Biomedical Ultrasound: Fundamentals of Imaging and Micromachined Transducers

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Lecture: 04

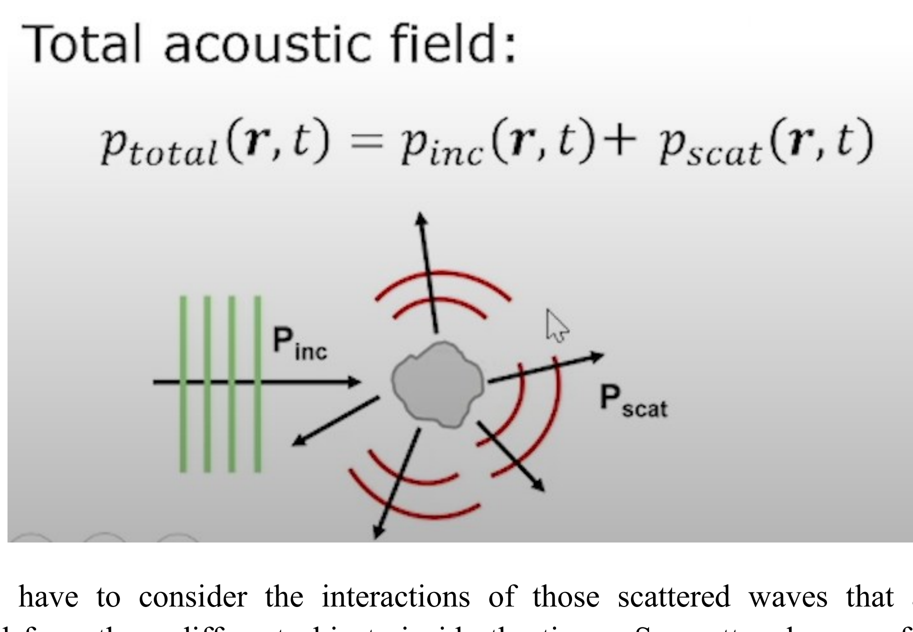
Scattering and acoustic wave equation

Hello, I am Professor Karla Mercado Shekhar and in today's lecture, we will discuss about scattering Now what is scattering? Scattering is any redirection or disturbance of the incident wave that is propagating in a medium We talked about reflection last time and reflection is actually a special case of scattering. Scattering is useful in ultrasound images because it allows us to visualize objects that are in the image

Here is, an equation of an incident wave in a homogeneous medium which satisfied the wave equation.

$$\nabla^2 p_{inc}(r,t) - \frac{1}{c^2}\frac{d^2 p_{inc}(r,t)}{dt^2} = 0$$

This you would have seen in a previous lecture And, the total acoustic field that is being generated in this medium, is the summation of the incident wave here that is propagating towards an object, and as well as the scattered wave that is being sent back towards the transducer as well as scattered into different directions, depending upon the scattering regime.



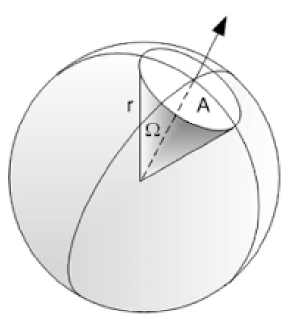
We also have to consider the interactions of those scattered waves that are being emanated from these different objects inside the tissue. So scattered waves from these tissue structures either combine coherently or incoherently You might have remembered from your previous physics class, about light and optics, where there's two two types of scattering: coherent and incoherent scattering.

In coherent scattering, the particles are very close to each other relative to the wavelength scale. So in this case, if the scattered waves are in phase they will constructively interfere with one another and if the scattered waves are out of phase they will destructively interfere. In terms of incoherent scattering, the particles are far away from each other relative to the ultrasound wavelength. Here, the scattered energies can be simply added, and constructive and destructive interference does not need to be considered in this case.

Coherent scattering	Incoherent scattering
• Particles are *very close together* relative to the wavelength scale • Phase is important • Constructive and destructive interference	• Particles are *far away from each other* relative to the wavelength scale • Scattered energies can be simply added • Constructive or destructive interference does not need to be considered

Now there is a metric that is used to quantify how much of the backscattered signal is being sent back to the transducer from the tissue And, this metric is called backscattered coefficient. The definition of the backscatter coefficient is the differential scattering cross-section, and that is the power scattered per unit solid angle in the 180 degree direction.

$$Differential\ Scattering\ cross\ section = \frac{power\ scattered\ per\ unit\ solid\ angle\ in\ 180^{o}\ direction}{incident\ wave\ intensity}$$



When we talk about scattering, the incident wave is actually propagating in the 0^{o} direction whereas the signals that are being scattered back into the transducers from the tissue is in the π direction or 180^{o} direction. And, in this formula, it's also divided by the incident wave intensity. Now we talk about per unit solid angle. So, in this case, when we talk about a two-dimensional circle, we look at the angle in terms of radians. A solid angle is a three-dimensional angle which looks into the amount of field of view of that object that is coming from a certain location in space.

The backscatter coefficient can be quantified by the following expression, wherein you have the back scattered power in numerator, divided by the solid angle times the incident intensity of the wave as well as the scattering volume.

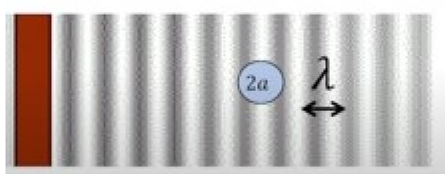
$$Back\ Scatter\ Coefficient(BSC) = \frac{backscattered\ power}{solid\ angle * incident\ intensity * scattering\ volume}$$

The scattering volume just represents the region of interest in your tissue that you were looking at, from which you are getting the backscatter signal.

Now, there are three scattering regimes that one can consider based on the size of the object that you are looking at. It depends on the object size relative to the ultrasound wavelength, and that will be quantified by this term, ka, where, k is the wave number, and that's inversely proportional to the wavelength. And, a is the radius of the scatterer.

$$ka = \frac{2\pi a}{\lambda}$$

So you can look into this schematic right here, wherein you have a wave field that is propagating where the dark regions correspond to the compressional part of the wave, and the white regions correspond to the rarefactional part of the wave. You can see here what one wavelength looks like. And, if the particle is of a particular size relative to the wavelength, it defines what the scattering regimes are.



So there are three scattering regimes that we will look at here.

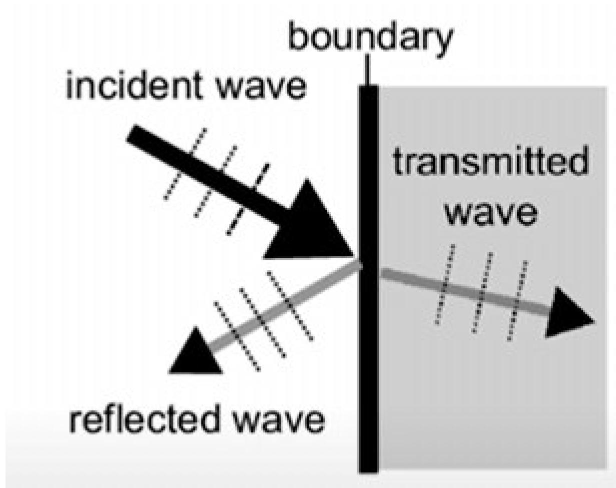
If ka >> 1, then we have specular scattering   In this case, typically what happens is that, there is reflection from smooth planar interfaces

The next regime is the Reighley scattering regime, in which case, ka << 1. In this case, you get scattering from really small particles in the object in the medium.

And finally, the diffractive or the Mie scattering regime, where ka ~ 1. In this case, the size of the object is basically approaching the size of the wavelength.

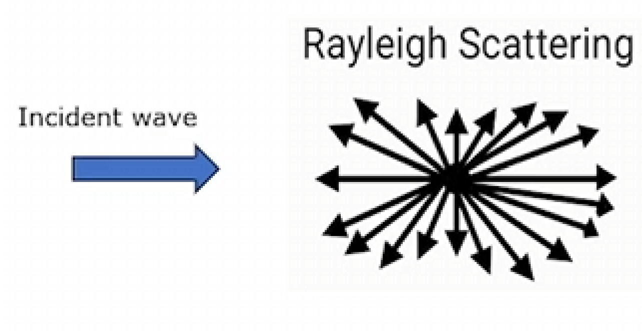
Now let's look at each of the regimes more closely. So for specular scattering, the object is much much larger than the ultrasound wavelength. So here, our ka parameter is again much much larger than 1. Here we show an ultrasound image of a liver and typically you will also see the diaphragm here in the field of view.

As we know, diaphragm is a muscle that helps us breathe. And what you can see is a muscle diaphragm that appears fairly bright in some cases, and slightly darker in a few of the cases. So, when we talk about the size of this diaphragm relative to the wavelength, it is fairly large. And, here you can see that the object appears good resolution, and typically for a specular scatterer, the detected size is pretty much accurate as you would see in the image. Also, what you would have noticed is that the strength of the signal varies depending on the orientation of the object. So for instance in the above image the transducer would be viewing from the top of the image and as it images through the tissue you can see that regions that are more flat in this image appear brighter than regions that are more angled, and that is because in specular scattering. The strength of the signal has a strong orientation dependence if you see the schematic below, you can imagine that the boundary is like a diaphragm. If the incident wave coming from the transducer is going in this particular angle then the reflected wave will also reflect back at an angle that is equal to the incident wave, but on the other direction So in this case not much of the reflected wave is actually coming back to the transducer and being considered as the backscattered signal.



So here's just an example of how the orientation of the specular scatterer would be affected and seen in an ultrasound image.

Next, we discuss Reighley scattering and this Reighley scattering phenomenon was actually derived by Lord Reighley who we have previously discussed before. He had published a book on the theory of of sound in the late 1800s. In this case the scatterers or the objects are considered much much smaller than the ultrasound wavelength. So in this case ka is $<<1$. What typically happens in Reighley scattering is that you have an incident wave that is going from the transducer, it then interacts with the object and scatters in multiple directions. Now this scattered signal typically has a weak angular dependence and typically these objects or the structures in the tissue cannot be visualized.



However such small structures can contribute to the background texture of the image. What kind of Reighley scatterers are can be found in an ultrasound image? Well, we know that tissues contain cells ,extracellular matrix proteins that are on the order of micron scale. So when we talk about ultrasound imaging at the clinical imaging frequencies the wavelength are on the order of 0.3 to 1 mm and so these cells and extracellular matrix proteins are quite small, much much smaller than the ultrasound wavelength, So they are considered Reighley scatterers.

Now here is an example of a scattering from a rigid sphere and this is the mathematical formulation of what a Reighley scattering would be if ka <<1.

$$\frac{I_{scat}}{I_{inc}} = \frac{k^4 a^6}{9r^2}(1 - \frac{3cos\theta}{2})$$

a = scatterer radius
r = radial distance from scatterer
θ = angle relative to incident beam

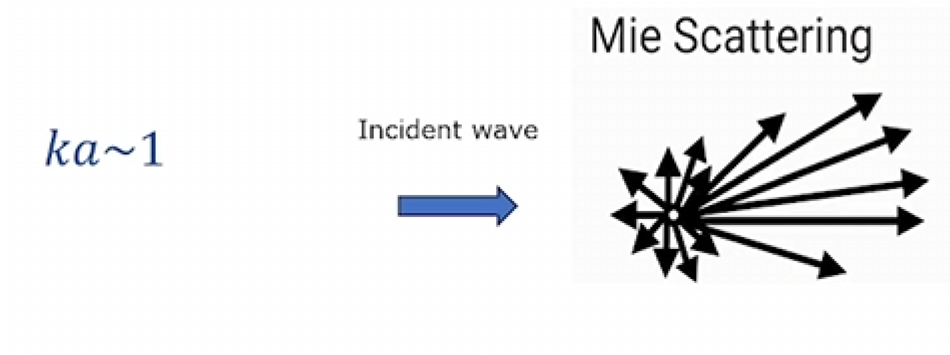
For backscatter, $\theta = \pi$

Although most parts of the body are not so rigid the rigid sphere approximation fairly holds in terms of the relationship of the intensity and the back scattered wave. So we can see here that we have a fraction of the scattered intensity over the incident intensity. So as you can see here, k is to the fourth power, it means that 1/λ is to the fourth power. So as the wavelength of sound decreases, the intensity of the backscattering would increase to the fourth power. So one example that we can think of in everyday life is our blue sky. We know that there are gas molecules in the atmosphere. They undergo Rayleigh scattering of light in this case. And we also know that in visible light, blue color, has the shortest wavelength. So this is the wavelength that would scatter the most, hence having the blue color in our sky.

In the body, we have red blood cells, cells that have an approximate diameter of about 7 microns, and if you use 5 MHz ultrasound with a 300 micron wavelength to view these red blood cells, then you would consider the red blood cells as being Reighley scatterers because they are much, much smaller in dimension compared to the ultrasound wavelength.

Now let's talk about the final scattering regime, which is the diffractive or the Mie scattering regime. This theory was developed by Gustav Mie, and the relationship between the ka value is on the order of one, meaning that the size of the object is of the order of one wavelength.

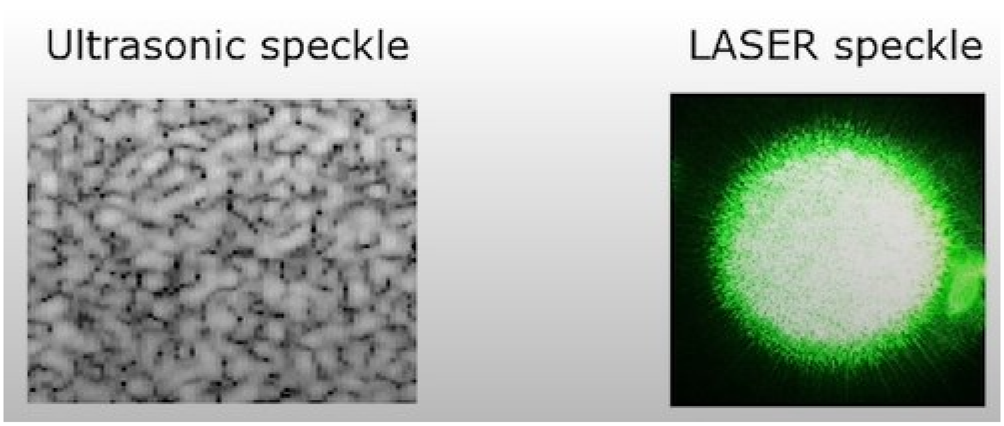
You can see here that the incident wave and a scatterer that undergoes the Mie scattering regime,where you can see that these scattering has a stronger angular dependence compared to the Rayleigh scattering regime.



So when we talk about Mie scatterers, the object size and its shape in the ultrasound image may not match with the actual dimensions that it has. For instance, if you view a Rayleigh scatterer, which can be a point scatterer, then you will see the same shape in an ultrasound image. But for Mie scatterers, this directional dependence of scattering actually impacts what the object will look like in an ultrasound image. So basically structures in the body that have a size on the order of a wavelength or in the order of millimeters can cause Mie scattering.

So for instance, when we talk about the atmosphere again, we have droplets in the clouds that undergo Mie scattering of light. So some of these clouds are gray or white, and almost all the wavelengths are scattered equally. So in this case, we know that visible light is almost white, so then we see some tinge of gray as well in the clouds, depending on the sizes of these droplets.

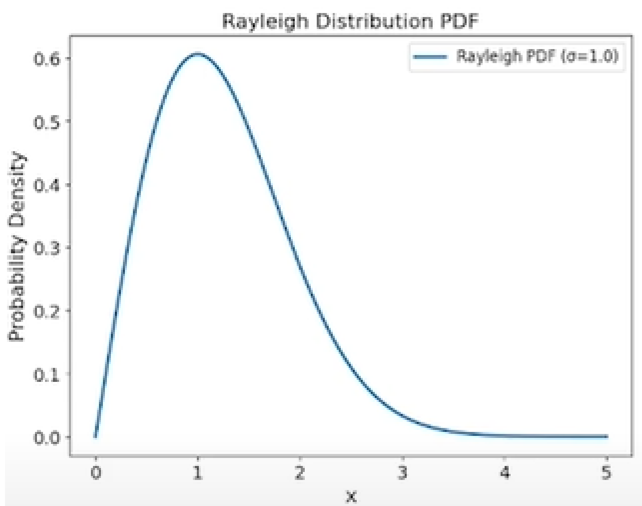
When we have all these scattered waves together interfering with one another, it forms what's called a speckle pattern. So this is inherent in any wave-based imaging modalities. For instance, when you have a laser pointer and you shine it onto a rough surface, you will see this granular pattern, similar to the image below. So the same thing happens in ultrasound as well . Ultrasound speckle looks like the image below on the left. So it's caused by constructive, destructive interference of the sound between the scattered signals from these very small scatters on the order of sub wavelengths.



And as mentioned earlier, this speckle can contribute to background texture of the ultrasound image. And the features of this texture also depends on the amount of scatterers that are in that tissue region that you are imaging. So the scatterer density also depends on the size of the scatterer as well as any acoustic impedance changes within those scattering interfaces.

Let us now discuss about fully developed speckle. So when the number of scatterers exceeds a critical value, inside a resolution cell, then the speckle pattern will turn into what's called a fully developed speckle.

In this case, the amplitude of this tissue scatter will follow a Rayleigh probability density function. And this density function looks like the figure below.



We can quantify this density function using the signal-to-noise ratio. Basically looking into the speckle signal-to-noise ratio, which is a function of the mean of the amplitude within that region of interest, divided by the standard deviation of that amplitude within the same region of interest.

$$SNR = \frac{Mean(Amplitude\ within\ region\ of\ interest)}{STD(Amplitude\ within\ region\ of\ interest)} = 1.91$$

And for a fully developed speckle it follows a Rayleigh probability density function, this SNR value would be 1.91.

So some speckle features also depend on the ultrasound frequency and beam properties and not on the macroscopic tissue anatomy. And also depending on the SNR, the organization of the tissue structures can affect what the SNR is. So we will later talk about the different types of speckle statistics.

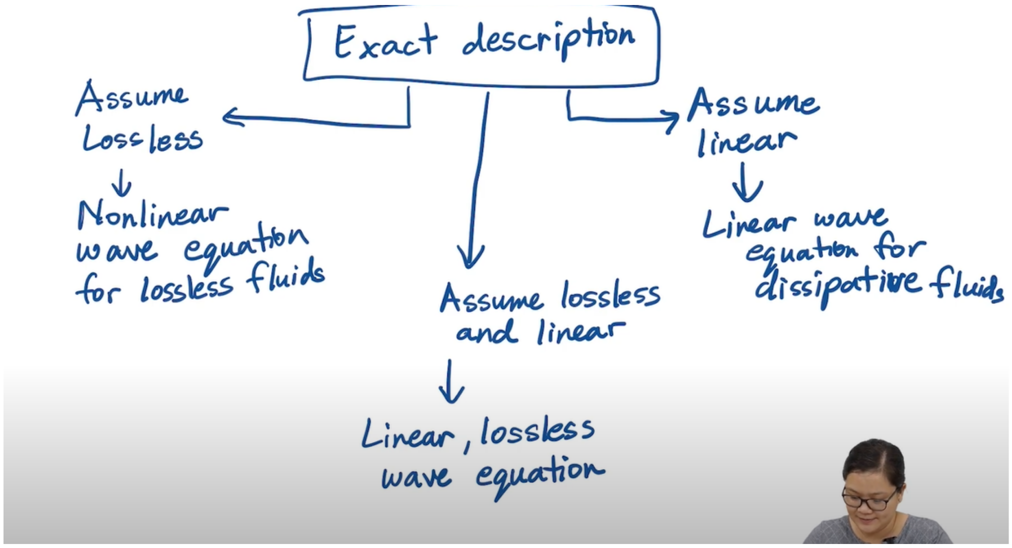
Speckle can sometimes be characterized as noise. But in ultrasound imaging, speckle is deterministic and not random like in electronic noise. The speckle can affect image quality, such as the contrast and resolution of the image, which we will discuss more in a subsequent lecture. But it can also be useful in some cases, such as when you're doing motion tracking, or when you are doing Doppler imaging and tracking the motion of certain structures in the tissue. Also in elastography, when you're inducing some shear waves that will then be used to track the motion of the tissue, and then getting some elastic property or mechanical property of the tissue.

Figure below shows an ultrasound image of a kidney. And you can see that most of the image has speckle in them. This speckle is important for assessing the background texture of the tissue. So in summary, we've talked about the scattering phenomenon. We looked at different scattering regimes, such as specular scattering, Rayleigh scattering, and Mie scattering, which depends on the size of the object or the structure in the tissue relative to the ultrasound wavelength. We also looked into speckle and ultrasound images. Now this lecture will form the foundation for the next stages when we look at ultrasound imaging.

Now let's change track slightly. In a previous lecture, we talked about the wave equation. Now let's derive the wave equation. So let's start out with several assumptions that we use to derive this equation. So if we assume that the medium is lossless, meaning that there is no decrease in pressure amplitude with distance of propagation, this will give us a nonlinear wave equation for lossless fluids.

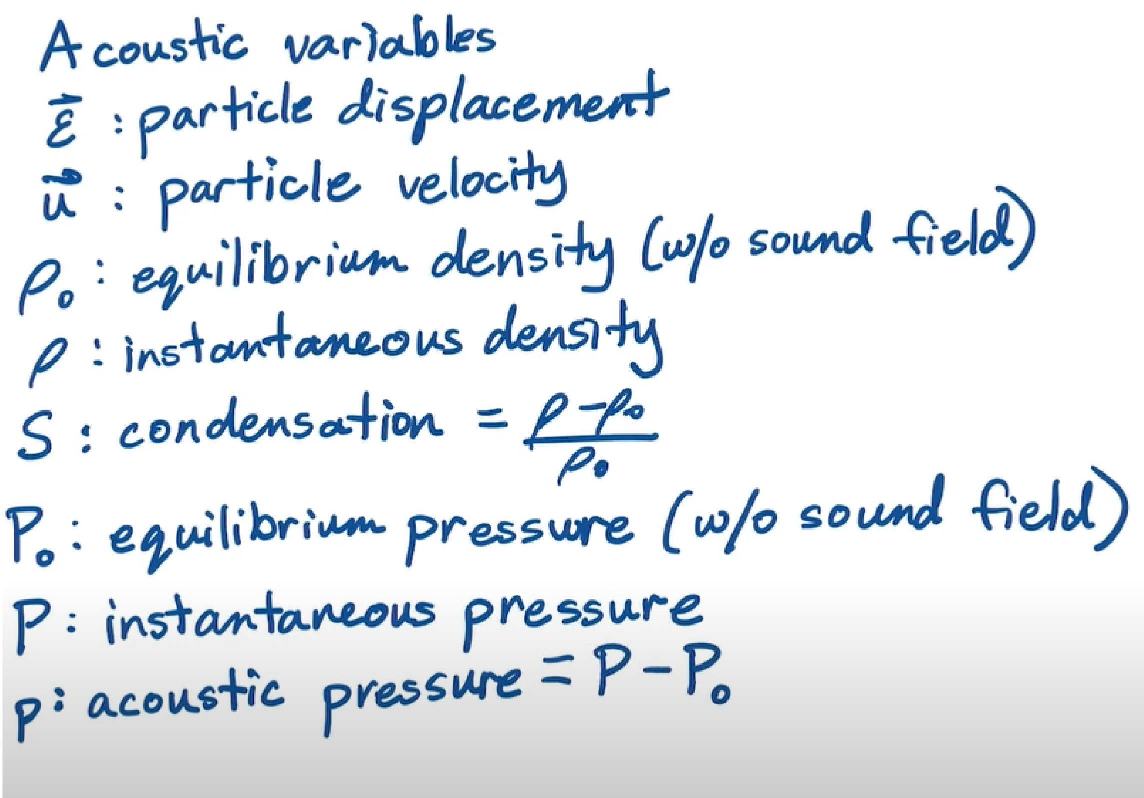
Another assumption would be to assume that the amplitudes of the acoustic pressure variations in the medium are negligible then we assume linear. And this will lead us to the linear wave equation for dissipative fluids.

Now if we assume both lossless and linear, then we would get our linear lossless wave equation.



Now there are several acoustic variables to keep in mind, and some of which we have already discussed in other lectures. First we have the particle displacement(ε), which is a vector. Then we have the particle velocity, which also a vector denoted by u. We have our equilibrium density that's denoted by ρ_0. The equilibrium density here is without the sound field. And then we have our instantaneous density, when the sound field propagates through the material (ρ). We also define a condensation parameter, which is denoted by S. The condensation parameter is defined by

$$S = \frac{\rho - \rho_0}{\rho_0}$$



We also define equilibrium pressure(P_0), without the sound field. This is the pressure of the material at its resting state without the ultrasound field. Then we also define the instantaneous pressure, which is denoted by P. And that's the instantaneous pressure as the sound field propagates through the medium. Then we define the acoustic pressure, which we have looked into before, and that is denoted by the p.

$$p = P - P_0$$

The longitudinal speed of sound is denoted by c, and the bulk modulus, which is a material property of the medium is denoted by B. And you will see that B is very much related to the speed of sound. In a previous lecture, we have looked into three equations which was used to derive the acoustic wave equation. So I'll just go over these three equations here

We talked about the equation of state, that relates the acoustic pressure, to the density of the medium. We also have the equation of continuity that relates the particle velocity, u, to the density via the conservation of mass. And finally, we have the equation of momentum, which relates the pressure, P, to the particle velocity via the Newton's second law. So we will use these three equations.

We're not going to derive exactly how these equations were derived, but we'll use this, and put them together to derive the acoustic wave equation. Depending on several assumptions, we'll give an approximate version of each of these equations. So let's look at the equation of state first. We assume small amplitude variations in the medium, And this is after linearizing the exact description. Then the equation of state becomes,

$$p = BS$$

Next, the equation of continuity. Assuming that the condensation S is small, the instantaneous density is approximately similar to the equilibrium density $\rho \sim \rho_0$. And so the equation becomes,

$$\rho_0 \frac{\partial S}{\partial t} + \nabla.(\rho_0 \vec{u}) = 0$$

Now finally the equation of momentum. Here we assume small amplitudes, and the equation of momentum becomes in the form,

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla \mathrm{p}$$

We will use these three equations to help us derive the acoustic wave equation. First, we take the divergence of both sides in the equation of momentum and this is what we would get.

$$\nabla \left(\rho_0 \frac{\partial \vec{u}}{\partial t} \right) = -\nabla^2 \mathrm{p} \qquad -(1)$$

Next, let's take a look at the equation of continuity. We will take the time derivative of both sides in that equation of continuity.

$$\rho_0 \frac{\partial^2 S}{\partial t^2} + \nabla. \left(\rho_0 \frac{\partial \vec{u}}{\partial t} \right) = 0 \qquad -(2)$$

Now note that the density here is not dependent on time. So we don't do partial derivatives of the ρ_0 .

Now if we combine this first and second equations, then we would get the following equation.

$$\rho_0 \frac{\partial^2 S}{\partial t^2} = \nabla^2 \mathrm{p} \qquad (3)$$

The last step is to include the equation of state

$$S = \frac{p}{B}$$

Also,

$$B = \rho_0 \, c^2 \qquad (4)$$

So equation 3 becomes,

$$\frac{\rho_0}{B}\frac{\partial^2 p}{\partial t^2} = \nabla^2 \mathrm{p}$$

Introducing the speed of sound from equation (4), the final linear lossless wave equation is:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 \mathrm{p}$$

This equation describes how acoustic pressure changes in space and time. See you in the next lecture. Thank you.