Networks and Systems

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Lecture-58
Inverse Laplace Transform

In the last lecture, we were considering finding out the inverse Laplace transform, through the method of partial fraction expansion.

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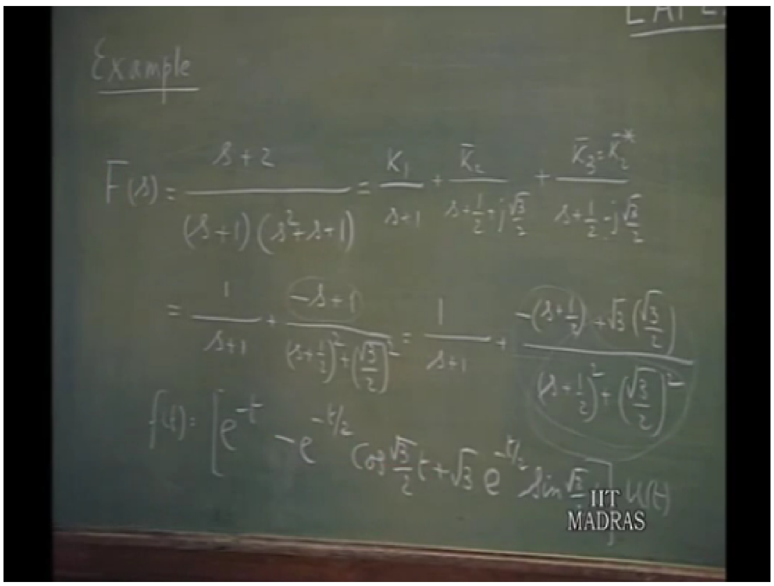

We shall continue the discussion with an example. Let us consider f of s rational function which is s plus 2 over s plus 1 s square plus s plus 1. This are; obviously, three poles; one at minus 1 and pair of complex conjugate poles at minus half plus r minus j root 3 up on 2. So, one way of handling this would be, to make it partial fraction expansion corresponding to the 3 poles.

So, k 2 over s plus half plus j root 3 upon 2, and another term corresponding to the conjugate pole minus j root 3 up on 2. Now, it tells out the pole is complex, this will also the complex number, k 2 is a complex number, and since the pole, this pole is the complex conjugate of this, it terms of the k 3 will be k 2 conjugate.

So, one way of filing this would be to find out the 3 residues, corresponding to the 3 poles, find k 1 which is simple, k 2 which is of course complex, and k 3 could have to spend extra time to find out k 3, because once you know k 2 the k 3 will be k 2 conjugate provide a coefficients in the rational function, coefficients of various powers of face in the rational function are real.

So, in making this in partial fashion once you find k 2 k 3 can be immediately formed out. But then this leads little bit complex algebra manipulation, and alternate way of doing this would be, to combine these 2 terms and get the result as a fraction because after all finally, when you find the inverse Laplace transformation, you will find out that when you have a function quadratic factor in denominator.

It is convenient to identify this, with the Laplace transform of e to the power of minus alpha t cos omega t type of function. So, to do that, let us write this partial fraction expansion as; s plus 1 corresponding to k 1, and the second term would be the combination of this 2, it also that will be s plus half whole square plus root 3 up on 2 whole square.

So, if we can express this quadratic factor in this form, it will be convenient, because we can associate this with a well known time function as we will see as go along. Now, to find out the residue of the pole s equals minus 1 multiply this by s plus 1 and substitute s equals minus 1 in the usual manner, and you will get the result s 1.

And once you have that, to find out the term corresponding to the numerator here. You can arrive the common denominator, and identify the numerator coefficients of s plus 2 and find out the numerator here as minus s plus 1.

So, this is straight forward I will not work out to give the complete details. After having down this you recall that, the Laplace transform of s plus alpha over s plus alpha whole square plus omega square, the inverse Laplace transform of that is e power of minus alpha t cos omega t.

Therefore, we should like to express this, as a linear combination of s plus half one term and root 3 by 2 as the other term. If you do that then we can identify this, with the Laplace transform of terms like e power minus alpha t cos omega t and e to the power of minus alpha t sin omega t. So, to do that, let me write 1 over s plus 1 alright, and the second term s plus half whole square plus root 3 up on whole square.

Now, I must express this as a linear combination of two terms like this. So, this is the minus s here. So, I will write it minus of s plus half; that is we can recognize this to be the Laplace transform of e power minus half t cos root 3 up on t 2 t with minus sign out in front.

But we have minus s plus 1; therefore instead of plus 1 you have minus half; therefore, you must introduce plus 3 by 2, but we must express this as a coefficient times root 3 up on 2. Therefore, this will be root 3 up on 2 multiplied by root 3. So, that is the linear combination of the two terms that we are looking for. Now, we have got the 3 terms which are known Laplace transform of particular time function.

Therefore, we can find out the inverse Laplace transform; that is the time function as corresponding to the first term you have e power minus t, corresponding to this term you have minus e power minus t upon 2 cos root 3 upon 2 t, and corresponding to this term you have plus root 3 e power minus t up on 2 sin root 3 up on 2 t. So, that is the of course the whole thing, is multiplied by u of t.

So, that is the inverse Laplace transform of this given rational function. So, now, we have used here a trick in the sense that we are not found out the residues of all poles physically, whenever we have complex conjugate poles, the pair of complex conjugate poles.

It would be convenient for as to find out a term in the partial fraction for both this terms together, with have a denominator quadratic factor with real coefficients, which case can identified with damped sinusoidal in the manner shown here.

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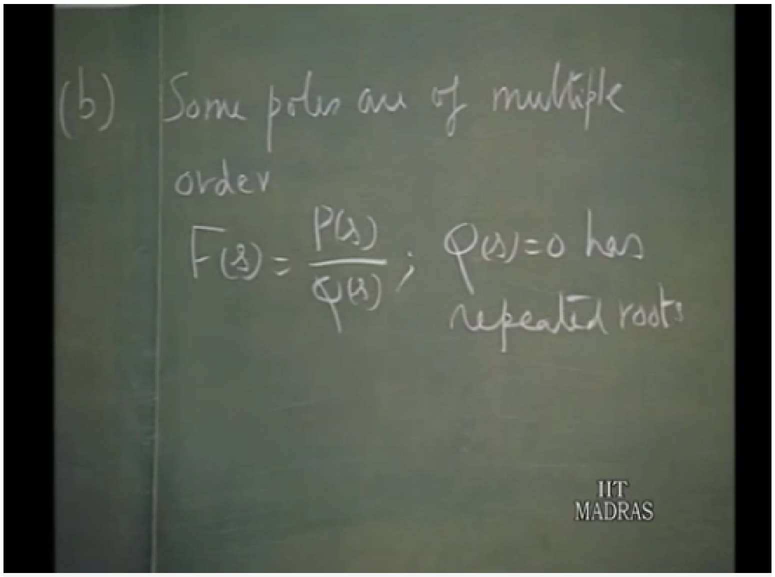

Now, let us continue our discussion with the situation, where poles are multiple order. So, far we have considered rational functions where the poles are simple. So, what happen in some poles are multiple orders. In other words if f of s is p s or q s q s equal 0 as repeated roots. So, some factors are repeated, then we will say the factor f of s has multiple order poles, depending up on the number repudiations.

Suppose is repeated twice, it is a pole of order 2. If a particular root is repeated 3 times, then we say a pole of order 3. We illustrate this again by means of example, before we work out the give the general rule.

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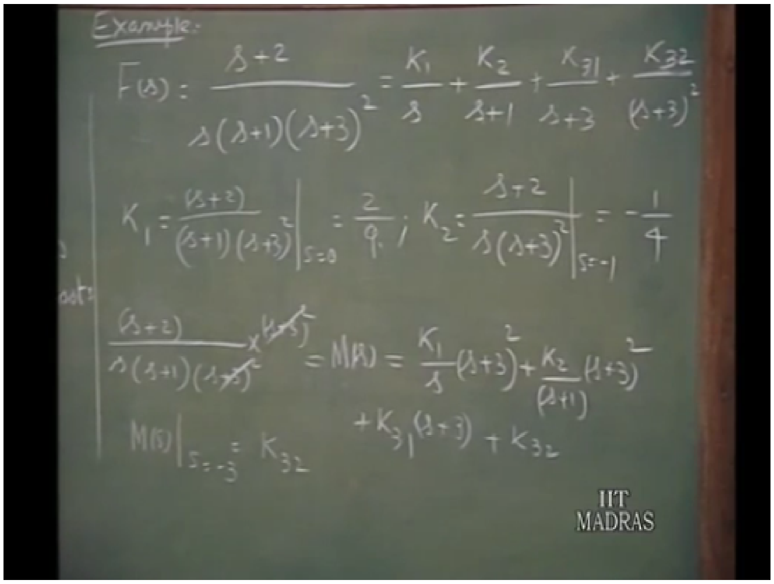

Suppose we have f of s equals s plus 2 divided by s times s plus 1 times s plus 3 whole square. So, this is an example. We will talk about the example first, then we get the general rule. Now, here we have 3 distinct poles; one at the origin, one at minus 1, and second pole is equals minus 3.

Therefore, we have k 1 over s plus k 2 over s plus 1, and corresponding to this multiple order s plus 3 whole square the denominator, you can explain this as k 3 plus s plus k 4, are alternately it suits as to have express this as k 3 over s plus s plus 3 plus k 4 over s plus 3 whole square, because these are easily identified with the Laplace transform of unknown type functions.

So, it would be convenient for us to express this in this manner. To streamline this work for extend this a general case we will put it this way. This pole at 3 has got 2 repeated roots; therefore, we have k 3 1 and k 3 2. I will put in this manner k 3 1 and k 3 2. Now, it is to find out k 1, we proceed in the same style forward fashion multiply this by s and substitute s equals 0.

Therefore, s plus 2 over s plus 1 times s plus 3 whole square with the substitution s equals 0, and that gives me value 2 by 9. No problem at all. Similarly, if you want to find k 2, you multiply this rational function by s plus 1 which leads me to s plus 2 times s times s plus 3 whole square and substitute s equals minus 1, because that is the pole that which you want to find the residue, and that becomes minus 1 4, so no problem about for this concern.

Now you have to find out k 3 1 and k 3 2 of this k 3 t is easily found out, by multiplying this rational function by s plus 3 whole square. So, that we unsecured s equals minus 3, all the other terms vanished. Therefore, k 3 k 2 can be found out as, or maybe we will illustrate this in general way that is. Suppose I write here multiply this s plus 2 over s times s plus 1 times s plus 3 whole square. I multiply this by s plus 3 whole square.

So, that this term will be canceled out, and we will call this m of s. So, this is m of s that is on multiplying this by s plus 3 whole square, I get m of s. So, on the other side you have k 1 s times s plus 3 whole square plus k 2 over s plus 1 times s plus 3 whole square plus k 3 1 times s plus 1 are times k 3 1 plus s plus 3 plus k 3 2. So, multiplying this by s plus 3 whole square the terms correspond to the partial fraction each terms get multiplied by s plus 3 whole square.

Now, if substitute s equals 3 minus 3 in this, we observe that this vanishes, this vanishes and leaves k 3 2. So, k 3 2 is found out in a again in a straight forward manner, by multiplying this by s plus 3 whole square and substitute s equals minus 3 that will k 3 2.
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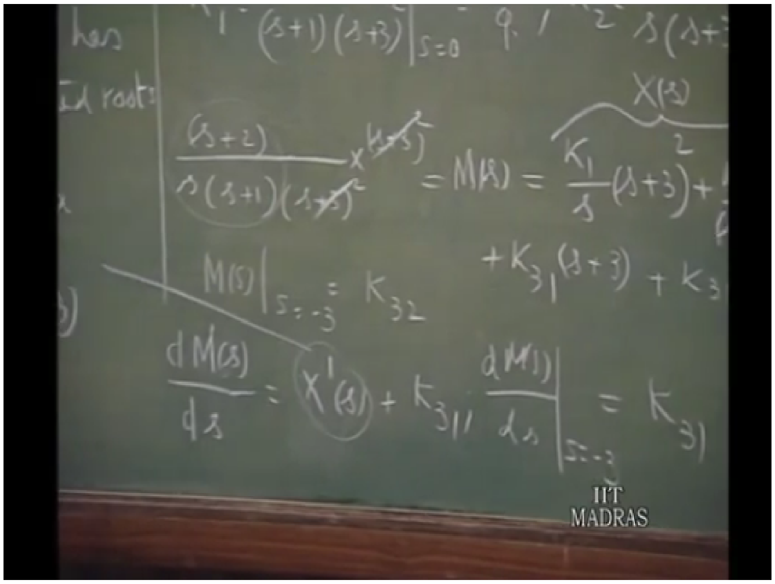

So, we have found out this residue, this residue and this residue. When we have multiple order poles, is actually the terms is associate with linear term that is called the residue. This is really the residue pole at s equals minus 3, not this 1 in mathematical literature this is called the residual the pole is equals minus 3, and that is the one which is order to find correspond compare.

Now, how do you find k 3 1. Now the trick in this a, we have got m of s is equal to s. now suppose you take the derivative of m of s with respect to s. Then what happens is, this term drops out and this leaves you k 3 1, as far as the other terms are concerned since s plus 3 whole square is a factor here.

When you take the derivative of m of s with respect to s, s plus 3 continuous be a factor, because it is a quadratic factor once you take the derivative s plus 3 continuous to be a factor, in both this terms. Therefore, we take d m s; the derivative of m s with respect to s d derivative of m of s with respect to s equals k 1.

Suppose I call this x of s, I have x prime of s plus k 3 1. And now x prime of s has a factor s plus 3; therefore, we have when you substitute d m s over d s as s equals minus 3 will give me k 3 1. And what is d m s; derivative of ms with respect to s, this is after all m of s.

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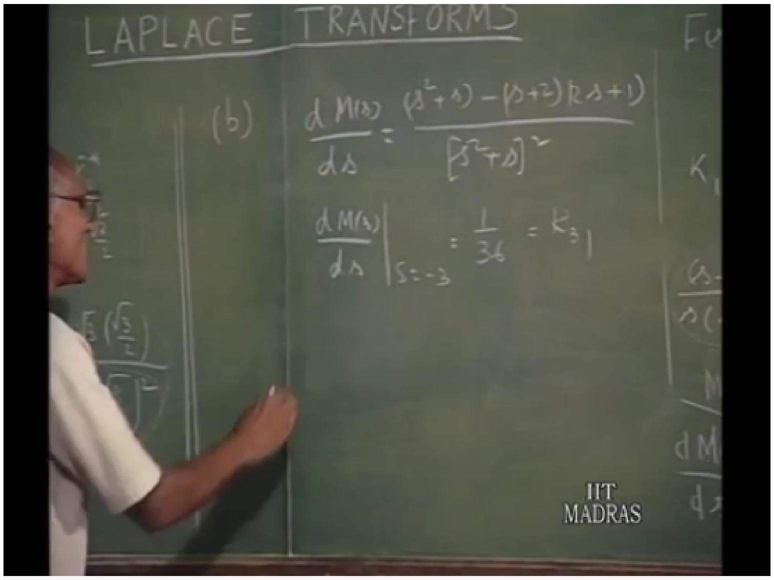

So, if you calculate that, you have derivative of m of s with d s equals derivatives of this, s square plus s times 1 minus s plus 2 times s square plus 2; that is 2 s plus 1 divided by s square plus s and that is what we have d m s by s, and s square plus s whole square. So, d m s over d s substitution s equals minus 3 will give me, after 1 by 36, and that of course k 3 1.

So, the abstract of this is, the partial fraction, required partial fraction expansion will be; k 1 over s k 2 over s plus 1 k 3 1 over s plus 3 k 3 2 over s plus 3 whole square, where all the residuals are found out.

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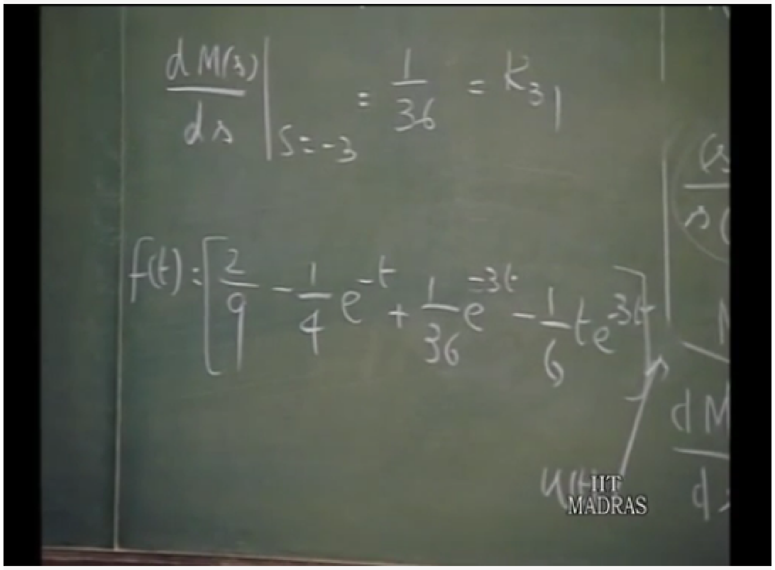

Therefore, f of t will be equal to corresponding to the first term k 1; that is 2 up on 9, 2 up on 9 or s, so for the inverse Laplace transform is 2 up on 9 times u t. u t will come again anywhere minus 1 fourth e power minus t corresponding to this term.

And corresponding to k 3 1 we have 1 up on 36 e power minus 3 t and corresponding to k 3 2 or s plus 3 whole square k 3 2 is found out as we have not work it out k 3 2 will be equal to, when we substitute s equals minus 3 here k 3 2 is found out as minus one sixth; therefore, this becomes minus 1 by 6 and 6 is k 3 2 over s plus 3 whole square, inverse Laplace transform is t multiplied by e to the power minus 3 t.

The whole thing is multiplied by u of t. So, this final answer is 2 up on 9 minus one fourth e power minus t plus 1 by 36 e power minus 3 t minus one sixth t e minus t t times e power minus 3 t multiplied by u of t that is the final results. Now, after having worked out in example, let us know, discuss this more general way, when you have multiple order poles, how do you find the various terms and the partial fraction expansion. Let us look at it in a more general way, rather than given example.