Networks and Systems
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Lecture-60
Application of Laplace Transform to Network Transients

After having consider the various properties of Laplace Transforms. We shall now take up Applications of the Laplace Transform technique to the solution of transients in Networks and Systems in general. To start with, let us consider solution of transient problems in networks, using the Laplace transformation method.

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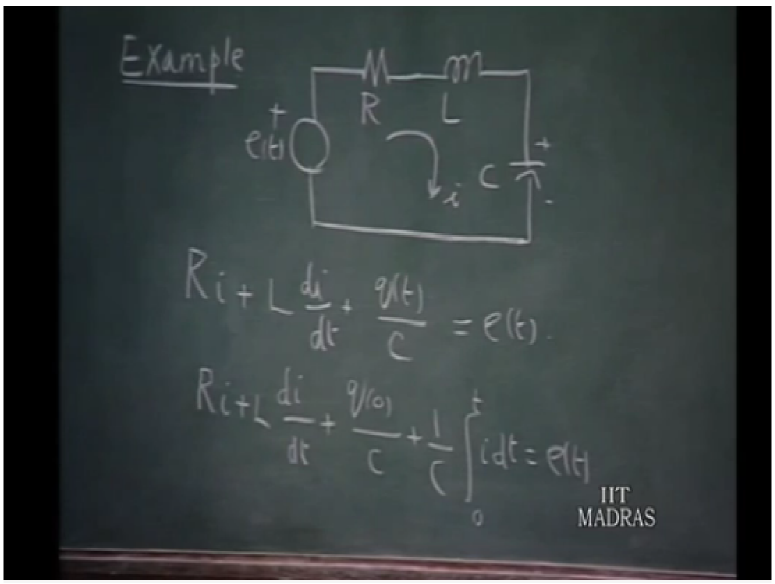

In considering the network transients, we can first form the differential equation pertaining to the network. And then, transform the differential equation in to the Laplace transform domain and arrive at the solution. This is the method that suggest by itself and this is a more fundamental method.

Later on, we discuss another technique, which avoids some steps to have to go through in the differential equation approach. So, let us start with differential equation approach. The best way to do this is to go through an example. Let us consider a circuit in which, there is a voltage excitation.

There is a resistance, an inductance and a capacitance, in series circuit comprising R L and C. Now, you are interest in finding out, the current of the circuit I. So, the differential equation pertaining to the solution for the current given this forcing function of e of t is given as the voltage drop across resistances R i plus L, d i d t.

That is the voltage drop across the inductor plus the voltage across the capacitor. That is V c. That can be written as q of t over c. That is the charge on the capacitor at time t divided by c. That is equal to e of t. This can further be written as, if you take the charge q of t to be L, d i by d t plus, there is the initial charge in the capacitor, let us say q 0 divided by c plus the additional charge that has been conduct to the capacitor plates in the interval starting from 0 up to the point t.

That means 1 over c integral 0 to t, i d t. That is equal to e of t. So what you have done here is, the q of t has been thought of as comprising two parts. One is the initial charge on the capacitor, will this as positive and this as negative. And the additional charge, that is been put at the capacitor place in the interval 0 to t integral I d t divide by c.

So, q 0 plus the additional charge divided by c, that is the expression for q t by c is broken up into this 2 parts. Now, in solving in second order differential equation of this type, it will turn out to be second order will see in a moment. We need to have know 2 initial conditions. So, the initial conditions are associated with active elements. So, one is the inductance and other is the capacitance.

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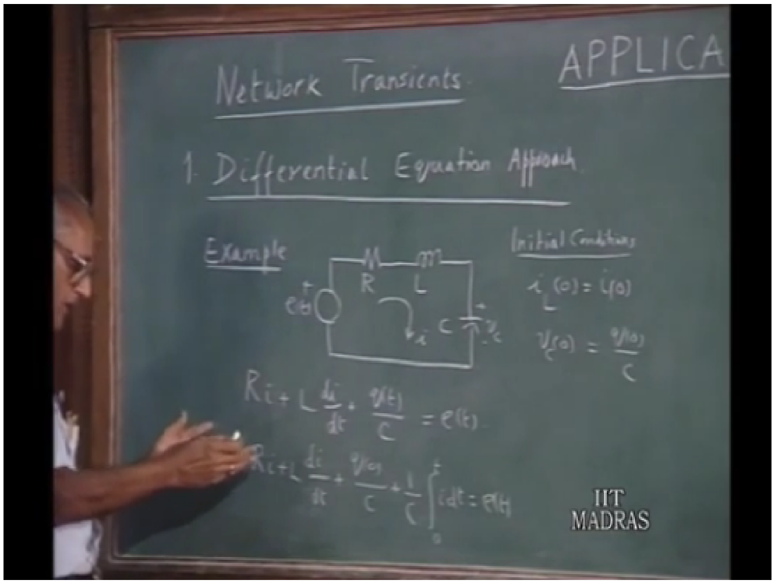

So, the initial conditions, we need to know to solve this problem are first of all something to do energy storage in the inductor to start with. So, i L 0 is to be found out is to be known and secondly V c 0. This is V c 0, i L 0 is same as i 0. Because, after all the same current is flow to inductance first, this is V c 0, this is q 0 over c. So, the knowledge of these two initial conditions are required in order to solve this differential equation.

Now, to find out the solution for the differential equation, we what we do is, we take the Laplace transform of each term on the left hand side and the term of the right hand side. So, each one of the terms in the equation a Laplace transform are taken.

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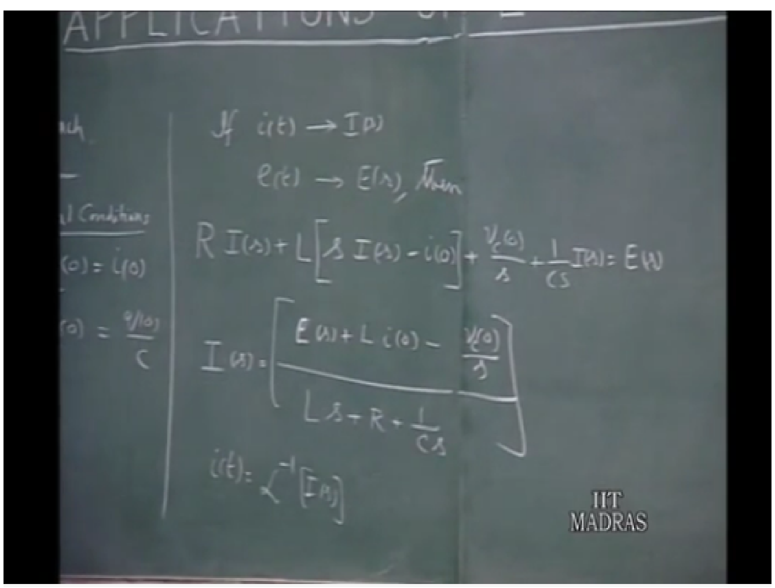

So, if the Laplace transform of i of t is capital I s. Then, R I; when we take R I; the Laplace transform of this would be R i the Laplace transform of R times I of s. The Laplace transform of this is L times d i by d t. The Laplace transform of d i by d t is s I s minus I 0. Laplace transform of q 0 by c is constant.

Therefore, this q 0 over c s and the Laplace transform of this 1 over c, integral 0 to t, i d t is the Laplace transform I s over s. So, this is 1 over c s times I s. The Laplace transform of e of t is suppose of e f, capital E of s, then this will be capital E of s. So, term by term we do the Laplace transformation.

And let us say the e of t has the Laplace transform E of s. Then, term by term transformation of this term will yield as R times I of s plus L times. The Laplace transform of d i by d t is s times I of s minus I 0, plus q 0 over c. That is q 0 over c as, that is V c 0 over s, q 0 over c is V c 0 over s, plus 1 over c s times I s. That is equal to forcing function E of s.

So, we have an equation now in I of s, which has to be solved for in terms of the more quantities. So, I can write this as I of s equals, E of s Laplace transform of the source function plus L i 0 minus V c 0 over s, divided by, the coefficient of I of s on the left hand side, which is L s plus R plus 1 over c s.

So, what we have done is, we have got an expression for the Laplace transform of the current. In terms of the known quantities, e of t is presumably know, therefore E of s can be found out. The initial conditions in the induct as regards the inductor current of the capacitor voltage are given, therefore this are found out.

And L, R and c r parameters the circuit, therefore we find I of s. And once, we have I of s, i t can be found out as the inverse Laplace transform of I of s. This is the approach that will have to taken. Now, a few commons on this, if you are interested in finding out the behavior of the current starting from t equal 0 minus onwards.

Then, the initial currents in the inductor and the initial voltage across the capacitor, you can have 0 minus values. On the other hand, if you want the solution to be valid from t equals 0 plus onwards. Then, you can substitute 0 plus values and 0 plus values for both I 0 and V c 0. So depending up on what solution, you want you can substitute either 0 minus initial conditions are 0 plus initial conditions put that down here.

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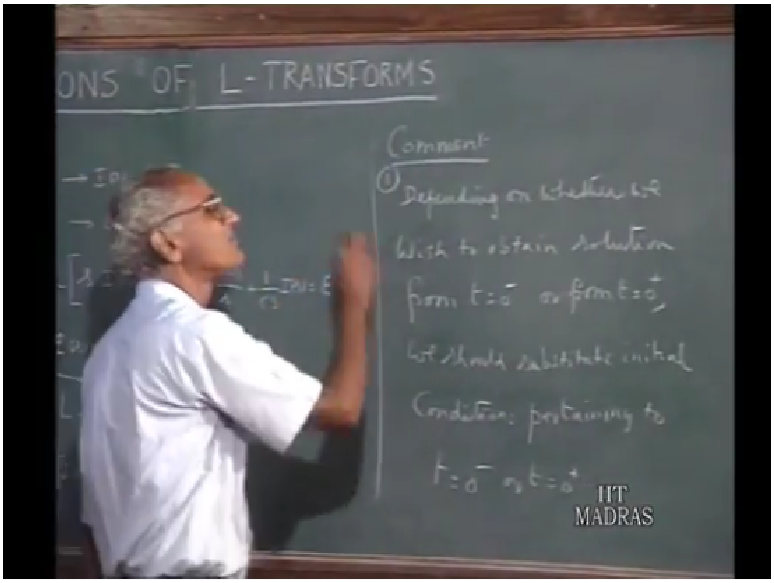

Comment, depending on, whether we wish to obtain solution from t equals 0 minus or from t equals 0 plus. We should substitute initial conditions pertaining to t equal 0 minus or t equal 0 plus. Remember, in the classical differential equation approach, we need to find out the conditions t equals 0 plus in order to solve this differential equation.

But, here you do not have to do that. If you know the 0 minus conditions and if 0 plus conditions are different from that, you need to substitute 0 minus conditions the complete solution is obtained. If 0 minus conditions and 0 plus conditions are one in the same, it does not matter, which are you substitute with make no difference.

But, if 0 minus 0 plus conditions are going to be different and you want the solution from 0 minus, you can substitute 0 minus conditions. In the classical differential equation, we need to perforce required to substitute the condition the 0 plus some times. If there is transition, when 0 minus to 0 plus, we have to make separate evaluation of the 0 plus conditions.

That is, avoid in the Laplace transform technique as we can see here. Now, second point to observe is, the differential equation is converted to into an algebraic equation. You are not doing any calculus part here, all the derivatives and integral sign are been converted into equivalent multiplicative factors. And so, the differential equation is converted into algebraic equation.

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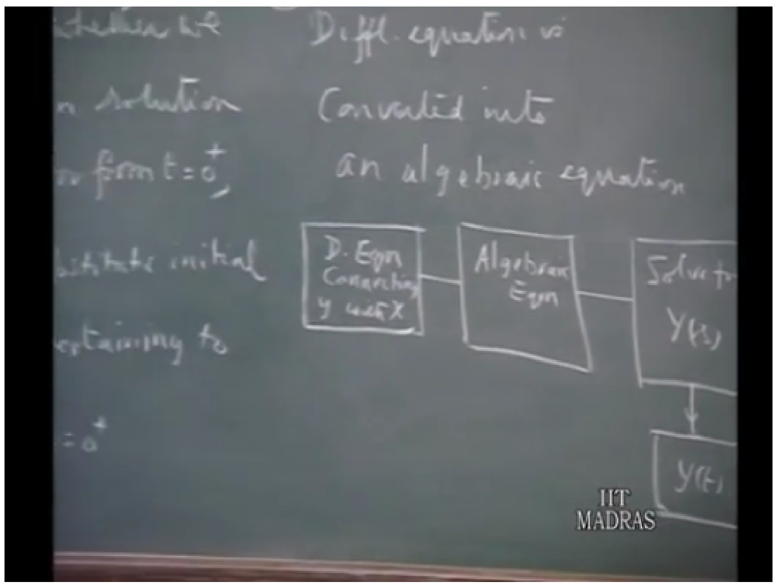

And where, trying to solve an algebraic equation really. So, the step that involved here is, first differential equation. Connecting an output quantity y with input quantity x. In general, output quantity y with x. Then, we convert this in algebraic equation, where the algebraic equation, it terms of the Laplace transforms of the output y of s, y t and input x t, so y of s in x of s.

From the algebraic equation, solve for the Laplace transform of the output quantity y of s. In this case, the current is output quantity. So, we solve for I of s. So, we solve for y of s. And once, you have got that, you find the inverse Laplace transform and then, get y t. So, these are the four steps that there are involved.

You form the differential equation; convert them into algebraic equation by Laplace transform in every term in the differential equation. Then, solve for the appropriate quantity, which are interested in this case. You have our case; it is I of s in general, it is a quantity y of s and find out it is inverse transform y of t.

The fact, that the initial conditions are substituted, even at the stage of finding out the algebraic equation. In the process, you have to using the initial conditions, means that there is no separate requirement for evaluation of initial conditions as you would have a classical differential approach.

The initial conditions are fed to the data at the beginning of the problem itself. And the evolution of the arbitrary constants, which is a troublesome feature in the differential equation of approach in completely avoided in the Laplace transform approach. So, this is the way, which one can handless.

However, you would like to simplify this further. Since, we are interested in finding out the electrical network transients. And this is the problem, which are quite frequently; who would like to streamline this procedures. And try to arrive at the formulation of the algebraic equation this manner, straightaway, without having to go through the intermediate differential equation. This is what, lead to us, what is called transform diagram approach.

That is what will take up later. Essentially, what we will try to do there is, try to write down this equations straightaway, without having to go to the preliminary of setting up the differential equation and trying to Laplace transform that. So, this is the approach that will take up. And that goes under the term, transform diagram approach and this is what will take up next.

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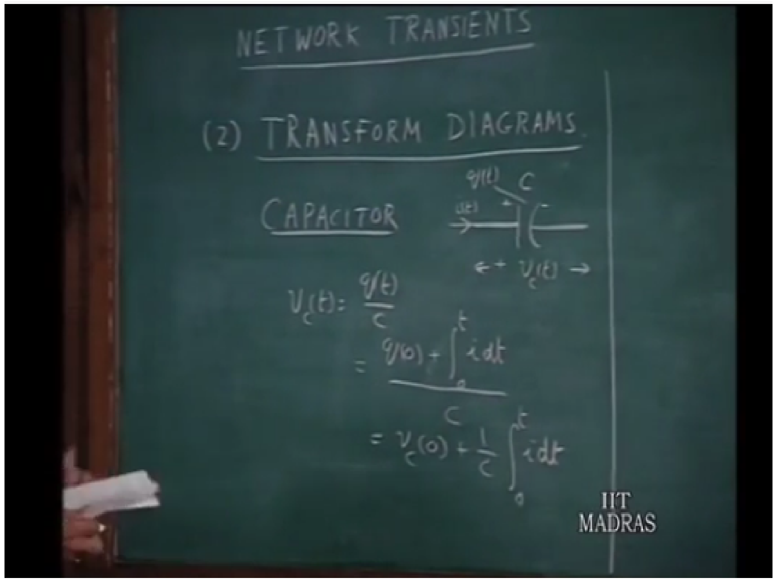

The second approach will take for the analysis of network transients is, what is the called the method of using transform diagrams. Before, we actually illustrate this method. Let us look at some preliminaries. To start with will take the capacitor, we know that the terminal equation is over capacitor can be put in the form, that v c of t is q of t over c. So, if the capacitor here, parameter c, this is v c of t. And the charge and the capacitor, suppose is q of t, the current is i of t.

Then, a ready point of time, v c of t is q of t over c. We can write this q of t as, the initial charge and the capacitor at time 0 plus the extra charge. That has been deposited on the capacitor place in the intervals from 0 to t. So, 0 to t of i d t; that is the additional charge on the capacitor in the interval from 0 to t.

That is the total charge 0 to t divided by c. Now, this can be further broken as q 0 over c is the initial voltage across the capacitor. Because, that the initial charge divided by the capacitor parameter c. Therefore, I can write this as V c 0 plus 1 over c integral 0 to t, i d t. So, in time domain, we have the capacitor voltage v c t is V c 0 plus 1 over c 0 to t, I d t this is the fundamental equation.

Now when where analyzing a circuit, which a capacitor is situated. We should like to look at the relation between the currents and voltages in the transform domain. Therefore, you like to express these quantities, not in terms of time expressions, but in terms of Laplace transformed variables. So, what we like to do is, transform these quantities into the corresponding Laplace transforms.

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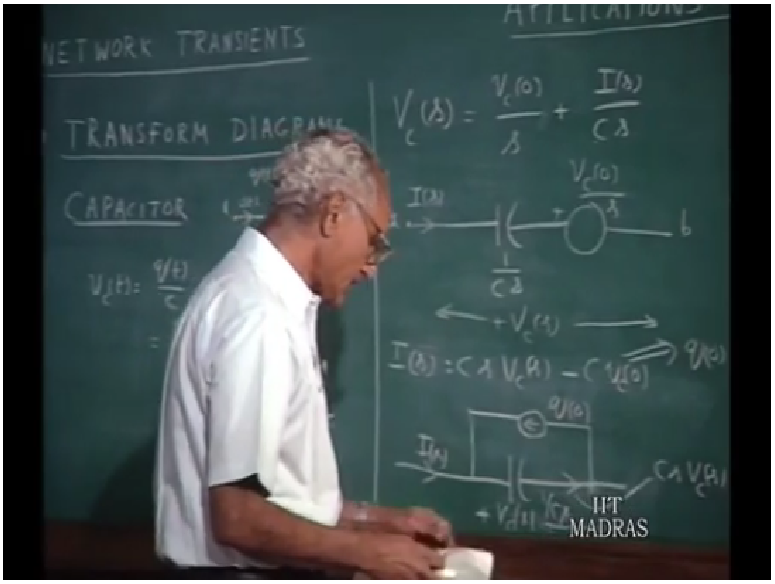

So v c of t, suppose as the Laplace transform of v c of s. Then, that is the Laplace transform on the left hand side. On the right hand side, V c 0, that is a constant. Therefore, it is Laplace transform will be V c 0 over s. Now, the second quantity 1 over c, the integral i d t, this is this is definite integral from 0 to t, which means, I of s is the Laplace transform of I.

Then, integral 0 to t of i d t of the Laplace transform I s up on s. And of course, we have 1 over c also here, therefore I of s over c s. So this is the equation in terms of the Laplace transform of the terminal current and the Laplace transform of the terminal voltage. We have an additional term here v c of s and I of s are connected, not only by the capacitor parameter.

But, also, we have an additional term depending upon the initial voltage across the capacitor. Now this can be modeled by means of a circuit representation like this. Suppose, I have a capacitor, let us say at the moment, we have the impedance 1 over c s and a current I of s is a passing through it.

Then, if this is the generalized impedance, it develops the voltage here, which is times 1 over c s. So, v c of s is I of s over c s plus also we have additional quantity, which does not depend on the current. Therefore this can be considered to be source. A voltage source, whose value is V c 0 over s and this, is the total terminal voltage across the capacitor.

So, we see observed that the terminals of the capacitor is suppose this a two terminal a and b. In the transform diagram in the domain, we can think of the capacitor to represent by means of a circuit like this. Where, this capacitor has generalized impedance 1 over c s in the sense.

The voltage across these capacitors in the transform domain divided by the current in the capacitor in the transform domain is 1 over c s. So, I of s passing through 1 over c s develops the voltage I s over c s. In addition, we have voltage source, V c 0 over s, together will constitute the capacitor voltage in the transform diagram. We can express the current in the capacitor in terms of the voltage.

So, from the same equation you can write I of s equal c s times v c of s, minus c V c 0. So, the current is now expressed terms of the voltage like this. This can be represented by a simple diagram like this. You have a capacitor, which voltage is v c of s. Let me create a little more room here, I will write this as I of s, c s times v c of s minus c V c 0, which of course is also q 0.

Therefore, I can represent this equation by means of the circuit like this. Whose impedance is 1 over c s and in parallel like this; we have a current source, whose value is q 0 in the transform domain. And the current here I of s and the voltage across this is v c of s. So, a voltage v c of s occurring across an impedance 1 over c s, will have a current here, which is c s times v c of s. So that c s times v c of s minus q 0 is the total current time, the current I of s. And this is the circuit representation of the capacitor in the transform domain.

You are observe that, we are now a capacitor in terms of a generalized impedance 1 over c s. Just as we have in the ac circuit analysis, you talk about the impedance of the capacitors 1 over j omega c. In the transform domain, we have thinking of the impedance the capacitor to be 1 over c s more about this.

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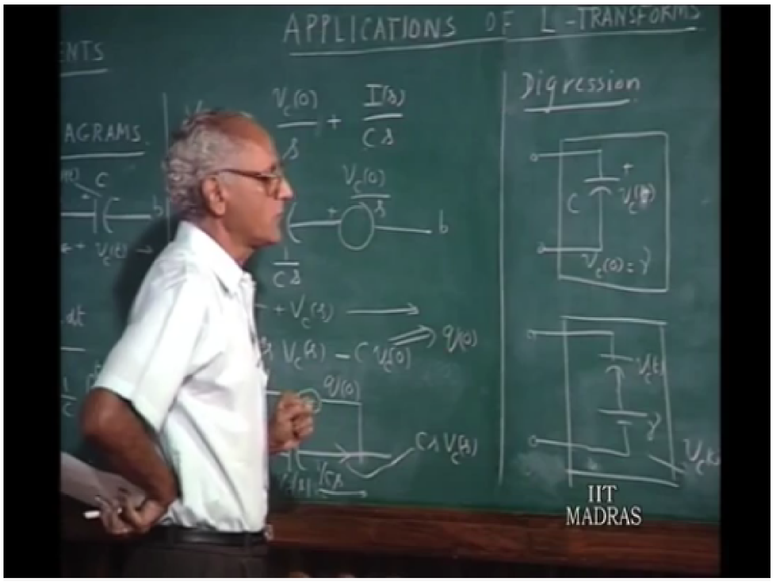

Now what is the corresponding situation, before for an inductor? But, before that, let me take a small digression here to stress a particular point. Suppose, I have a capacitor, which is charge, suppose I have the capacitor here, the initial charge on the capacitor, this is v c of t.

The initial charge of the capacitor, suppose is gamma. I enclose this capacitor to the black box. Now let me take another situation, where we have an uncharged capacitor, in series with a voltage source equal to gamma, this voltages source. So here this is v c of t and we have v c of t is 0. In this case, V c 0 is 0.

That is initially uncharged V c 0 is 0. So, we have two situations, one a capacitor, which is charge initially a gamma volts. And another, a capacitor in series with the voltage source, whose value is gamma, but the capacitor itself, initially uncharged. So, if these two, a capacitors systems are given to you in two black boxes.

They provide identical terminal relationship into whatever to circuit in which it are connected. Or, in other words, if you for any external measurement, you will not be able to distinguish between these two. You have the identical terminal relation for all possible experimental situations. In other words, these two are equivalent for every possible consumable extensile.

And therefore, if you have a capacitor, which is carrying an initial charge and therefore, initial voltage gamma, we can represent this an uncharged capacitor in series with a source. And if you take the Laplace transform of this and this capacitor c as 1 over c s and the dimension for generalize impedance and a source, which is gamma over s.

So, that is an exactly this equation that you are having. Initial voltage for the capacitor can be represented of that source and that source; v c source will have a transform like this. So, this two apply this. Now, we can also think of an equivalent current source representation that will lead to this but we will not do that.

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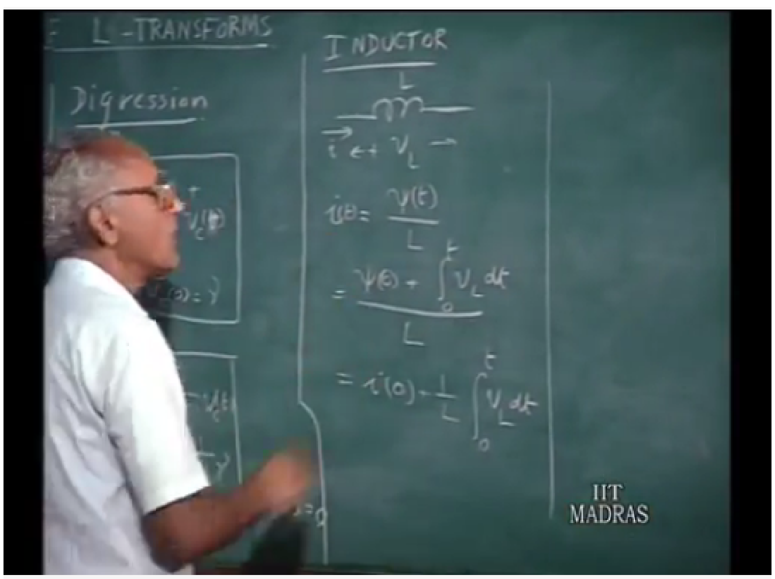

Now, let us look at the situation with reference to an inductor. We have an inductor L, carrying a current i and let us say, the voltage across this V L. We know that, the current in inductor, say i equals flux linkages in the inductor at any point of time divided by L. That is the differing parameter in inductors.

See, the psi of t, the flux linkages can be written as the initial flux linkages plus the additional flux linkages that come up in the inductor as the results of an applied voltage, which is V L d t divided by L. After all voltage is the rate of change of flux linkages, therefore flux linkages are is related to voltage by an integral relation like this.

This can be written as, psi 0 over L, that is the initial value of the current in the inductor. So, I 0 plus 1 over L integral 0 to t, V L d t. So, that is the terminal relation of an inductor, the current in terms of the applied voltage and the initial value of the current that we have here. Now, you would like to represent this by means of a circuit diagram, similar to what we had in the capacitors.

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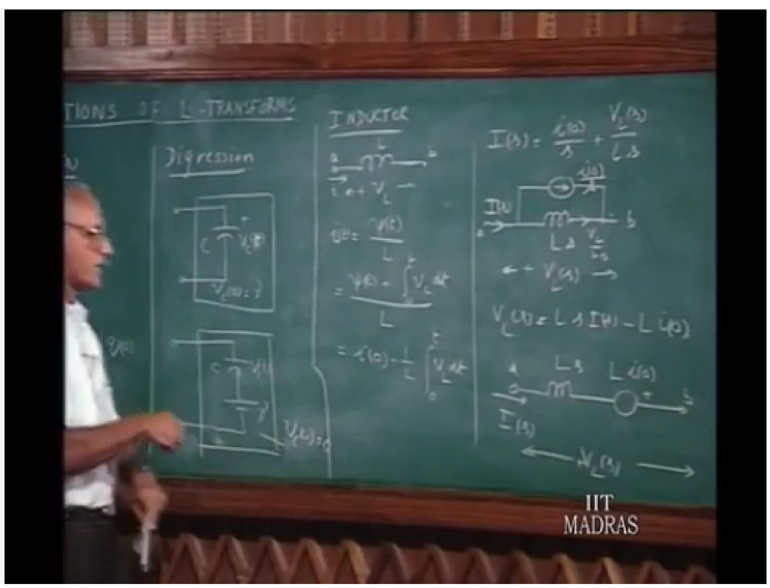

So, taking Laplace transform of this quantities, we have I of s, the Laplace transform of i t is I of s, i 0 is the constant. Therefore, I write this I 0 over s, that is the transform of this. And again, we have a definite integral from 0 to V L d t, therefore in V L d t has the Laplace transform of V L of s. This will be V L of s over s and we have 1 over L also.

Therefore, we have V L of s over L S. That is the equation. That describe the terminal relations of an inductor in the transformable. Obviously, therefore, you have an equivalent circuit like this. In the transform diagram, which represent this equation, suppose we say the inductor has generalize impedance L of s. L s and then, terminal voltage is V L of s.

So, the current in this portion is V L over L S. But, the terminal current is different from by an additional current here. Therefore, we have a current source, which is equal to I 0 over s and this is the terminal current I of s. So, terminal current I s is related to V L of s in this manner.

You have parallel circuit, one contains the impedance the inductor, who is generalize impedance is L s. Therefore, V L of s divided by L S the current here. In addition, we have this source current here. Now, on the other hand, if you like to express V L of s in terms of I s, I can write this as V L of s equals from the same equation L as times I of s, minus L times I 0.

So, this equation let us itself to an interpretation in the transform diagram in this manner. So, L s is the generalize impedance. You have current I of s. So, that develops the voltage, L s times I of s. In addition, you have another voltage, which is independent of I of s, therefore we can think of this as a currents voltage source L i 0 with this pollard.

So, the total voltage is V L of s. So, we have V L of s is I of s passing through L s generating through voltage, L s times I of s minus L i 0, what we are having. Now, just as we have an inductor capacitor, we can initially charge can be represented by uncharged capacitor is with the voltage source.

We can also interrupt at inductor current to be an uncharged. An inductor is does not carry an initial current in parallel with the current source. So, that is leave it as you for you to work it out. And you can do the same interpretation in the case as well. Now, we therefore for both the capacitor and the inductor, we have reach two equivalent representations, one in terms of voltage source and other in terms of current sources.

Similarly, here in terms of voltage source, in terms of current source. We can use either of these equivalents for either of these equations. And depending up on the total circuit computation, one of the other will be found to be more advantages in many instances. For example, you have low current basis.

Then, you would like to reduce the number of loops. Then, a voltage source representation may be for favorable. On the other hand, if you are using the node voltage method, a current source, like this. So, the total number of nodes 2 here, it will be 3. So node voltage that in that situation, you may find use of the current source more favorable, that of course depends up on the particular problem.

Now, few another common theories, that if you have a and b the terminals of the inductor. Here, also this must be the terminal a and b. You should not think that, this of the two terminals of the inductor. If these are the terminals of the inductor in the original circuit, the transform diagram. We should not think these are the two terminals of the entire branch, including the voltage source, which transfer the inductor in the transform diagram.

Few commons about, what you have done so for. We see that, unlike the Ac circuit analysis, where we have impedances, j omega L and 1 over j omega c. We have not only such impedance generalize impedances. But, also we have sources. If you did not have the sources represents the initial condition, the inductance and the capacitor are represented by impedance L s and 1 over c s respectively.

So, it is the initial sources which coming to the picture, initial conditions of the reactive elements, which are equivalent to sources. On the other hand, we can say that, if the capacitor has an initial charge, inductor has an initial current. They can be replaced by equivalent sources and therefore, here afterwards, we can think of in the circuit analysis problem.

The effect of the excitations or effect of source on otherwise the circuit is what we are looking for. Because, initial currents in inductor an initial charges the capacitor can be represented by equivalent sources also you have seen. Therefore, without any loss of generality, we can think of a situation, where we are considering the affect of sources on circuit elements, which do not carry any initial current or initial charge.

Because, those initial condition can be replaced by equivalent sources. And once, we have that, then the terminal voltage and the terminal current of an inductance capacitance transform domain. Have proportional to relationship this V c of s equal I s over c s and V L of s Ls times I of s.

So, we have the concept of generalized impedances for both the inductor and capacitor in the transform domain. The transform of the voltage has the proportional relationship the transform the current and so on and so forth. So, this are called generalized impedances and that what we have to make note on. This generalize impedances are after all, extensions of the impedances that we have in the Ac circuit analysis.

We know that in the Ac circuit is 1 over j omega c is the impedance of the capacitor; j omega L is the impedance of the inductor. Here, is the j omega, we have s in the Ac circuits, we are thinking of an excitation function, which is e power j omega t. It is on that basis, we have derived the impedances in the Ac circuit analysis. But, here it turns out that thinking of excitation function the type, e power s t.

Therefore, it is not surprising, therefore we land up an impedance function, generalize in a way has 1 over c s and L s respectively. So, this is the generalization of the impedance that we use in the case of Ac circuit analysis. A third point which I like to make note of is, when you take the initial conditions. You can liberty to choose either 0 plus conditions are 0 minus conditions.

That is why; in all this analysis, I did not commit myself to stay, whether it is 0, 0 plus or 0 minus. You can use 0 plus or 0 minus, if you want to find out the system behavior, starting from 0 minus onwards. Then, you have to use all the initial conditions pertaining to 0 minus. On the other hand, if you are satisfy with the determination of the network response starting from 0 plus onwards.

You can substitute 0 plus conditions and carry out the analysis. So, in all this conditions I 0 V c 0 and so on, can be either 0 plus or 0 minus consistently write through. After having set that, we know that in transient problem some switching takes place, let us say t equal 0.

And we are given or at least we can find out the conditions in the circuit, before the switching operation takes place. That means 0 minus conditions are known to us in most of the situations. Therefore, it will be convenient for us in substitute 0 minus conditions. And the solution that we get will include the transition between 0 minus to 0 plus.

We do not have to purchase additional trouble, to find out the 0 plus conditions, because they are not normally known to us before hands. Unless, this continuity between 0 minus and 0 plus. So, we do not have to take extra effect to calculate 0 plus conditions and substitute in the equations.

It is enough to take the 0 plus conditions. And if there is any transient between 0 minus conditions and 0 plus conditions, that is automatically coming out in the final solution and this is one of the advantages of the Laplace transform technique as you will get to know, when you workout circuit problems.