Networks and Systems
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Lecture-57
Laplace Transform Examples

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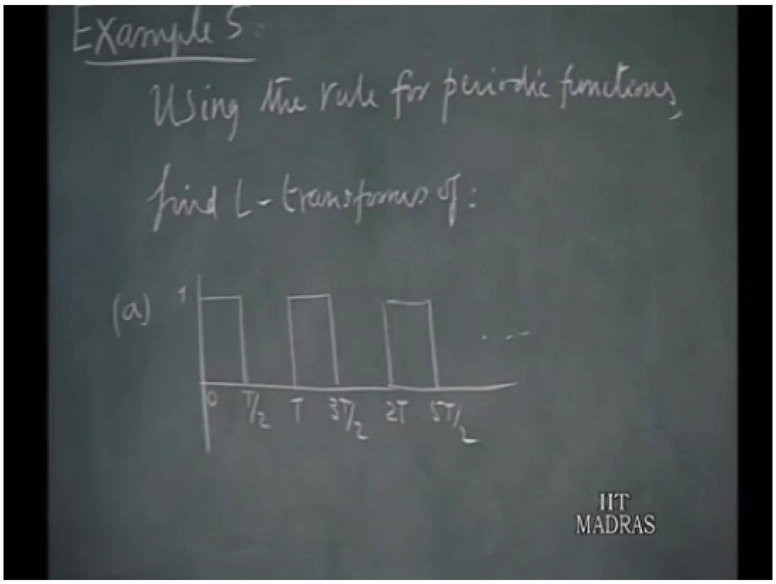

Let us consider one more example; using the rule for periodic functions, find Laplace transform of two functions. A, a pulse train like this, like that it goes; 0 p upon 2 t 3 t upon 2 2 t 5 t up on 2 extra, it is an unit amplitude.

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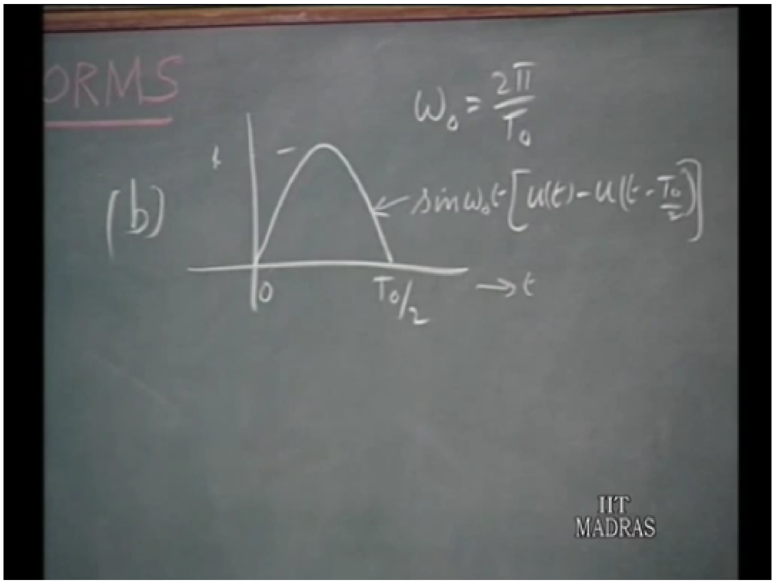

B, a single pulse of a sinusoidal 0 to t upon t not upon 2, where omega 0 equals 2 pie up on t 0. So, this will be; suppose this is 1, this is sin omega not t, in the interval 0 t not up on 2. So, I can write this as ut minus ut minus t 0 upon 2. This is something which we already discussed earlier, but you would like to arrive at the results in a different fashion.

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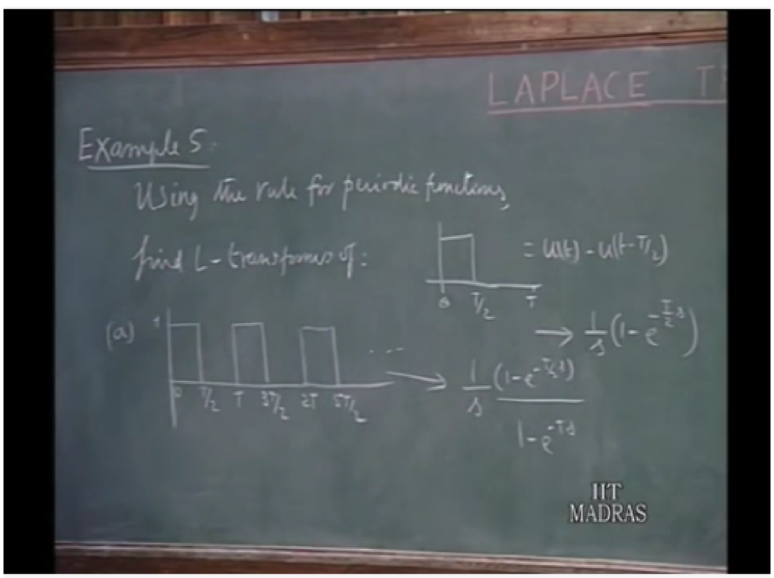

Now take this, we have a pulse train here. So, if you know Laplace transform of 1 single pulse, we can find out the Laplace transform of the pulse train, because if the periodic repetition of the same event, whatever is happening from 0 to T, is repeating itself. So, if you want to find out the Laplace transform of, just this portion.

We know that this is equal to u of t minus u of t minus t up on 2. This is the function is simply a step function minus delayed step function, staring at small t equals t upon 2. Therefore, Laplace transform of this is 1 over s. This is the Laplace transform of this is 1 over s. The Laplace transforms of this the same 1 over s, but multiplied by e to the power of minus t upon 2 times s; that is the Laplace transform of this.

Now, all we are having is, the same pulse is repeating itself, identically every capital t seconds. Therefore, this is the periodic phenomena, and we found out the Laplace transform of the phenomenon in one period. So, the entire periodic function will be obtained by 1 over s 1 minus e to the power of minus t by 2 s. The basic function, divided by 1 minus e to the power of minus ts, according to the rule that we had earlier derived for a periodic case.

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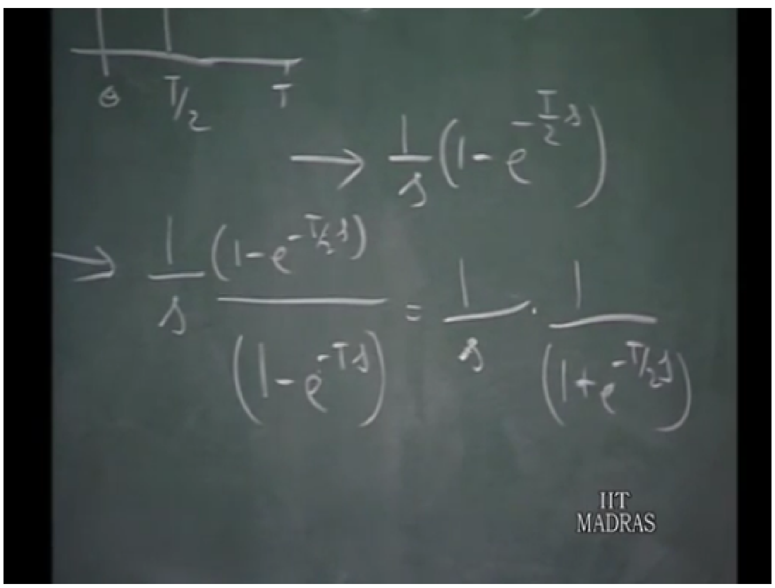

So, the answer is 1 over s, you can put it in the form if you wish or 1 over s times 1 over 1 plus e to the power of minus t by 2 s, because this is a factor here. So, that is the answer for this the Laplace transform of this periodic pulse train.

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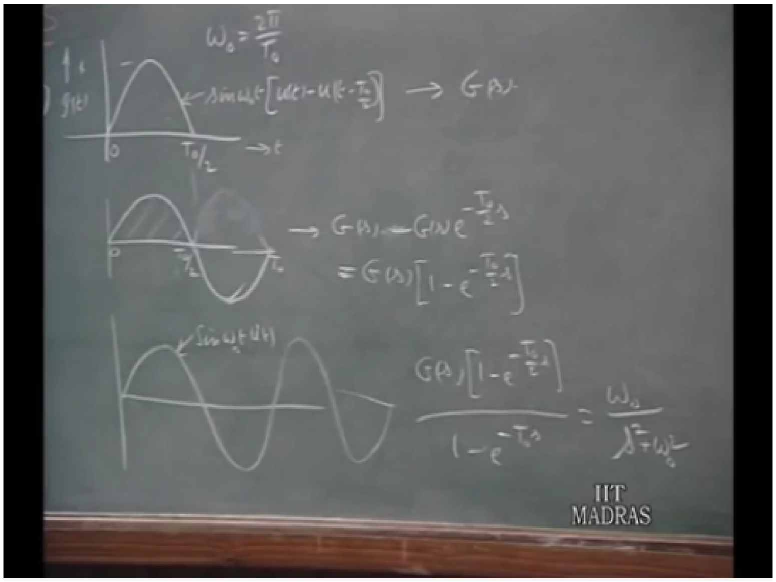

Now, when you go to B, we have derived this in a different fashion, but now I would like to, derive it using the property of the periodic function. Suppose I call it g of t. Then let me say that this as Laplace transform g of s. Now, suppose I extend this like this. I complete this sign function for one complete period.

If the side Laplace transform g of s, what would be the Laplace transform of this. This portion could have the Laplace transform g of s. Suppose this portion can be clipped over like this. Its Laplace transform would have been g of s multiplied by e to the power of minus t 0 by 2 s, because the same thing is delayed by t 0 by 2 second.

So, the Laplace transform of the simple, a loop like this would have been g of s times e to the power of minus t not upon 2 s. But know instead of this you are having a negative going thing. Therefore, this will be plus minus g of s times e to the power of minus t not by 2 times s, in other words the Laplace transform of this 1 period of the sine way would have been g of s times 1 minus e to the power of minus t not by 2 times s; that would have been the Laplace transform of this, in terms of the Laplace transform of this.

Now, suppose the sinusoidal is repeatedly numbered. So, that is the a complete sin omega not t ut, repeating endlessly. Since we know the Laplace transform of one period. You can find out the Laplace transform of the entire periodic function by dividing this by 1 minus e to the power of minus t 0 s.

This is the Laplace transform of this. But we know the Laplace transform of this, this is after all sin omega not t ut, and we know the sin omega not ut as the Laplace transform. Omega not over s square plus omega not square. So, the Laplace transform of this which derived in terms of the Laplace transform of the single loop, must be equal to omega not over a s square plus omega not square. Therefore, using equating this 2, we get the result that g of s equals.

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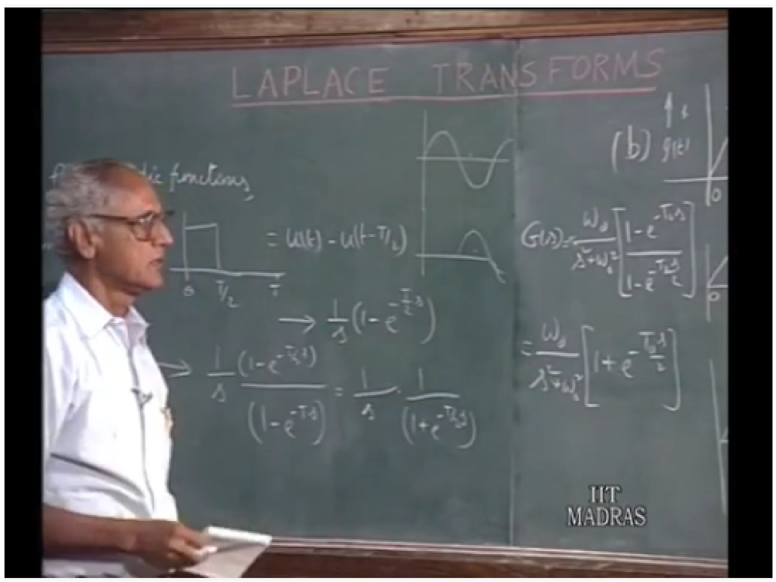

Omega not over s square plus omega not square, multiplied by 1 minus e to the power of t not s divided by 1 minus e to the power of minus t not s up on 2, which is simply omega not over s square plus omega not square into 1 plus e to the power of minus t not s up on 2. The result which we already derived in the last lecture also, using a different property what we did was, if you had a sin wave like this, and another sin wave which is delayed t not of seconds.

If added this 2 you produce this pulse. So, from this we are arrived the same result in a different way, but now we got the same result using the property of the periodic functions. We had. So, for talked about the transformation in the forward direction; that is given function of time, we are trying to find out the Laplace transforms. But in the solution of network transient, we also have a occasion, to have to find out the inverse Laplace transformation.

Given f of s you should like to find out what the f of t the corresponds to it. This is called the inverse Laplace transformation. Now, this inverse Laplace transformation can be approach from 2 points of u; 1 is what is called the partial fraction expansion; that is what will discuss primarily.

It can also be approached through the defining integral relation, of f of t, getting f of t from f of s of the inverse transform integral relationship, which goes as remember 1 over 2 pie j integral from c minus j infinity to c plus j infinity of f of s e to the power of st ds. The second approach is, little complicated, and it can be used only for a special occasions arise. But for ordinary purposes, its enough we know how to find the inverse Laplace transformation to the partial fraction expansion.

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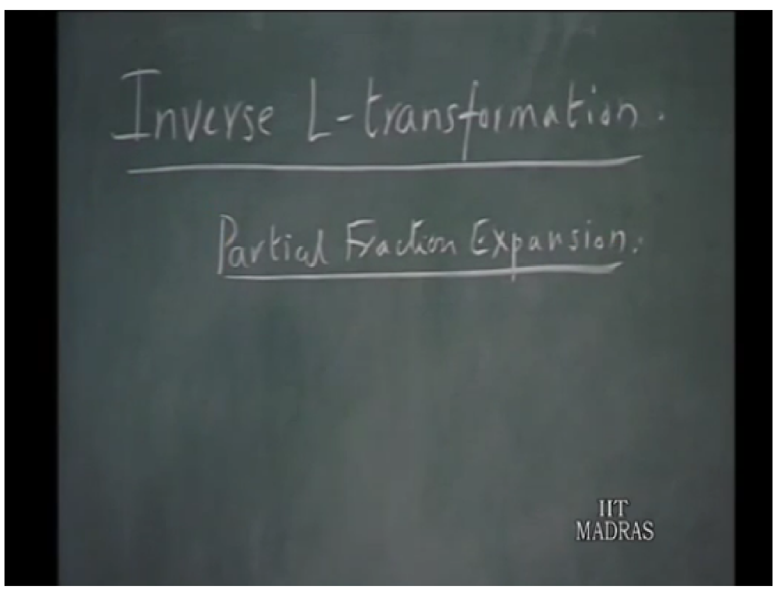

So, let us consider the partial fraction expansion as the preferred method of finding out the inverse Laplace transformation. The approach that we take up here is somewhat analog as what we do, when we take up the integration of functions. You know when we have to integrate something the integrant is split up into components which are recognized to be the derivatives of some functions.

So, once we know recognize the derivatives, then the function whose derivatives they are we know, and then some of all those functions will be the integral of the various derivative function that we know, more over the same approach is take up here also. In other words, normally the f of s that we have to deal with is the rational function.

So, we split up this rational function, as the sum of small elementary functions, which are Laplace transform of known time functions. So, if you do that, then the sum of all this elementary function, will have the inverse Laplace transform which is the sum of the corresponding time functions which we recognize as the made of the elementary functions; that is what we do.

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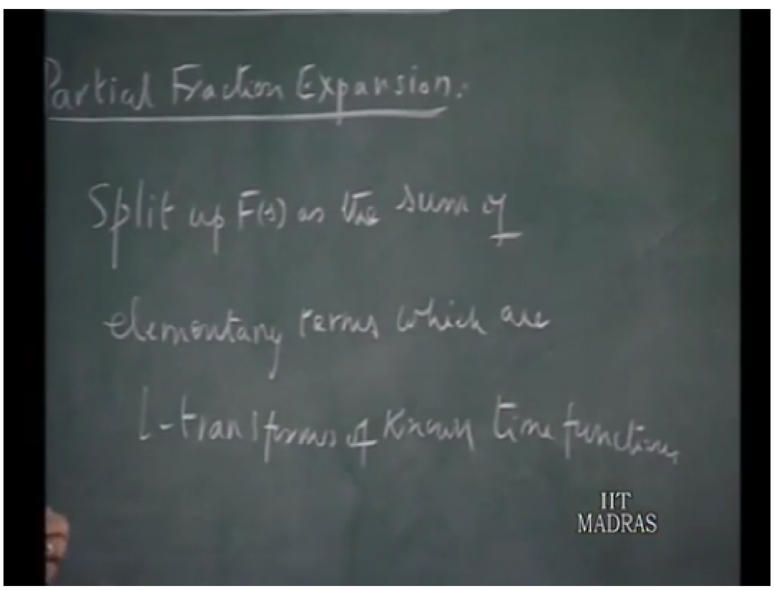

So, split up f of s; the philosophy is split up f of s as the sum of elementary terms, which are Laplace transforms of known time functions. This is the philosophy.

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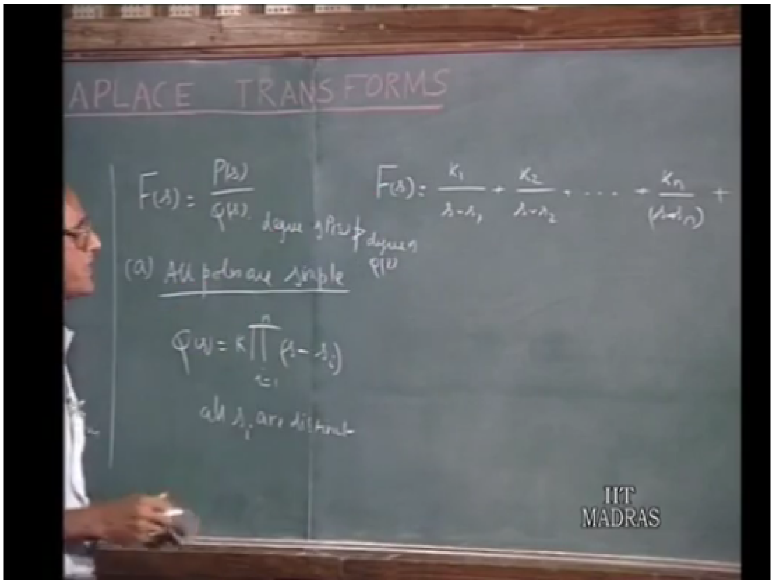

So, let me illustrate this; let us take f of s as the rational function, which is the ratio of 2 polynomials in s qs. And we know the values of s which make the numerator 0 are called the zeros of this function. And the values of which make the denominator 0, are called the poles of this rational function.

So, let us take the case all poles are simple. in other words q of s is the product of some constant k times s minus si where all si are distinct; that means, if you take i from 1 to n whatever the factors here, all of them are different values. No two repeat themselves none of this poles repeats itself; that means, no 2 si are the same. Now, to handle this, we can write this f of s as; k 1 up on s minus s 1 plus k 2 up on s minus s 2 down the line kn over s minus sn.

If there are n such factors in q of s; say i from 1 to n, then there are n such factors. And if the numerator degree same as the denominator degree you also have a constant term, k 0. We will assume in our case that the degree of p of s, is not greater than the degree of q of s, which is usually the case, so we do not have to worry about cases where you have stopped with k 0, you do not have the terms like ks ks square and so on.

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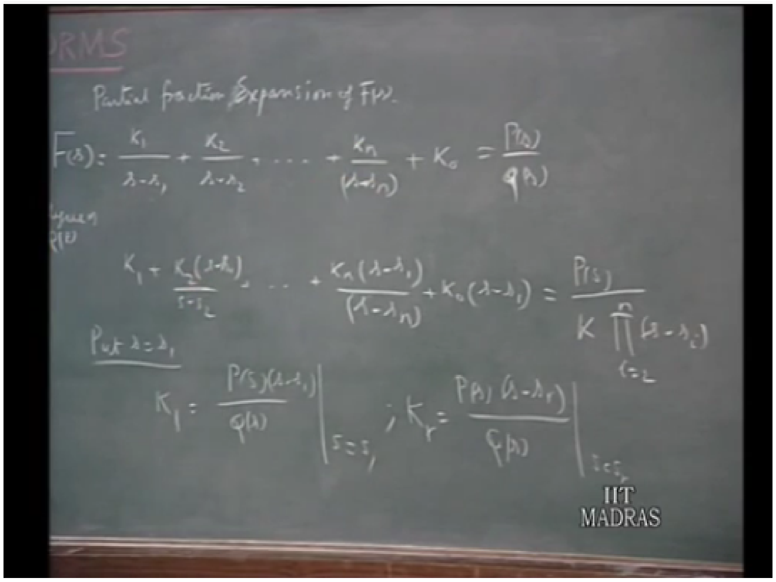

So, this is f of s. now if this is the partial fraction expansion of f of s, where all poles are simple. So, given qs we have to put this in this form. If you do that, then we can find out the inverse Laplace transform of each one of these elementary term quit conveniently. So, our first job is, to express ps over qs to be like this. Now how do you find the values k 1 to k n.

Now, you observe suppose I multiply all these terms, this side the equation and this side the equation by s minus s 1, then I get k 1 plus k 2 over s minus s 2 times s minus s 1 etc plus k n times s minus s 1 over s minus s n plus k 0 times s minus s 1 equals p s over. As far as denominator is concerned, because s minus s 1 is the multiplying factor the numerator, one particular terms get dropped out.

So, you have some k the product of s minus s i i starting from 2 to n only, because s minus s 1 got dropped out. Now, if in this put s is equal to s 1. Suppose substitute s is equal to s 1 in this, then all this will vanish, and on this side only k 1, and on the other side you have therefore, p s over q s what you have really done is, we have multiplied by this s minus s 1; therefore, this s minus s 1 got cancelled out in q of s and that left this.

So, I can put this in this form p s times s minus s 1 by q of s with the substitution s is equal to s 1. So, that is how you can evaluate k 1. in general any one of this suppose you want to find k r, all you have to do is p s multiplied by s minus s r divided by q of s; that means, this s minus s r term is cancelled out q of s. Symbolically you represented in this fashion, then amplitude s is equal to s 1.

So, that is how we can find out all this factors. These are called the residue of this poles k 1 is the residue pole it equal to s 1, k 2 as the residue as the pole it s 2, k n is the residue at the pole in s n. So, we can find out all this residues, and once we have all this residues, we can find out the corresponding f of t. So, once we have this f of s, corresponding f of t, is easily obtained, by finding the inverse transform of each one of this.

We recall the k 1 over s minus s 1, as corresponds the time function which is equal to k 1 e to the power of s 1 t. Similarly, k 2 e to the power of s 2 t and so on and so forth.

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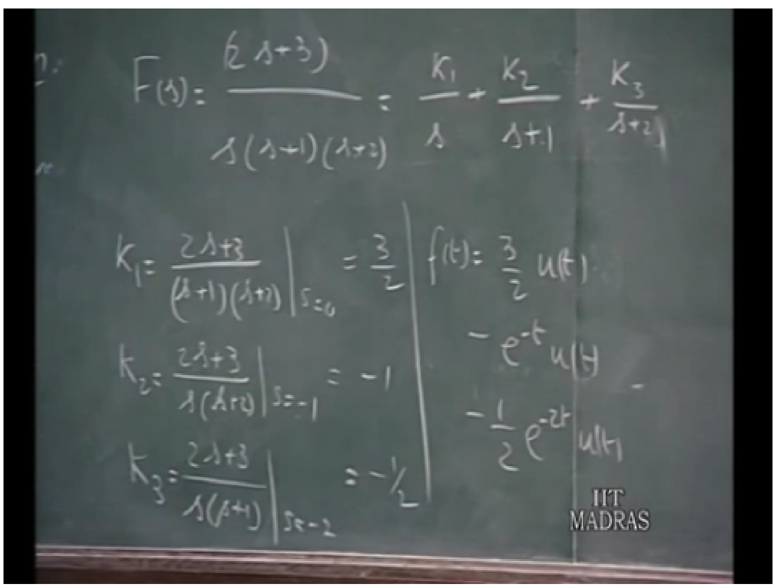

One quick example we illustrate this, and then we will be done with that, as for this lecture this concerned. Example, suppose f of s equals 2 s plus 3 over s times s plus 1 times s plus 2. So, it has got 3 poles. So, I will have k 1 over s plus k 2 over s plus 1 plus k 3 over s plus 2.

So, these are the three terms that evaluate. To find out k 1, you multiply this function by s. So, once you multiply this function by s, you have 2 s plus 3 over s plus 1 times s plus

2 and substitute the value s equal 0, because pole is at equal to 0, this is 3 upon 2. Now, as far as the k 2 is concerned, you multiply this function by s plus 1.

So, 2 s plus 3 over s times s plus 2 substitute s is equal to minus 1, because s plus 1 made equal to 0. So, s is equal to minus 1. If substitute s is equal to minus 1 in this, you get the answer minus 1. And like wise k 3, you multiply this by s plus 2; therefore, we have left with 2 s plus 3 divided by s times s plus 1 and we substitute s is equal to minus 2, and the answer for that happens to be minus half.

So, the partial fraction of expansion of f of s leads to this three terms; k 1 k 2 k 3 which are evaluated like this. So, from this f of t can be obtained as k 1 by s 3 by 2 up on s, it corresponds to 3 by 2 u t plus k 2 up on s plus 1. This corresponds to k 2 we to the power of minus t k 2 is minus 1; therefore, e to the power of minus t u t; that is the inverse Laplace transform of k 2 up on s plus 1.

The inverse Laplace transform of this is, k 3 e to the power of minus 2 t k 3 is minus half. So, minus half e to the power of minus 2 t u t; that is the final result 3 upon 2 minus e to the power of minus t minus half e to the power of minus t u t. So, this is how one can find out the partial fraction expansion in the case of simple poles, and from that you can find out the time function, to which the original given f of s corresponds to.

So, in this lecture, we worked out various examples, to illustrate the properties of Laplace transform that we studied earlier, and then we started a discussion on the, whether of finding out the inverse Laplace transformation of a given function of s; that is to find out identify f of t to is the given f of s corresponds, and in this direction we started out in the partial fraction expansion of the given function of time, given function of s.

And as the first case we have taken the situation where all poles are distinct and simple; that means, no poles is repeated, and we can find out the rescues of the poles in every compact way, by multiplying ps up on qs the rational function, by s minus s i where s i is the pole, at which you have to find out the residue. So, once these residues are find out.

All the k factors are found out, the inverse Laplace transformation is quite straight forward, and that is what we have illustrated by means of an example. We will continue this discussion of finding out the inverse Laplace transformation by partial fraction of expansion, in the next lecture where we will consider, the situation where some poles are repeated, may be for the 2 or 3 as the case may be.