for you to have a look at it; n is equal to three, this is b_3 c_0 minus b_2 $c_{1.}$ This is c_1 square; this is c_0 c_2 minus $b_{2.}$ Now let us first evaluate it so that we can have the other quantity. So this comes out to be; the numerator is 0.651 and the denominator is 6.22, so I have this value as – 0.1047. Now I would use the quantities for evaluating $h_{2.}$ For a moment let me just put this h_2 over here.

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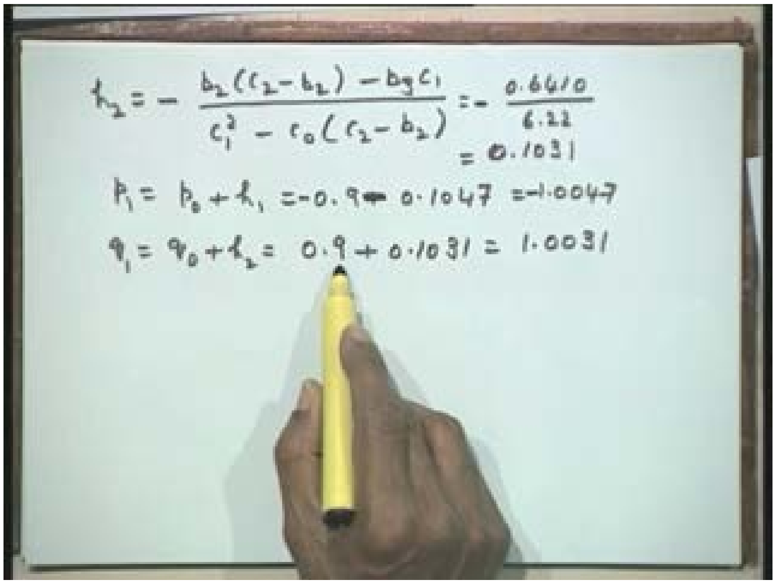

I am putting here n is equal to three in all these quantities and deriving this particular quantity. I will have here minus b_2 into c_2 minus b_2 minus b_3 c_1; denominator is the same c_1 square minus c_0 into c_2 minus b_2. So if I evaluate this the denominator is 6.22 and the numerator comes out to be 0.6410; so that this is 0.1031. So these values have been used and the values that we have here is b_0, b_2, b_3, c_0, c_1, c_2. So these are the quantities that we are using over here. Now I need the next approximation that is equal to p_1 is equal to p_0 plus h_1 that is – 0.9 – 0.1047 that is equal to – 1.0047. Then q_1 is q_0 plus h_2 and q_0 was 0.9, this is 0.1031 and this gives me 1.003. So h_1 is – 0.1007 and h_2 is 0.1031, so this p_1 is – 0.9 and q_1 is equal to 0.9. So we add up and we have this required approximation. Now we would perform the synthetic division with respect to these two quantities. So let us again write these coefficients.

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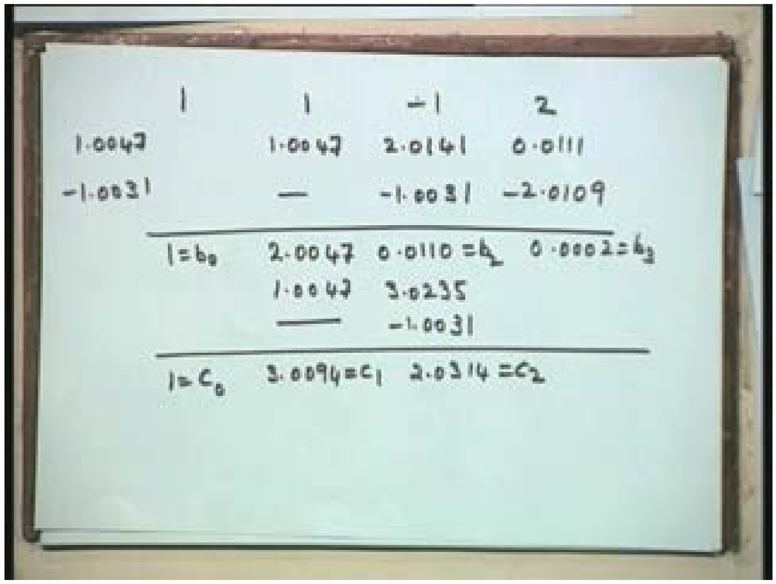

So let us again write this as 1, 1, - 1 and 2; this is 0.9 and - 0.9 and we are taking it as minus $p_{1,}$ that is 1.0047 and – 1.0031. So with that we will have again 1 that is equal to b_0 and multiply these two 1.0047, there is nothing over here and this is 2.0047; multiply these two quantities 2.0141; 1 into this that is – 1.0031; so add up all the three, I produce 0.010 that is equal to $b_{2.}$ I multiply this and this to give me simply 0.011 and I am multiplying these two quantities to give me – 2.0109; we add up to give us 0.02. Now I proceed to the next step. I will have 1 that is equal to c_0; 1 into this 1.0047; there is no quantity here that is equal to 3.0094, that's equal to $c_{1.}$ I am multiplying 3.0094 by 1.0047 that is equal to 3.0235; 1into this that is -1.0031, we add up to give us this quantity that is equal to $c_{2.}$

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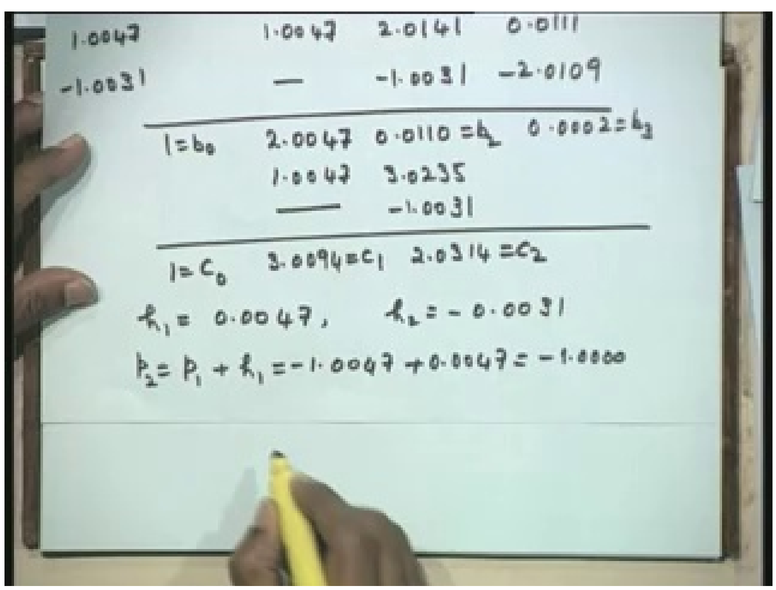

Now we obtain this quantity c_0, c_1, c_2, b_0, b_1, b_2, b_3. I will substitute in the previous values of h_1 h_2 and I would drop the computation, you can just verify that. I would get the values of h_1 as 0.0047. I get the value of h_2 as – 0.0031. So the next step approximation would be p_2 is p_1 plus h_1 that is – 1.0047 plus 0.0047 is equal to – 1.0000. Let me write down the next value.

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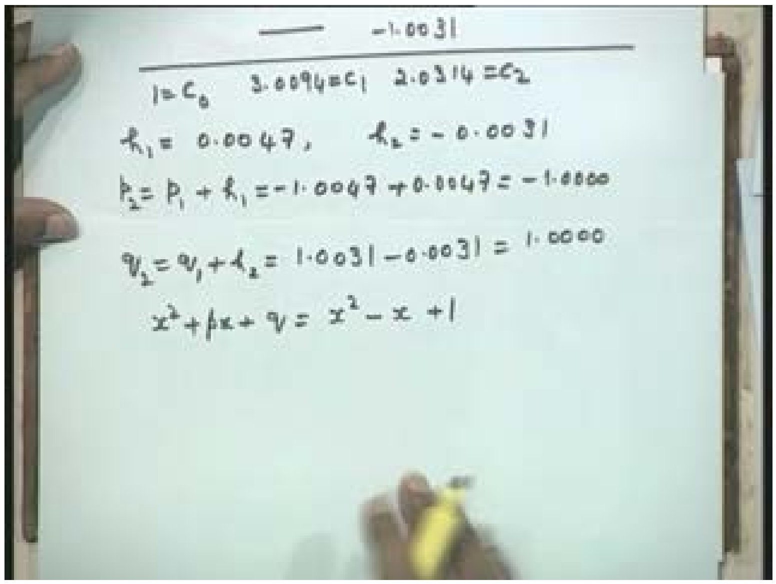

Now we have q_2 is q_1 plus h_2 and q_1 is 1.0031 and h_2 is – 0.0031 and that is 1.0000. Therefore the quadratic factor that is extracted is x squared plus px plus q that is x squared minus x plus one and it so happens that this is also the exact quadratic factor. Here we are able to get the accuracy because the h_1 plus h_2, the required accuracy has been obtained and the exact factor of the problem is x square minus x plus one.

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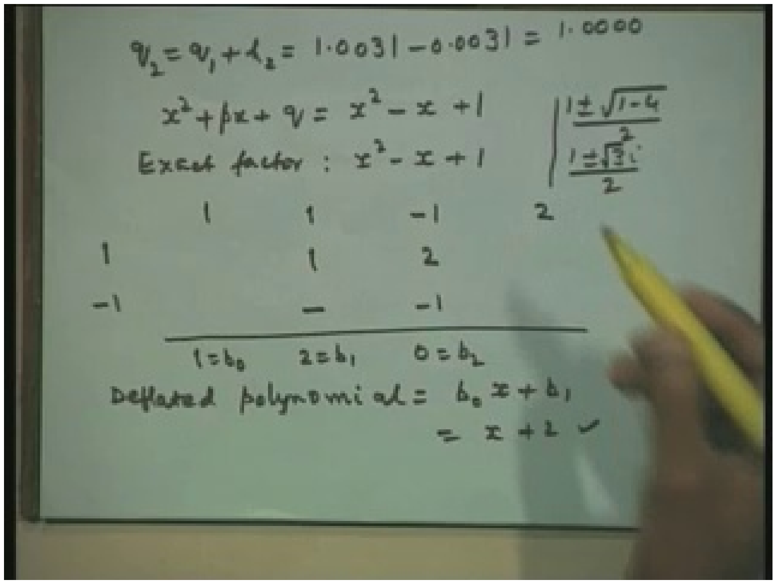

Now let us look at the second part of the coefficient to get the deflated polynomial. So let us write down first step of the synthetic division. So I will have this is 1, 1, - 1 and 2 as the coefficients; then minus p that is equal to 1 and minus q that's equal to - .1, this is minus p minus q. So this is $b_{0;}$ 1 into1; 2, that is equal to $b_{1;}$ 1 into 2, that is 2 - 1 that is equal to 0, that is equal to b_2. We don't need the last part because the deflated polynomial will be b_0 that is equal to b_0 x plus $b_{1;}$ that is equal to x plus 2. The deflated polynomial is simply x plus 2. So we have now extracted all the three factors. The linear factor has been obtained, so that the root x is equal to minus two is also obtained and this gives us a quadratic factor. Therefore this gives us the two roots. We can find them in the normal way. This is a complex pair.

So what is this complex pair; this one plus minus root of one minus four by two. So what we are really getting here, is the complex pair from this particular value.
So once we obtain the quadratic factor, the complex pair that is obtained from here and whatever is left out would be the deflated polynomial on which we can again use the same method to get the quadratic factor. But if it is finally an odd polynomial we land up into a linear polynomial. This is how we can apply the Bairstow's method to extract the polynomial. In fact this is the most commonly used method in the software that is available currently on your computers. The Bairstow method is the most popular method. B_2 is a cubic equation and we extracted a quadratic factor, therefore what is left out will be a linear polynomial. So it will be simply b_0x plus b_1 one will be the deflated polynomial; two units less than the degree that we started with.

Now we would give a direct method which of course is not available on the computer software but it is interesting to know a how a method can be applied sometimes when you have a real and distinct root case and it is easy to apply that particular method.

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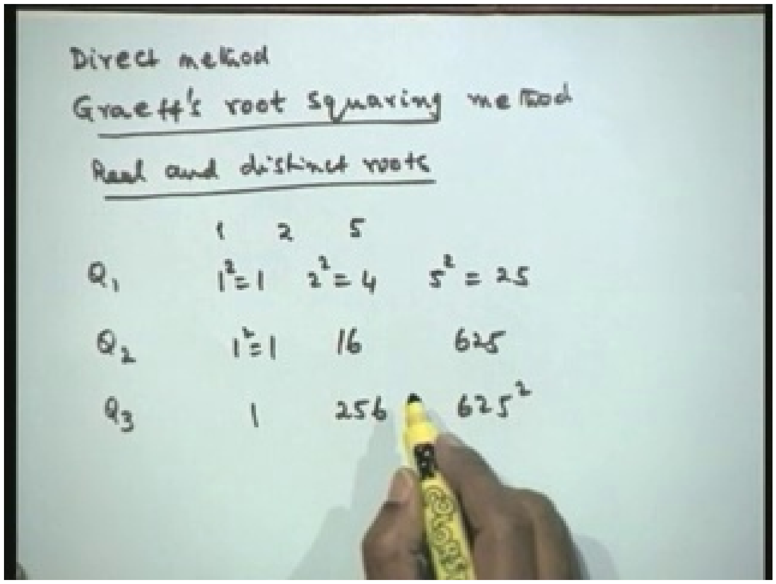

This is a direct method and the method is called the Graeff's root squaring method. The idea behind the Graeff's root squaring method is very simple. Suppose we will consider the case of only real and distinct roots. Suppose that a polynomial has got roots say 1, 2 and .5 Let us take this simple example, then we shall determine a polynomial whose roots are squares of these roots. So we will construct a new polynomial say Q_1 whose roots are squares of this; 1 square that is one; 2 square that is 4; 5 square that is equal to 25. We will see that we are going to put a negative sign for each one because of the reason of deriving it but essentially you can talk of magnitude of the squares. Then I will find another polynomial using the same procedure whose roots are squares of this previous one, which is now 1 square that is 1; 4 square that is 16; 25 square which is 625. Then I go on doing this root squaring procedure; what happens is after a few steps of this root squaring procedure the roots are separated by a large amount. The ratio of the leading one is very large compare to ratio of these ones. Therefore we will make this simple rule of finding the roots of a polynomial given that sum of roots is known; quantity product of two roots taken at a time is known; we know the simple formula of a polynomial. We shall just use that to obtain the approximation from this polynomial and that is because the roots gets separated so much that the lower ratios will be insignificant compared to the ratios that we have.

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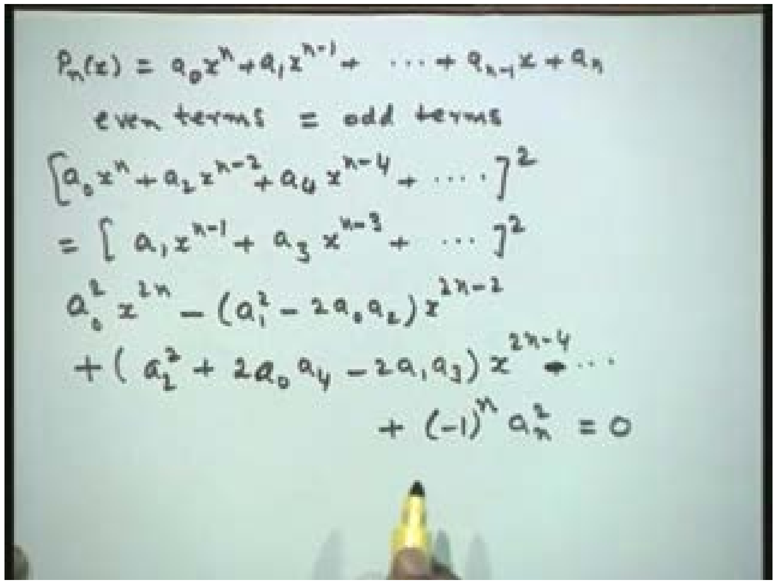

Now let me illustrate how we get this particular thing. So let us start with our polynomial Pnx. So that is your a_0x to the power of n; a_1x to the power of n minus one plus… a_n minus one x plus a_n. I will take all the even terms to the left, all the odd terms to the right or odd terms to the left and even terms to the right, one of the ways. So you will have the even terms on the left hand side and we keep odd terms on the right hand side, alternatively we can keep odd terms on the left and even terms on the right and that is starting with the first one. This is n, it is even. Let us say x to the power of n is even, x to the power of n minus two, x to the power of n minus four, all these will be on one side; and x to the power of n minus one, x to the power of n minus three, they will all be on odd, and they go to the other side. So what we will therefore have here is a_0 x to the power of n plus a_2x to the power of n minus two, a_4x to the power of n minus four plus so on and then I square both sides. So I do squaring on both sides. So this is a_1x to the power of n minus one, a_3x to the power of n minus three plus so on, square. So I take the even and odd terms to the right hand side and left hand side and then square on both sides. Then I simplify it, bringing now everything to the left hand side. So I would therefore have $a_0$square x to the power of two n that is first term; then we will have x to the power of two x because we have squared it; x to the power of two n minus two is the next term; x to the power of two n minus two will be contributing from here, the square of this and the cross product of these two will give you x to the power of two n minus two. That means I will have here on the left hand side two times a_0 and a_2; from the right hand side I will have minus a_1 square. So what I would therefore have is a_1 square minus twice a_0 a_2x to the power of two n minus two. So I am collecting the coefficient of x to the power of two n minus two on both sides. So the contribution will come only from these two from the left and this from the right.

Then I would write the next term x to the power of two n minus four. Now you can see two n minus four comes from the square of this as we move from the left to right, so this will give you two n minus four. This gives you two n minus four, so one term on the right hand side has come in to the left. This cross product a_0, a_4 is two n minus four. So there will be two terms from the left and one term from the right. The terms are $a_2$square, two n minus four; cross product, two times a_0, a_4 and this product coming to the left minus twice a_1 a_3x to the power of two n minus four plus so on; alternatively positive negative and finally we will land up with minus one to the power of n, an square is equal to zero.

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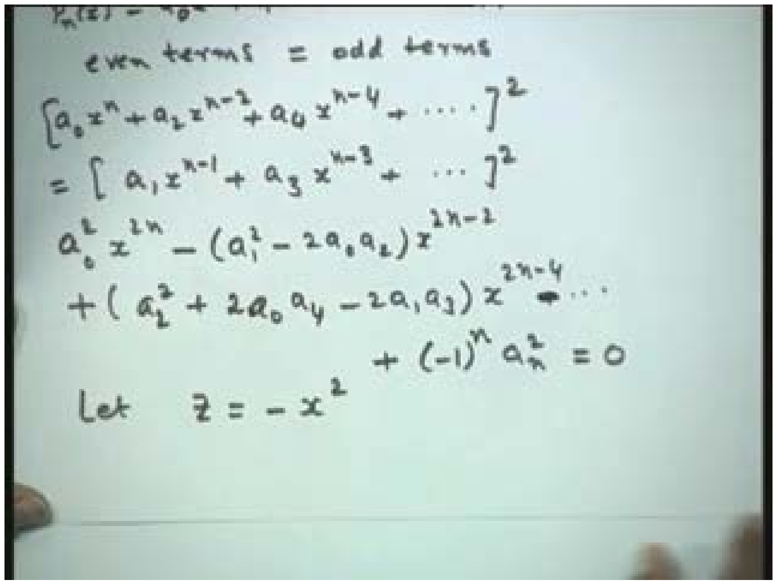

Now we have put minus one whole square, a_n square depending on whether we have got even terms on the left or odd terms on right, the suitable sign of this is taken care of. What we do here is, we said that I need a polynomial whose roots are the squares of the previous roots. Therefore in this I will substitute, I will let z is equal to minus x square. In fact I have now taken the opposite sign of the square of the roots. So in magnitude it will be square of roots. So I substitute z is minus x square. If I do that we are substituting z is equal to minus x square in this. So this will be x square to the power of n. So x squared to the power of n means minus z to the power of n, so a_0 square into minus one to the power of n, z to the power of n.

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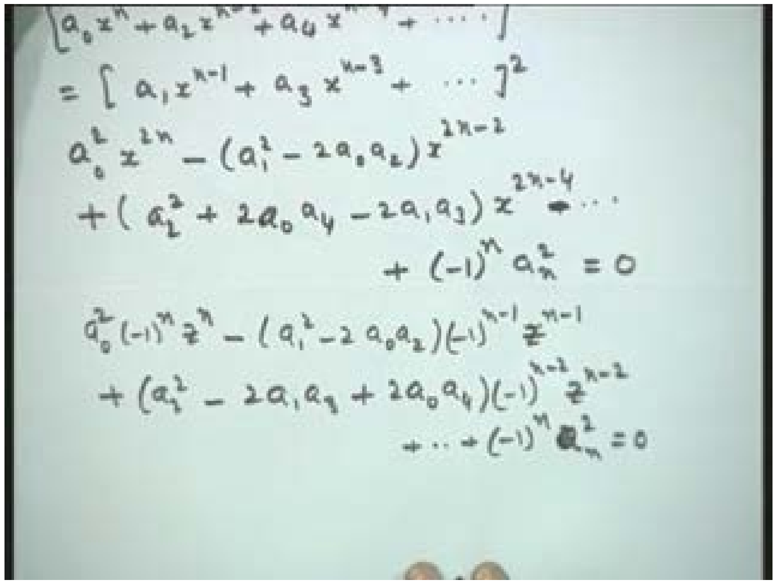

So the first term will read as a_0 square minus one to the power of n, z to the power of n; so I have substituted x square is equal to minus z. Therefore I will have minus one to the power of n coming from here, then I will have minus a_1 square, twice a_0, a_2, this will be x square to the power of n minus one, so I will have minus one to the power of n minus one, z to the power of n minus one. This is minus one to the power of n minus one, z to the power of n minus one, a_1 square minus two a_0. So the next term will be a_2 square minus twice a_1 a_3, I am writing first twice a_0 a_4. I will have minus one to the power of n minus two, z to the power of n minus two plus so on minus one to the power of n, a_n square is equal to zero. Now I can remove minus one to the power of n throughout. You can see this, minus one combines with this to become minus one to the power of n. This is same as minus one to the power of n minus one to the power of n is common throughout. So I can remove that.

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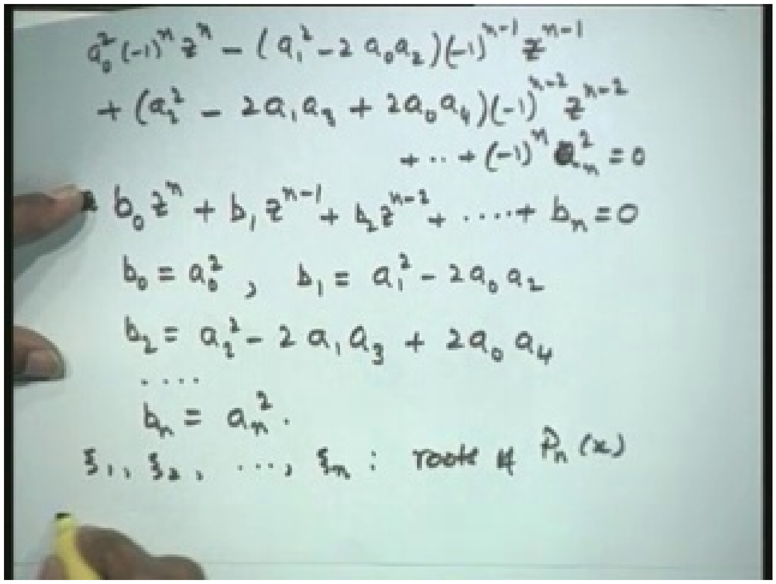

Let us use a notation b_0z to the power of n plus b_1z to the power of n minus one plus b_2 z to the power of n minus two plus… plus b_n is equal to zero. So that b_0 is equal to a_0 square, b_1 is a_1 square minus twice a_0 $a_{2,}$ b_2 is a_2 square minus twice a_1 a_3 plus twice a_0 a_4 and so on b_n is a_n square. Now this is a polynomial of degree n whose roots are the squares; we have substituted z is equal to minus x square, the roots of this are the squares of the previous root with an opposite sign. Therefore if the original equation has got roots xi_1, xi_2; if you take xi_1, xi2, xi_n as roots of Pnx, the original equation, then the roots of the new equation are minus xi_1 square, minus xi_2 square so on, minus xi_n square.

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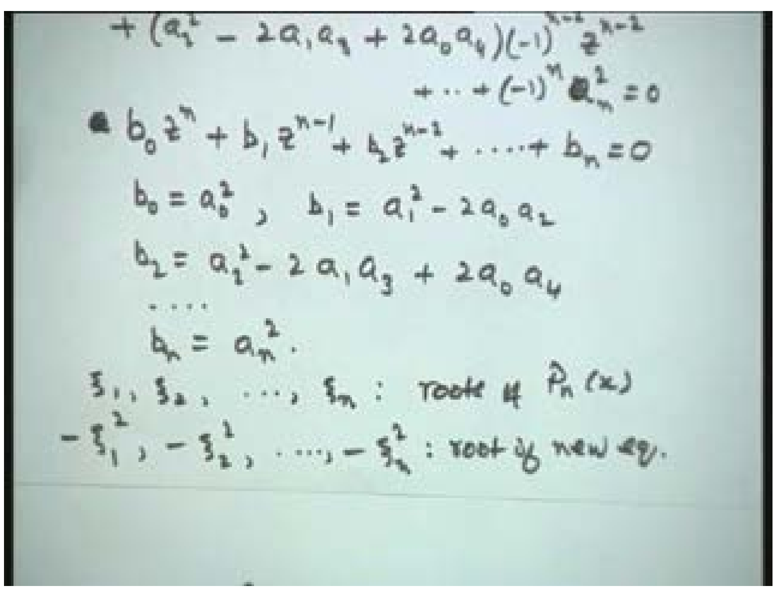

Now interestingly just as we have put the synthetic division procedure in a formal way, here we will put it in a tabular form and show how we can get all these coefficients without really looking into the polynomial. Let me write down the coefficient, and then I will explain the logic behind how we are going to write it. Let us first write down all the coefficients in the formula.

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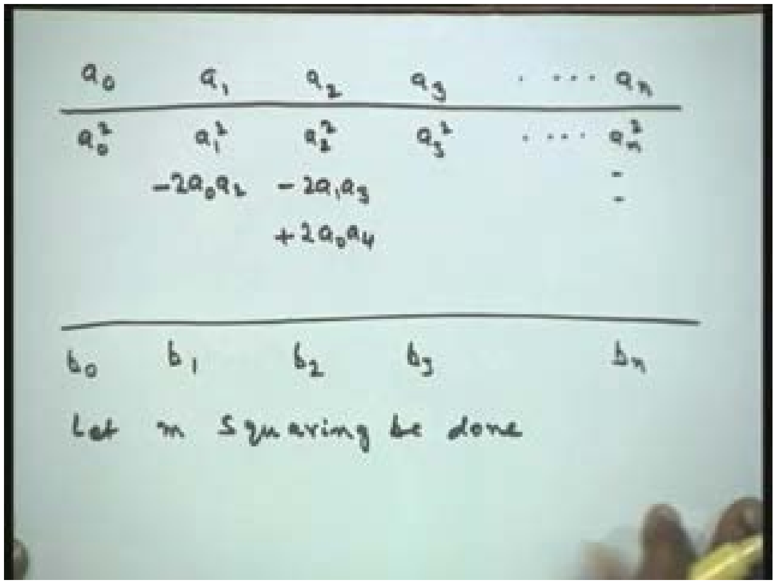

Let me write it here, it is equal to a_0, a_1, a_2, a_3… a_n. All these coefficients have got the squares of the previous coefficients in the first term. So I will have here a_0 square, a_1 square, a_2 square, a_3 square, a_n square. Now if you look at this next coefficient, this is the product; you take any particular term (that means location here) we take the nearest neighbors of that, multiply it with an opposite sign with a 2. So if I have a_0 square, there is no neighbor on the left hand side. Therefore there will be no term below that. Here I have a neighbor on this side a_0, a neighbor on this side a_2. Therefore I will multiply this and an opposite sign to it, I will put minus twice a_0, a_2. I will put minus twice a_0, $a_{2.}$ Now there is no other neighbor available to me on this side, so there will be no more term below this. I take a_2; a_2 has got neighbor a_1 and $a_{3.}$ So it will have twice a_1 a_3 coming from here. Then the next neighbor available to me is a_0 on this side, a_4 on this side, but they all go with an alternative sign plus, minus, plus, minus and so on; so the next one will be a positive sign, two times a_0 a_4. The next neighbor is available. So for there is no next neighbor now available on the left, so I go on putting this sequence in this fashion and here when I reach the last one there is no neighbor for it at all, so there is nothing. So I would just add it up. This will be b_0, this will be b_1, and this will be b_2, b_3 and so on b_n.

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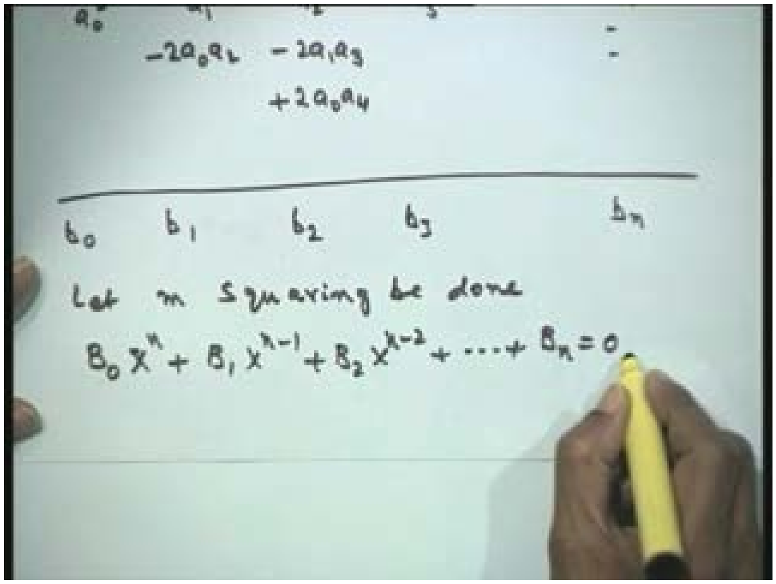

Now you can see for this roots squaring procedure we really do not have to look into it. Just take that particular location, look at the neighbors, what are the available neighbors, write down its cross products with plus or the suitable sign, add them up and that will give you b_0, b_1, b_2. Now let us say we have done m squarings. Now the final polynomial; let us take it as B_0 without loss of generality let us put back the coefficient as some x to the power of n, B_1 x to the power of n minus one, B_2x to the power of n minus two, plus so on plus B_n is equal to zero. So we are doing the last one.

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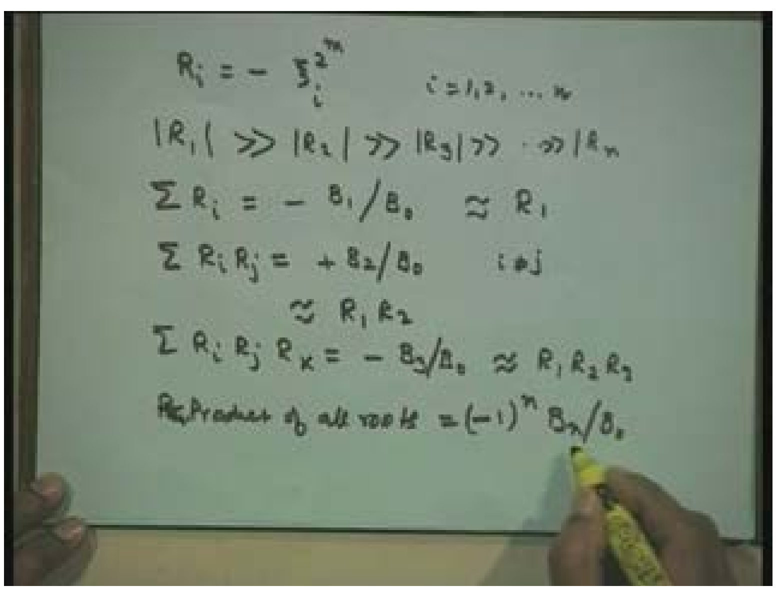

Now since we have done the m squarings, each time roots have become squares of each one. So after m squaring the root will become xi. This xi is two to the power of m with an opposite sign. Therefore the root of this, if you called it as a Ri; if you take this roots of this equation as this then these roots should be as follows; Ri must be equal to opposite sign xi_i two to the power of m, xi_i to the power of n; i is equal to 1, 2, 3, … n. Now since the roots are real and distinct if I take the magnitudes of this root, I can arrange the roots in magnitude from highest to lowest. So this root will be much larger than this root; this root will be much larger than this root and this root is being much larger than Rn. So we are arranging them in the descending order with the largest magnitude at the top and this for example, we have taken 1, 2, 5; we find it after two iterations 625, 16, 1. So we are now arranging the roots of that new equation finally in the order of R_1, R_2, and R3 of that final equation. I would now use the simple idea that the sum of roots in a polynomial is equal to minus B_1 by $B_{0,}$ which is our formula. Therefore the sum of the roots R_i is equal to minus B_1 divided by $B_{0.}$ Then we know that the sum of the roots when taken two at a time will be B_2 by $B_{0.}$ So this will be sum of R_i R_j is equal to plus B_2 by B_0; i_0 is equal to j of course taken roots, taken two at a time. Now we will write the remaining ones, since the roots are far separated I would approximate this as R_1, the leading root. In the example which I have taken 1, 16, 625, I will approximate the sums of the roots as 625, that's what we mean. So when we do few more squarings, the last root the, biggest root is the order of some ten to the power of fifteen; the next root is order of the ten to power of five or six; so the ratio of this is so big that we will approximate it as this. We will approximate R_1, R_2. For example R_i, R_j and R_k will be minus B_3 by B_0 and this I will approximate R_1 R_2 R_3 and the last one will be the product of all the roots. The product of all roots is equal to minus one to the power of B_n. We have to insert a suitable sign, based on whether it is positive or whether it is even or odd polynomial; so minus one to the power of n B_n divided by B_0.

From this we have a very simple rule of getting my root. R_1 is approximated by the first one. Now let us take the ratio of the successive coefficients. R_1 by R_2 two divided by $R_{1,}$ that will be R_2 and what is that ratio, B_2 divided by B_1 one with an opposite sign. Now I will take these two, if I take the ratio of these two, R_1 R_2 gets cancelled and I will get R_3. R_3 will be minus B_3 divided by $B_{2.}$ So I am getting my approximation for R_1 R_2 R_3 sequentially.

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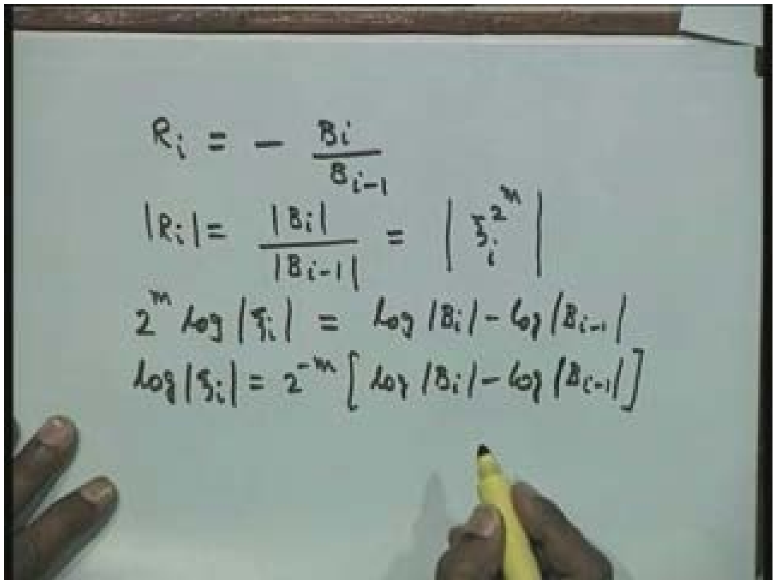

Therefore I am getting the ratio R_i will be equal to minus B_i by B_i minus one, that is simply theses coefficients with an opposite sign would give you the approximation. I will put this previous slide here; from this equation now we will be able to straight away write down what will be my required approximations for this. Once I write down this one I will find out what its magnitude is? Magnitude of this is magnitude of B_i by magnitude of B_i minus one and this will be magnitude of xi_i to the power of m; each root is xi_i to the power of two to the power of m. So this is the simple equation from which I must find out what is the value of the xi_i approximation. So let us take the logarithm on both sides. If I take the logarithm on both sides I will write it first, two to the power of m outside, logarithm of this. So I will have two to the power of m log of magnitude of xi_i and logarithm of this is log of magnitude of B_i minus log magnitude of B_i minus one and alternatively logarithm of magnitude of xi_i is two to the power of minus m into logarithm of magnitude of B_i minus logarithm of B_i minus one.

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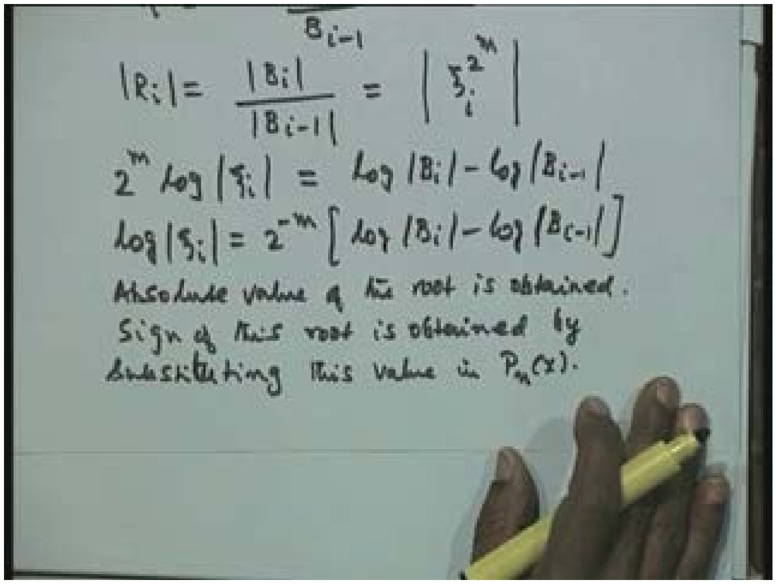

Therefore the root is obtained; it is going to give us the absolute value of the root. So we are getting the absolute value of the root. We need to find its sign; the sign is obtained by substituting the original equation. The sign of this root is obtained by substituting in the given equation; by substituting this value in Pnx.

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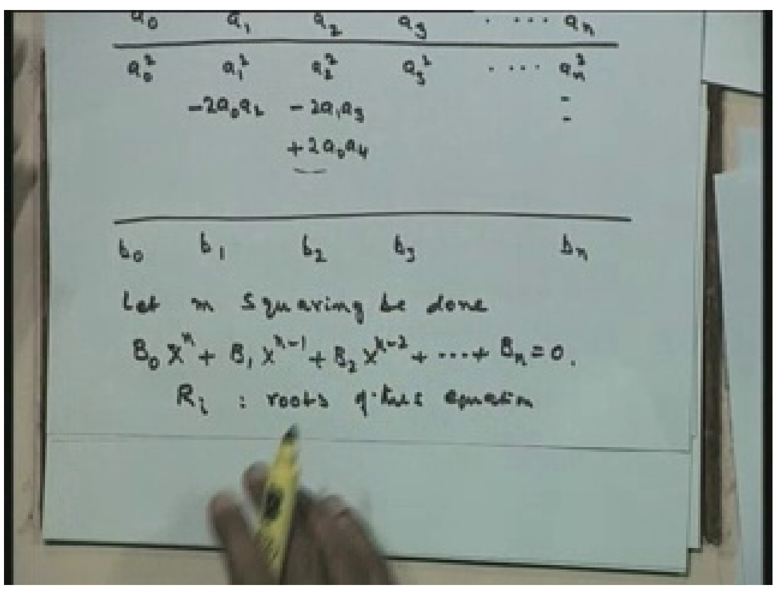

Now an indicator as to when you should stop, of course, we can do some two squaring three squarings and so on; an indication when the root squaring procedure should be stopped would be; if you look at that this particular table we have formed, when the contribution of these cross products becomes small compared to this a two squared, then this equation is almost identically same as the previous equation because the contribution of this is almost negligible. That means the stopping procedure is that we can see that the contribution of the cross products is negligible compared to the previous square.

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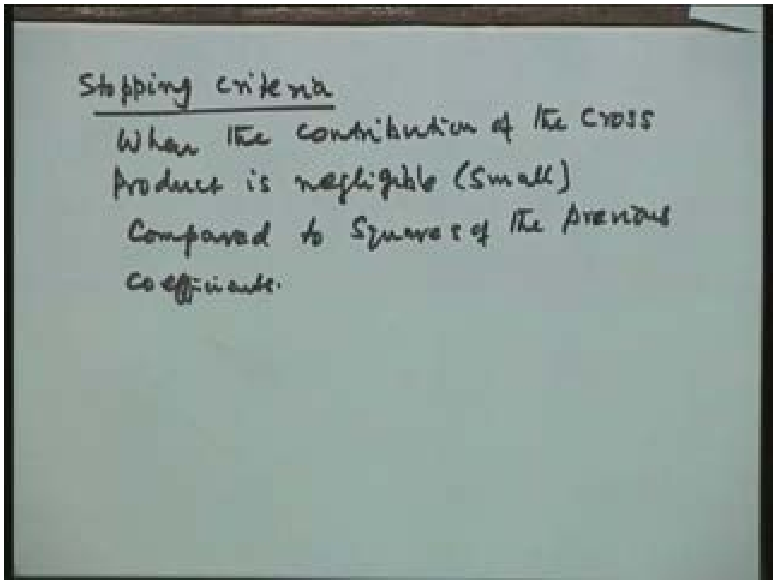

Stopping criteria is when the contributions of the cross products are negligible (or let us just call it as small) compared to squares of the previous coefficients.

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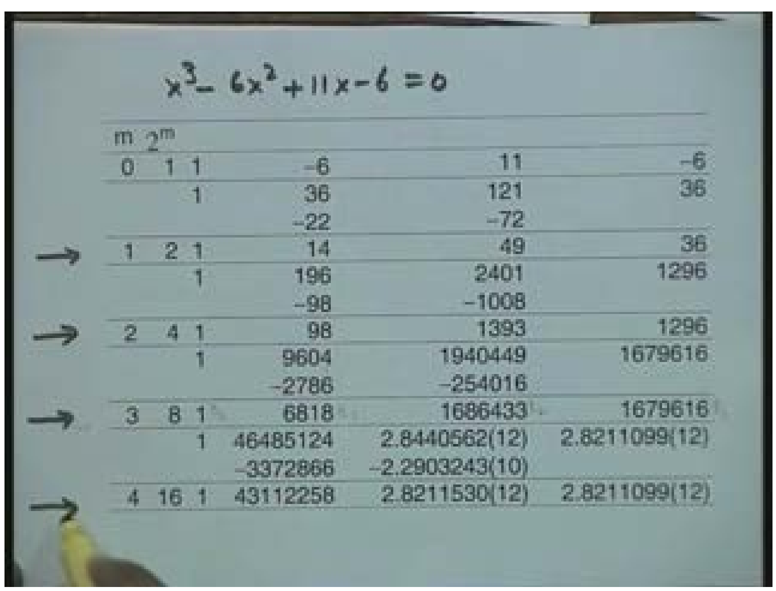

m	2^m				
0	1	1	–6	11	–6
		1	36	121	36
			–22	–72	
1	2	1	14	49	36
		1	196	2401	1296
			–98	–1008	
2	4	1	98	1393	1296
		1	9604	1940449	1679616
			–2786	–254016	
3	8	1	6818	1686433	1679616
		1	46485124	2.8440562(12)	2.8211099(12)
			–3372866	–2.2903243(10)	
4	16	1	43112258	2.8211530(12)	2.8211099(12)

I would just show how the terms would come out. The equation that I am considering here is eleven x minus six is equal to zero; this is the equation that is given here. This is your a_1; - 6, 11, - 6 are the coefficients, and then I will have 1 square, 36, 121, and 36. The cross product is – 22, this cross product is - 72 with an opposite sign, add up, this is the first squaring that is done. Then I square this, square this196; square 49; square 36; then the cross product for this is the product of these two and this two times 49 with an opposite sign; two times 14 into 36 gives this one and this is the next value, so this is your second squaring. I square 98, square this, square this, and this is the fourth squaring. Now you can see as I was saying, as we as go further, this number of squaring, these numbers are going to be enormous and small and they get separated very fast. Now what I would therefore require is the formula that we had written. I use this particular formula straight away on these coefficients; this is my B_0, B_1, B_2, and B_3. So I would now take the ratios of the required ratios. Depending on the number of squaring that we have done; here we have done squaring four, so I will have here two to the power of minus four xi_1 log B_1 minus log B_0. So I will take the logarithm of this minus logarithm of this divided by sixteen will give me the logarithm of the absolute value of the first root.

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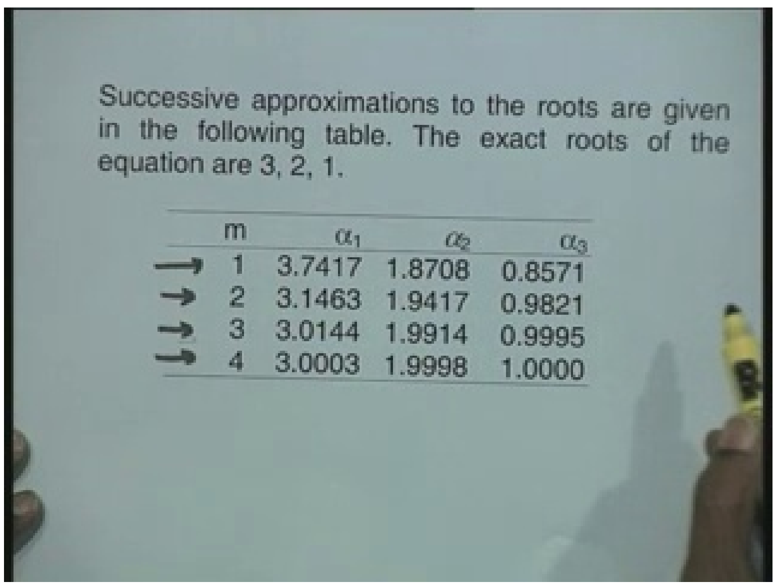

So with this if I do that, what I am getting is this particular value of the table. This is the final value that we have; the successive approximation to the roots, this is the first squaring, second squaring, third squaring and fourth squaring. This is first root alpha one, alpha two and alpha three. As I said this is equal to B_1 by B_0; this is B_2 by zero divided by sixteen and this is B_3 by B_2 divided by sixteen in the last case. In the first squaring we are dividing by two to power one that is two; next squaring we are dividing by two square two that is four; in the third squaring we are writing two cubed that is eight; in the last squaring we are dividing by two to the power of minus four that is two to the power of minus sixteen. So I am talking about this particular quantity, division of this particular quantity. So this table value is obtained and you can see the exact root for this is 3, 2, 1, it is the exact solution of this particular thing. For writing it manually it is okay but if you go to the computer, once these larger numbers are encountered there will be lot of round off error, there will be problems. Therefore this direct method is not usually in the computer software whereas the Bairstow is the most popular method that is used in the software. If you take any software and obtain the roots of the equation, all the roots of the equations the method will say that they have used the Bairstow method.

Now before closing let us make few comments on what we have done so far on this particular topic. So what we have done here is few methods for finding the roots of nonlinear equations. The first method we have done is bisection method which would never fail. We are never using the value of the function in bisection method, we are using only its signs but it is very slow. What is the accuracy that we obtain; each term we are getting a factor of division by two, which means in computer language we are getting a one bit of accuracy in each iteration. One bit of accuracy is too low for us and it is therefore it is very slow, even though it will never fail.

Then we have given Regula Falsi and secant methods. Regula Falsi method also would never fail because the root should lie in the interval that we are considering. Therefore the root is always is the in our interval, therefore we will never fail but it has got a linear rate of convergence. Therefore it is again very slow. The secant method or the Chebyshev method, Newton Raphson's, method multi pointed iteration methods, they all converge very fast. We know that Newton Raphson has quadratic convergence, Chebyshev has cubic convergence, multi point has got cubic convergence but there must be a little compromise between the two and the best compromise will be if you are not sure what is happening in the solution of the problem, use few iterations of the bisection or the Regula Falsi; so we are sure that we have now reduced the interval to sufficient lag. Suppose you started with a length of interval as sixteen, if you have used three times bisection, first bisection length will be reduced to eight, and second bisection will be reduced to four and the third bisection will be reduced to length two or even sufficient for us to do few two bisections or Regula Falsi. Then we jump over from there to a very powerful method like Newton Raphson or Chebyshev method, so that the convergence would be very fast. That is one way of tackling the problem where you do not have any knowledge of what is happening in the solution of the problem.

Now if the all the roots are required in a polynomial then the only thing is, we have to do Bairstow method. Lots of experiments have been done with Bairstow method and try to find out when Bairstow method may fail. The Bairstow method fails only in one situation and that is a very artificial situation; and that is let us take the polynomial which has got only one real root but all complex pairs. Let us say for the polynomial degree 21, 20 are complex pairs and one is real. Now you are choosing your initial approximation to the quadratic factor coefficients in the quadratic factor x square plus px plus q. Now choose it artificially that x square plus px plus q vanishes at the root. Let us suppose one was one of the roots. Now choose your px and qx square plus px plus q such that it vanishes at x is equal to one. Now since x square plus px plus q is a quadratic factor it cannot extract a quadratic factor because you have forcibly made it to be zero at the only real root that exists. Therefore this is the only situation where Bairstow will fail but then we have a solution for it. The solution is multiply it by x that is make it as an even polynomial that means we multiply by x, you are providing one extra root x_0. Now make it an extra zero. Now apply the same approximation which we have done in the previous case. Now it will definitely converge because now we have introduced a real pair by introducing x is equal to zero; zero and one is a pair now. Therefore this x square plus px plus q will converge in just one or two iterations because it has now got a pair zero. So one and zero will come and it will never fail. Therefore the best way if you are doubtful whether the equation has got only one real root and your approximation may be becoming zero at that root just make the given polynomial as an even polynomial so that the method would never will fail. Therefore as I was saying the software that is available to you often uses this particular idea. They try to make it as an even one. Of course there are various other improvements in the software which has made it very powerful software but the basic idea is using the Bairstow method.