Networks and Systems
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Lecture-64
General LTI Systems and more about H(s)

In the last lecture we worked out several examples, illustrating the application of the Laplace transform technique, to the solutions of transients in networks. We also made special mention, of the particular virtues that Laplace transform technique brings, to the solution of such problems.

We compared the Laplace transform method the differential equation approach, and pointed out the various benefits that the Laplace transform technique will bring in. We also made a note that, where the initial conditions are specified at different points of time, not necessarily all t equals 0.

The differential equation approach will be probably more beneficial than the Laplace transform technique, because there is no provision in the Laplace transform of technique, to take care of specification of conditions, at different points of time. If you have such a situation then you once again land up in arbitrary constants in the Laplace transform solution, so there is no special advantage in that.

Today will start with another example of a network transient problem, in which a discontinue source is present, and then later on we move to a discussion, of the application of the Laplace transform technique through a general system.
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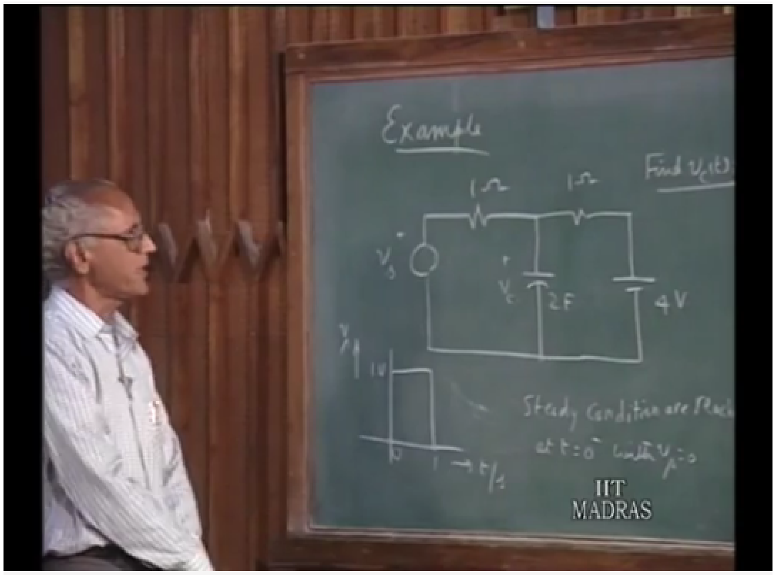

So, let this be a voltage source v s included in the circuit of this type. The capacitance of 2 F and dc source of 4 volts, and this v s is a discontinuing a source, having one volt a pulse lasting from 0 to 1 second. And we will also assume that steady conditions are reached at t equal 0 minus with v s equal 0.

So, before the pulse is switched on, v s is off course 0 and that time we assume the steady state conditions are reached. The question that is asked is, find v s find v ct for t greater than or equal to 0. This is the question that is asked. Now, we set up the transform diagram for this. Before doing that we need to know, because there is no reactive element the capacitor we should like to know; the 0 minus value the capacitor voltage.

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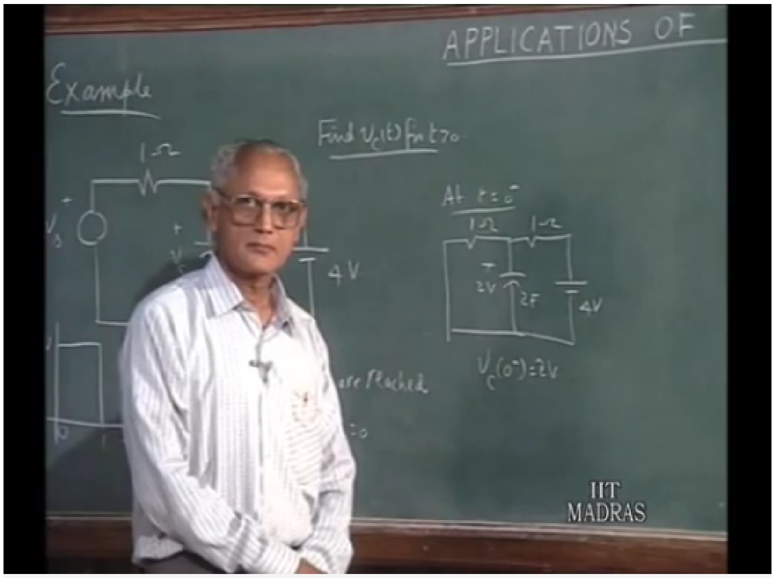

So, to do that let us see the system, the various parameters the system at various variables in the system at t equals 0 minus. At t equals 0 minus this voltage source is 0. You have a 1 ohm resistance, a capacitor, another 1 ohm resistance and a dc source. This is 4 volts, this is 1 ohm, this is 1 ohm, and this is the capacitor of 2 farads.

So obviously, if the steady state conditions have been reached in the circuit, this 4 volts will drive a current of 2 amperes to the two 1 ohm resistance in series, and there will be a 2 volts develop across the capacitor. Therefore, v c 0 minus is 2 volts. So, that is the voltage initial condition of the capacitor that you have to use in solving this problem.

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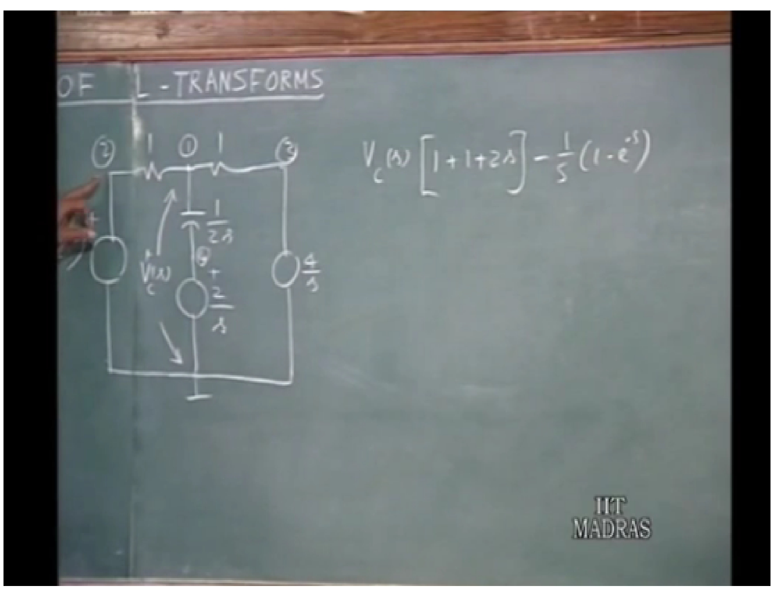

So, let us from the transform diagram of this. Transform diagram of this would be, for the voltage source, you have the Laplace transform of this voltage source. This is u t minus u of t minus 1, this is a pulse function. Therefore, the Laplace transform of this to be 1 by s times 1 minus e to the power of minus s; that is the voltage source.

In addition we have this resistance another resistance, so 1 ohm, and then a capacitance whose impedance is generalized impedance 1 over 2 s, and then a source representing the initial capacitor voltage this is 2 volt source. Therefore, this will be 2 up on s, and then you have the Laplace transform of this dc voltage source, which is 4 up on s.

Now, the capacitor voltage is what is required. So, it is the voltage across these 2 nodes. This is v c of s. As pointed out in the last lecture, when you interesting to finding the capacitor voltage, you should not try to measure the capacitor voltage across this portion only, because it is the totality of this 2 together, is what represents this 2 farads capacitors, including the initial condition source.

So, when you want the capacitor voltage, it is the voltage between these 2 points, which has to be calculated. Now, a simple way solving for this would be, take the load equation approach, take this as the datum node; and call this 1, this as 2, this as a 3, and perhaps this as 4, and therefore, if you find out the voltage of node 1 that will yield you a capacitor voltage which is the voltage of course with reference to datum node, node 1 represents to datum node.

So, write in the node equation at node one v c of s times. The sum of the admittance is connected to be this node; 1 plus 1 plus 2 s minus the coupling term with reference to voltage with other nodes. So, with reference to node 2, we have 1 by s times 1 minus e to the power of minus s; that is the voltage of node 2.

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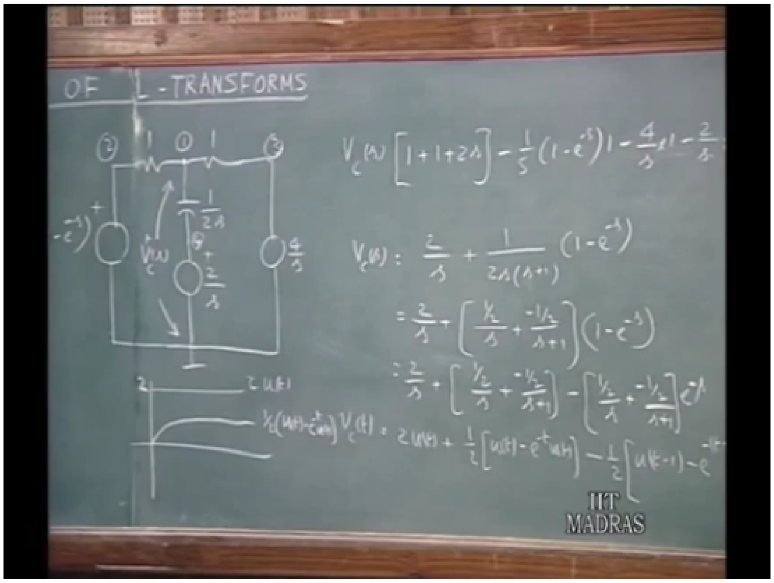

And the coupling admittance between node 1 and 2 is 1; therefore, times 1 minus node 3 has the voltage 4 up on s, and the admittance of the element connecting nodes 1 and 3 is 1 times 1 minus node 4 as the Laplace transform voltage 2 up on s, and the admittance between one and 4 is 2 s, generalized admittance equals to 0.

So, if you all the terms here are known expect v c of s. So, if you transfer these quantities on the other side, and find out the expression for the v c of s, it turns out to be 2 up on s plus one over 2 s times s plus 1 times 1 minus e to the power of minus s; that is what it will be. So, you can write this as 2 up on , and make the partial fraction expansion of this portion already, because later on we interpret, this one minus e to the power of minus s, how it reflects in the time domain.

So, will make the partial fraction expansion of this only, that will be a consisting of 2 terms half and minus half times 1 minus e to the power of minus s. So, essentially therefore, you have three quantities we deal with; 2 upon s is 1 plus half upon s plus minus half upon s plus 1; that is one group of terms, and the same group of terms multiplied by e to the power of minus s.

So, if you find the inverse Laplace transform of each one of this groups, and add them of that will be the v c of t. So, v c of t would be, this will be 2 times u t. Here you have half u t minus e to the power of minus t u t; that is the inverse Laplace transform of this, and you observe that this is the same thing as this except multiplied by e to the power of minus s, which means whatever time function we have here, is delayed by one second; that all means.

Therefore, it is minus half of u of t minus 1 minus t minus 1 u of t minus one, so that is what it could be. You can sketch this v c of t. So, v c of t would be. This 2 u t will be a stuff function like this. This is 2 times u of t, and this is u t minus e to the power of minus t u t. So, start with the 0, and exponentially claims to a value equal to half.

So, this is half of u t minus e to the power of minus t u t, there is what could be. As for as this portion is concerned, it has a negative value, is a negative of this, but it starts at t equals one.

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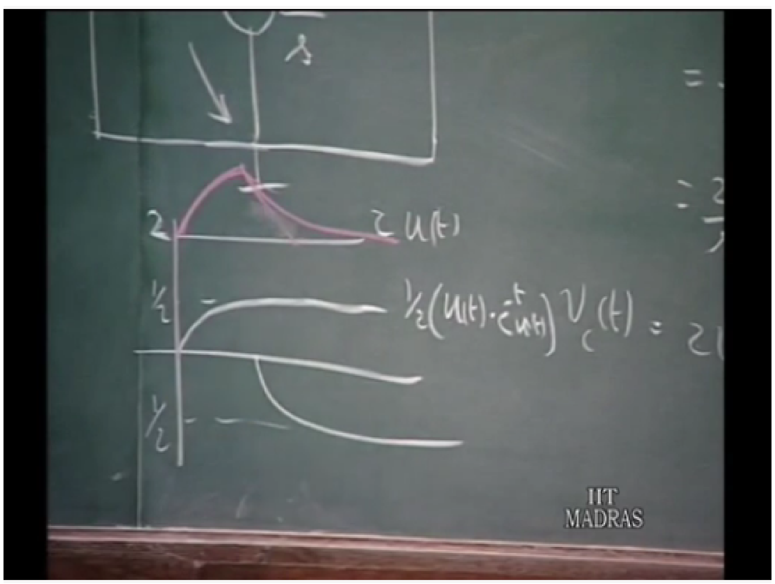

Therefore, it will be something like this, and asymptotically reaches half. So, if you add all this you will get, some curve like this, in the case. It has this 2 will add up to 0 ultimately. So, this is your total circuit. So, that is v c of t. So, this problem illustrates how you can handle the solution when there is a discontinuous time function.

And how the discontinuities or time function is of course this is discontinuous, and how the finding out inverse Laplace transform. When you have the e to the power of minus s are such terms are present, how you can find out inverse Laplace transform. Make using the property, that e to the power of minus s means, that particular function of time is delayed by one second. If it is the e to the power of minus t s is delayed by t second.

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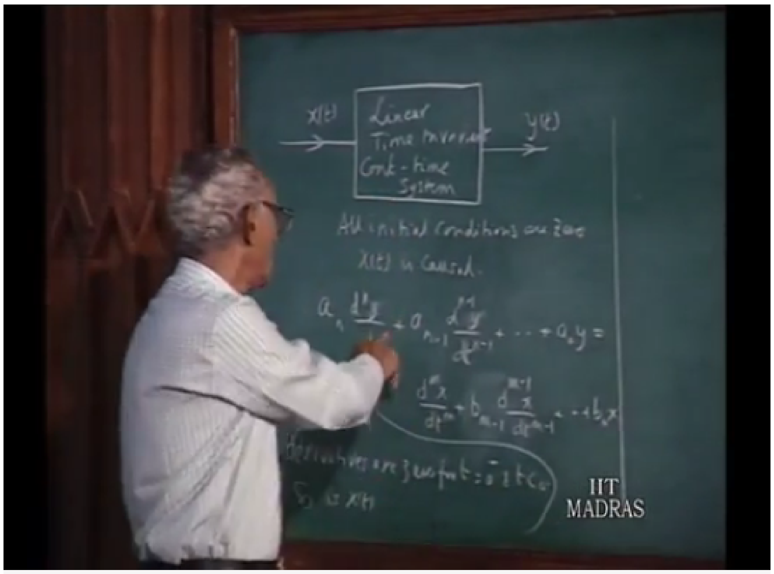

To discuss the application of the Laplace transform technique, to a general system. Let us consider a system represented by this black box. This is a linear time invariant, continuous time system with an input x t and an output y t. We shall assume that all initial conditions are 0; that is the output response quantity y t and all its derivatives are 0 prior to the application the input x t, will also assume that x t is causal; that means, x t is identically 0 t less than 0.

Now, we have taken a single input single output system, and if there are really a number of inputs multiple outputs. We can find out the response to each of the individual inputs in term and use the super position principle. Therefore, there is no loss of generality involved, in our taking a single input single output system.

We have seen that such a linear timing variant continuous time system, is in general described by differential equation of the type; a n d n x d t n plus an minus 1 the n minus

1 derivative, sorry this would be y output quantity y d n y d t n d n minus y d t n minus 1 on up to a not y equals dm x d t m plus d minus 1 x d t minus 1.

We have a coefficient d m minus 1 here d minus 1 x by d t m minus 1 up to b naught x. This is a n order differential equation with cost and coefficients. Now, we assume that y t and all derivatives are 0 for t equals 0 minus and t is less than 0; that means, identically the circuit is dead, as for as the response quantities is concerned, and so is x t, because x t is a caution function is identically 0 for all negative values of time.

So, under these assumptions, we can take the Laplace transform of the all the terms on the left hand side and the right hand side. in doing since all the initial conditions are 0, if the Laplace transform of y is y of s, the Laplace transform of d n y d t n is simply s to the power of n times y of s.

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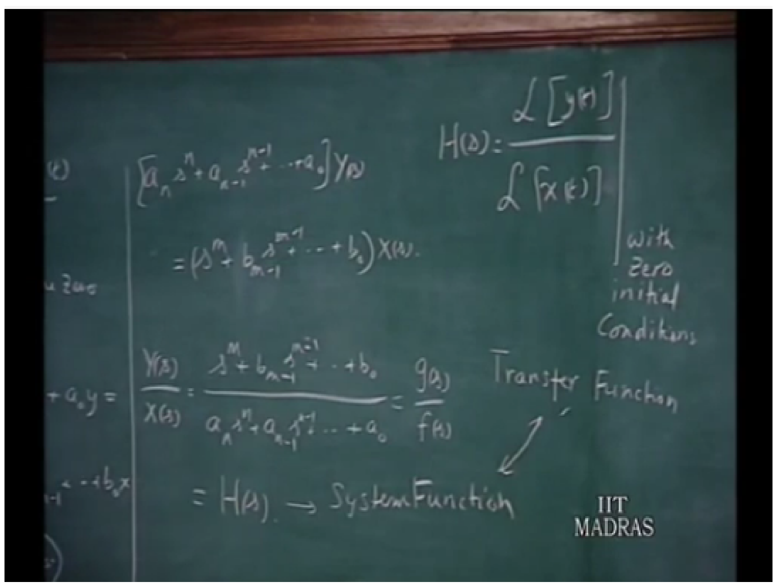

So, that makes the transforming this equation a very simple task. So, if you take the Laplace transform of that, you have a n s to the power of n plus a n minus 1 s to the power of n minus 1 times a not plus a not times y of s equals the Laplace transform of the side of equation, which is s to the power of m plus a b m minus 1 s to the power of m minus 1 on up to b not multiply x of s.

So, that is the differential equation, after transforming its after make in the Laplace transform of the differential equation is converted in algebraic equation, relating the transform of the response quantity to transform the important quantity.
Now, the ratio of y of s to x of s, is therefore, the ratio of 2 polynomials s to the power of m b m minus 1 s to the power of m minus 1 plus b naught divided by a n s to the power of n a n minis 1 s to the power of n minus 1 plus a naught. This we can put ratio of 2 polynomials g of s over f of s.

Now, I have taken the leading coefficient here to be 1. It does not entire any loss of generality, because even if there was leading coefficient here which is different from one. We can divide all the terms by the term and make this 1. Therefore, there is no loss for generality involved in taking the leading coefficient here to be equal to 1.

Now, this y of s or x of s is derived in this fashion called h of the system function. So, formally we can say the system function h of s, is the Laplace transform of the output quantity y of t by the Laplace transform the input quantity x of t with 0 initial conditions. What we mean by the 0 initial conditions is, that the response quantity and all its derivatives are going to be 0 and 0 minus, and the before the application is input x t.

So, in this particular case we have to extend to be a causal function which is identically 0 of t equals 0 minus, and then until the time the response quantity in all its derivatives are going to be 0. Therefore, each of the various transformed in to Laplace transform of domain d n y d t n goes as simply s to the power of n times y of s. The system function is also referred to as transfer function in literature.

Particularly in the context of networks, control systems and so on. This is also refers to as transfer function, another name for this. So, essentially when we talk about the system function in the Laplace transform situation like this. We are change that is the ratio of the Laplace transforms of the response quantity the Laplace transform of the input quantity, with 0 initial conditions appear to the application to the input.
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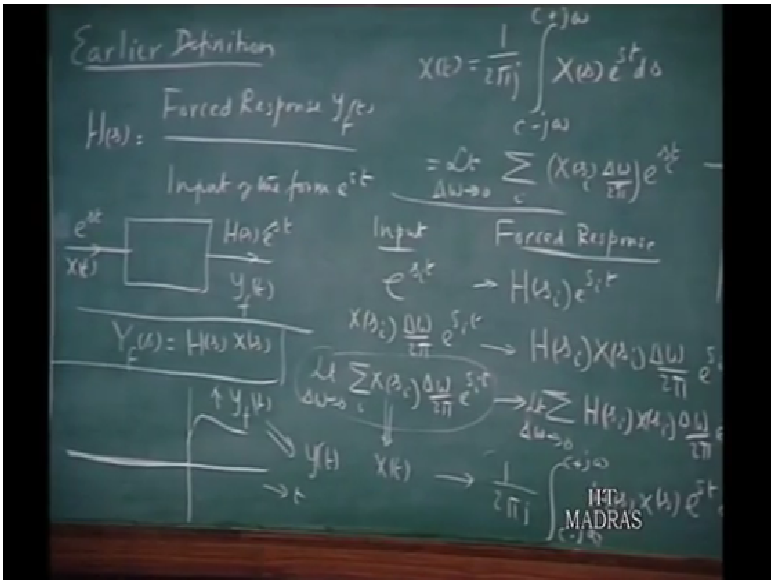

You recall that earlier we said in our introductory lectures that system function is the ratio of the forced response of the system, to the input when the input is an exponential signal. In other words earlier definition was h of s was defined as, the forced response y f of t to an input of the form e to the power of s t.

So, we said at the time, that if you had a system per which you have an input e s t, and then the x t is equal to this. Then the output, the forced response output will be h of s times e to the power of s t. Now, you defined a system function in different way, are they the same. So, you would like to now analyze this, and then tie them up, and see what to show both of them or indeed the same.

Now how do know about it. we know that x of t for the Laplace transformation theory, is related to x of s by 1 over 2 pie j c minus j infinity to c plus j infinity of x of s e power s t d s. What it means is, limit as delta omega goes to 0 of summation of elementary signals x of s delta omega 2 pie x of s is the coefficient density x of s can delta omega over 2 pie is the coefficient of the signal of this type e power s t summation over all such equals lead x t.

And how do we take this elementary component, you take which belong to contour here, and take any particular equation s i and then take the delta omega, and then if should take the exponential signals corresponding to each such element, that will add up to this. In other word we can write here s i of t, s i in one particular thing and summed up on i.

So, if this is the situation, then we can think of any arbitrary input x t as the sum of exponential signals of this time. So, let us see how it goes. Suppose we have the input, and the forced response. So, if the input has been e power s i t, according to our earlier discussion, the forced response will be h s i e power s i t.

Now, on the other hand, we have now not a single s i of t, but a whole lot of this s i of t, each we get by a coefficient like this. Therefore, if i have x s i delta omega over 2 pie e power s i t that would give me by linear the principle, because is the linear system. We have h s i x of s i delta omega over 2 pie e power s i t.

Now, if you this is one particular complex frequency signal. We can tiny amplitude like this. Now you have whole lot of this complex frequency signal. Therefore, we take the sum on i x s i delta omega over 2 pie e power s i t that should give me correspondingly, limit as delta omega goes to 0 of x s i x s i delta omega over 2 pie e power s i t.

This of course limit as delta omega goes to 0. And this is exactly what x t would be, x t we have seen from the Laplace transform theory is little omega goes to 0 of x s i delta omega over 2 pie e power s i t. So, all this tiny exponential signals, ranging from minus infinity to plus infinity along gone which contour.

If we add of them that will fetch you x of t; that means this is really x of t. So, if you have this x of t, and the combination of all such signals can be put in the integral form 1 over 2 pie j c minus j infinity to c plus j infinity of. Now s is a running variable no long discrete values; x of s times x of s e power s t d s, because d s is j times delta omega.

So, this is what it will be. So, from this we have observed, if you take this entire quantity if you call that y f of t. This y f of t has 1 over 2 pie j c minus j infinity c plus j infinity of x of s h of s e to the power of s t d s; that means, the Laplace transform of y f of t will be x as h s times x s; that means, we have y f s Laplace transform is h s times x s.

So, according to our earlier definition, we found that the sum of the forced responses will have a Laplace transform which is h of s times x of s. But in our Laplace transform theory, we are saying the total response y of s is equal to h of s times x of s. So, is it that is the total response y t is the same as y f of t that you have got obtained here.

The answer to this is, that they turn to be the same, because this Laplace transform inverse Laplace transform integral, provided you take s in the reason of convergence of the transform h of s times x of s. It turns out from the Laplace transform theory, that plot y f of t, this is going to be identically 0 for negative values of time, and it any some other value like this, for t greater than 0. This is y f of t.

So, according to the Laplace transform theory y f of t is going to be identically 0 for negative values of t and it have only values for positive value, non 0 values for positive values of t. And in our system, we also have y t which is identically 0 for negative values of time.

Therefore, y t aggresses with the y f of t identically for negative values of time. Therefore, there is not be any other term we introduced that spoil the initial conditions of the problem. Therefore, this y f of t is also equal to y t. Consequently there cannot be any other extra term that is present.

So, this y f of t is indeed equal to y t therefore, y f of s which is h s times x s, is indeed the same as y f of s h s times x s; therefore, y f of s obtained in this fashion is the same as y of t. So, the discussion therefore, gives us this information, that the sum of the forced response as calculated in this manner, will also yield the total response.

Total response in sense there cannot be any other term as far as the negative values of time is concerned. But as far as positive t is concerned, whatever you calculate for y f of t will include the finite number of transient terms, and the finite number of force frequency terms appropriate to the given h of s and x of s.

There are finite numbers and in the conventional sense what we talk about what we meant by transient term transient response, and forced response pertaining to x t and ht, as for as t greater than or equal to 0 will always be obtained y f of t. Even though your calling this forced response, the forced response y f of t includes the transient and the steady state responses, that terminate from t equals 0 onwards.

The reason why we are saying this forced response, because all this tiny exponentials here, start from minus infinitive onwards. So, if there all starting from minus infinitive all this exponentials are starting and each of them produced response, and the some other force responses, agrees with the total solution for negative values of times.

That means, there cannot be any other additional terms starting from t equals minus infinity onwards, but y f of t which is obtained here as for as this part of the solution is concerned, it includes both the transient and steady state terms, which we talk about next conventional fashion.

So, h of s that was described earlier h of s, is defined now one and the same therefore, we do not have to make a things between this 2. And the h of s place very important role in the system studies. It has got various useful properties and various interesting properties, and it will be our task now to take up a discussion, of the various important aspects of the system function h of s, and that will take up next.

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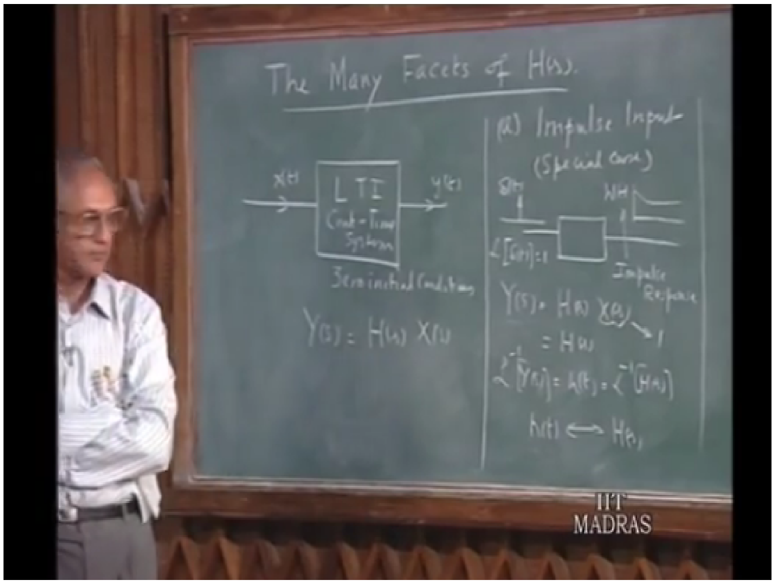

So, the system function h of s as many important connotations. So, it will be our task now to look at these connotations. So, there is a many facets to this the system function h of s, and that is what you like to steady. So, we are talking about a linear time invariant continuous time system with an input x t and an output y t.

And the system function enables us, to relate the output to the input, not by means of the differential equation, which is complicated, but by a simple algebraic equation in the Laplace transform domain. We also assume here that 0 initial conditions; that is important.

So, once you have the 0 initial conditions, the output and input are related by a pure algebraic equations, where h of s is in general a rational function; a ratio of 2 polynomials. And that differential equation is converted an algebraic equation of this type, is the central advantage of the transform technique, because which Laplace transform is very prominent technique.

Now, let us look at the various examples; a, suppose we have impulse input; a special case, not a general expect t, but let us say we have any impulse input. So, what we are talking about here is, input is delta t, and you get an output which perhaps like this, will call that h of t.

So, corresponding to a impulse input impulse at the origin, you get an output h of t h of t is called the impulse response, which we already we observed earlier. Now, applying our this formula here, y of s whatever output you get here, is related to the input that is now h of s the system function times x of s, x of s happens to be 1, because x t happens to be delta t x of s has 1. The Laplace transform of delta t as we know equals 1.

So, that means, the output is simply h of s, but the output you are calling h of t; therefore, Laplace transform, or I will write this y of s itself is h of s. So, the inverse Laplace transform of y of s, which around particular case is h of t, is the inverse Laplace transform of h of s.

In other words the impulse response h t and the system function form the Laplace transform pair, and it is anticipation of that, you use the symbol h for the impulse response, and the H for the system function. So, this is a very important result. The system function h of s of the Laplace transform of impulse response.

That means, you apply unit impulse to the system, is 0 additional conditions; unit impulse station like t equals 0. Whatever response we get call to the h of t, that will be the inverse Laplace transform of h of s, or if you know impulse response of h of t, you can find out h of by taking the Laplace transform of h t. This 2 form Laplace transform pair.
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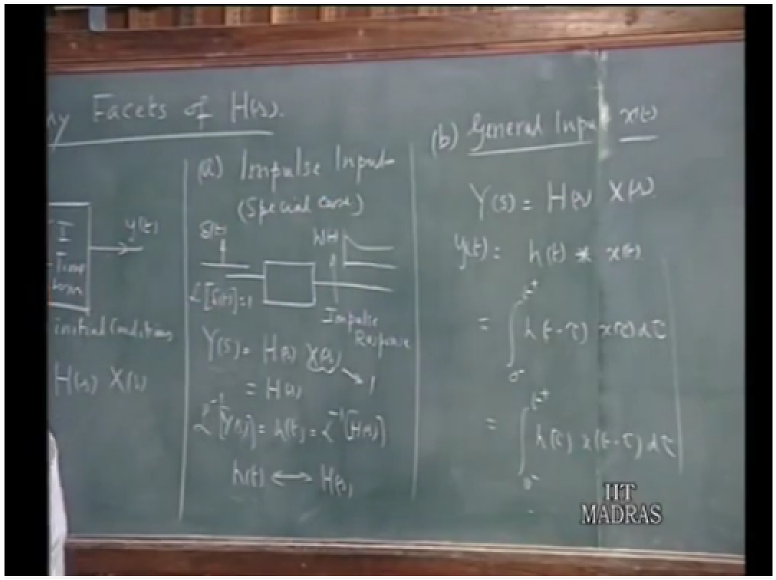

Now, using this information general solution in time domain, general input x t, suppose you are having, and you want to find out the output in time domain. We know that y of s equals h of s times x s. Now, by the convolution property, we know if y of s is the product of two quantities h of s and x of s.

Then y t we know, is the convolution of the time domain quantities which are the inverse Laplace transform of this two. The inverse Laplace transform of h of s is h t the inverse Laplace transform of x of s is your x t. So, in other words the output is the convolution of the impulse response, and the input quantity, which in fact you already observed in our preliminary discussion.

We already noted this. I am just trying to relate this to the Laplace transform equation in this manner. What does it mean this is the convolution integral. It means 0 to t h of t minus tau x tau d tau. Alternately if you wish, you can write as 0 to t h tau x of t minus tau d tau.

This are you can interchange roles of this, both this, or the equivalent to each other, and as I mentioned earlier on if it may turn out, that x of t you have impulse such origin h of t is the impulse the origin. So, to take care of the impulse of the origin, in general it would be advisable to take 0 minus to t plus. If to take care of the possibilities that x t are h t may have impulse.

If you have do not have impulse there is a n does not matter whether you take 0 to t a 0 plus or 0 minus. Now, these integrals are called convolution integrals. They are also called literature super position integrals, because after all your super posing the effect of x of t by treating them as a summation of the different impulses, something which have already discussed.

So, the significance of the convolution integral as a super position of, the input considered as a sum of impulse something which we already discussed.

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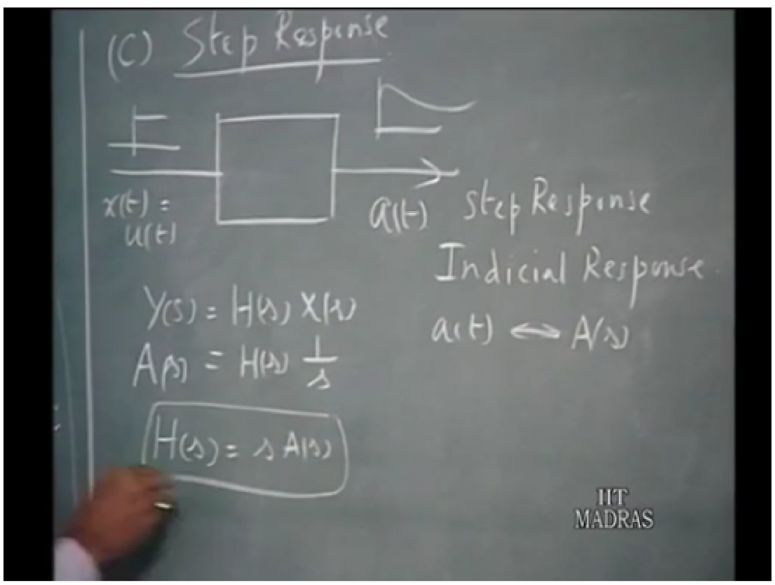

A third aspect; I also mentioned in our earlier discussion that step response is also a way of characterizing a linear system. So, let us see how it is related to a system function h of s. So, in step response we are thinking of unit step applied as a input. So, x t is u t, and the corresponding output whatever you are having here, is given a special symbol a t.

And this; a t is called the step response sometimes it is called indicial response. Now, using this general equation y s equals h s times x of s. So, in our particular case this h of s and x of s happens to be the 1 over s, because the Laplace transform of a unit step is 1 over s.

And y of s the particular Laplace transform of the step response that we are getting we will called that as a of s that means, if a t has the Laplace transform of a of s capital a of s, then a of s is h s times one over s or h of s equals s times a of s. So, the relation between Laplace transform of the step response and the Laplace transform of the impulse response is given like this; h of s times 1 over s is a of s, or s as times is equal to h of s.