Networks and Systems
Prof. V.G.K. Murti
Department of Electronics & Communication Engineering
Indian Institute of Technology – Madras

Lecture-50
Application and properties of Laplace transform

We have seen how differentiation in time domain carries over in the transform domain by multiplication of f of s by s and adding on to get the information according the initial value of the quantities being differentiated.
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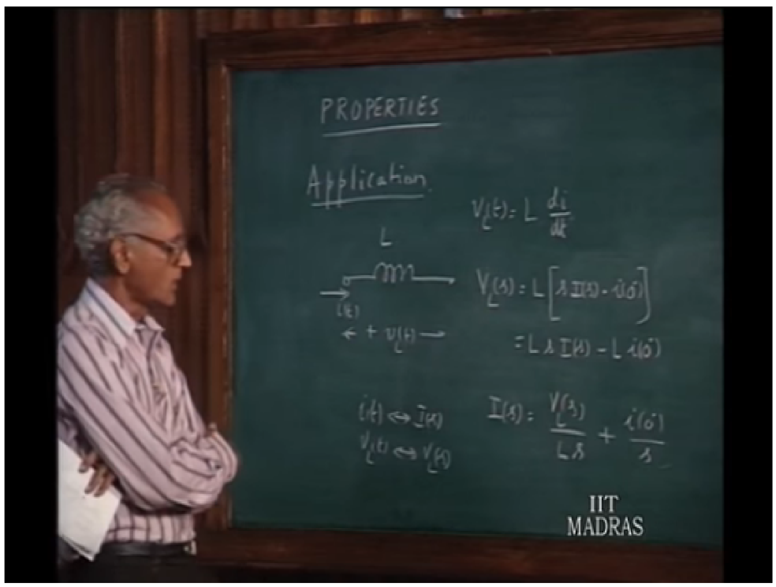

Let us look at the application of this. Application is important in the sense, that the result that are going to get here, will be used r and r again instead Froude analysis. Let us consider, an inductor of l henries carrying a current i t that the voltage across the inductance be VL. So, this are the 2 terminals of the inductor and you have the current i t passing through l generating a voltage VL.

The fundamental rule relating VL and it domain this is VL of t of course, VL t equals l times di dt. Now, if you know the Laplace transform for the current i t. Then we should be able to find out the Laplace transform of the voltage VL of t using the differentiation rule that was just now study. So let us, Laplace transform the left hand side and the right hand side assume that i of t has the Laplace transform I of s. So, if you make the Laplace transform for the left hand side and right hand side respectively.

And let VL of t have the Laplace transform VL of s then we have VL of s equals l time. The Laplace transform of di up on dt will be s times i of s minus i 0 minus. We can write this alternately as Ls times i of s minus l i 0 minus. Now, 2 points have to noted here. 1 time domain quantities are given lower case letters VL and i when you make the Laplace transform of that we use the capital letters VL of s and i of s this is the convention.

Secondly, we are interesting to finding out VL of t starting from 0 minus onwards. There is the possibility that i becomes to 0 minus to 0 plus. So, if you want to take the derivative of the current including the transition from 0 minus to 0 plus. Then we must take the initial condition here a 0 minus. On the other hand, you are interested in the behavior of the VL of t from 0 plus onwards only and you would like to consider all values of VL from before 0 minus then we have take this for 0 plus.

That we are going to illustrate this taking 0 minus values because that is more general situation and optionally you can just take 0 plus as well. So, this equation tells as that if you know the Laplace transform of current i of s because we calculate VL of s in this manner. But to do this must be the realist value to be current we can inward this expression and find out i of s in terms of VL of s as 1. So, if you do that then I of s can be written as VL of s divided by Ls plus you are going to transfer this to the other side get i 0 minus.

So, that gives you an alternative expression relating the terminal current and terminal voltage in the Laplace transform domain i of s is VL of s over Ls plus i 0 minus over s. So, both this equations now are the terminal equations bring that in the transform domain. This the terminal equation bring that in the time domain, this are the equations transform domain both are equal.

Notice 1 thing that, the derivative operator that remove this are purely algebra equations; however, the extra term that you need to have here initial conditions in the invert. Now, provide the initial conditions are 0 suppose i 0 minus is 0. There is the proportional relationship between VL of s and i of s VL of s is Ls i of s i of s is VL of s over Ls.

That means, there is a kind of generalized impedance to the current flow in the transform domain. If the current i of s passes through inductor the voltage is Ls times i of s. So, Ls can be regarded at the generalized impedance of the inductor in the transform the way. same thing here also, VL of s over Ls.

So, 1 over ls can be thought of the generalized remittance. So, the induct the impedance concept the proportional relationship between voltage and current whatever domain we are taking about transform domain or j omega domain, sinusoidal circuit take situation for all valid provided you don’t have to the extra term.

That means, if there are no initial condition there is a proportional relationship between voltage and current in the transform away. These initial conditions that spoil the correct. But will see how to take care of that relates, but now, this particular equation can be represented by means of circuit representation like this we have an inductance.

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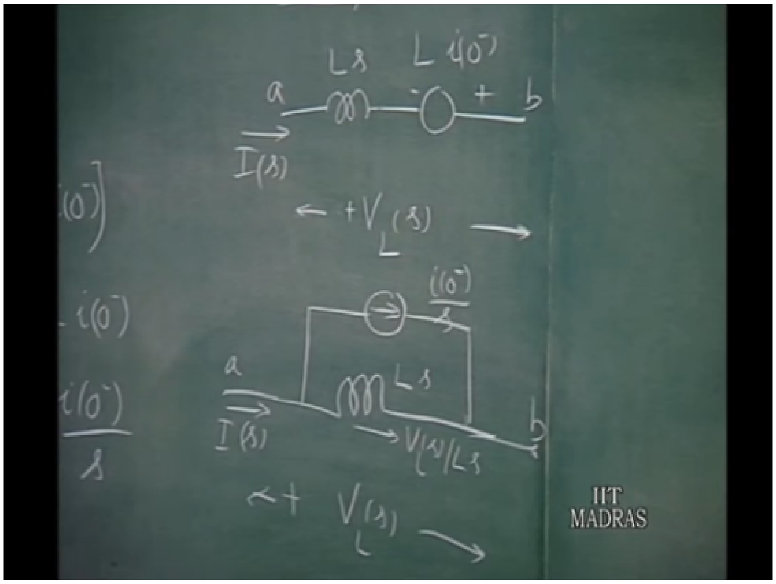

We can think of that having a generalization impedance Ls and we have a current passing through it not the raw current, but the transformed current Laplace transform variable and across the terminals of inductor we must away among voltage of VL of s. So, this is the voltage of the terminals of that we know, that VL of s is the Ls times i of s minus L i 0 minus.

So, this current is passing through Ls will developing voltage which is Ls times i of s in transform domain. But to complete the picture we must have traditional term here, minus Li 0 minus and this quantity is independent of i of s. So, Li 0 minus is independent of i of s therefore, there must be extra voltage here which is independent of i of s that is equivalent to voltage source.

So, you must have a voltage source in the transform domain which is L times i 0 minus like this. So, this description on such the sequence that is this circuit VL of s can be thought of as i of s passing through ls developing a voltage Ls times i of s to as minus Li 0 minus because this is the drop followed by arise.

Similarly, if you want to model this equation by means; of equivalent circuit representation. We have a element having a generalized impedance Ls with the terminal voltage VL of s. So, naturally the current here will be VL of s over Ls. But the terminal current is; is larger than this by additional value; therefore, you should have a current source which i s i 0 minus over s. So, the sum of this 2 currents will be your terminal current I of s.

So, this than is another equivalent representation of an inductor in the transform domain. We have, a inductor having generalize impedance L s in parallel with a current source of tend i 0 minus or s in the transform domain this the terminal relation. So, you have 2 equivalent circuits for an inductor, which incorporates initial value in the transform domain.

Both this are called transform diagrams, transform diagram for a particular inductor. One incorporates the voltage source the other incorporates a current source. So, depending up on your convenience when a circuit analysis problem. You favor the useable voltage source we use yes.

On the other hand, if you are a loud with the current source you can use this in particular if you have a network in which the load equations are preferred in whether of solution may be you need to have like have a current source. But both are equivalent the important thing to keep in mind is that if the terminals of the inductor in the actual circuits are a and b here, the equivalent the terminals in the transform diagram are of this a and b.

We should not thing that the terminals of the induct the transform diagram are this 2. Because the totality of the structure here represents the inductor carrying a initial current. So, that is be kept in mind what we have done here is representation of an inductor in the transform domain.

Where the variables in the circuit are the transformed variables are the functions of time where this is the Laplace transform of the voltage this is the Laplace transform the current. So, the variables are the transformed variables and the element tend the Ls this is called generalized impedance of an inductor. In the transform domain the generalized impedance of inductor transform domain is Ls.

In a similar fashion, we can give a similar representation of the capacitor with initial charge on that we will postpone a latter time we will not do this in greater detail at this point of time. Just as we have generalized impedance Ls for a inductor for a capacitance out of the generalized impedance is 1 of our cs.

We will do that later, but our point in presenting this at this stage is to show an application of the derivative rule in the Laplace transform theory. How you can apply that to evolve circuit models of this time this are called transform diagrams. So, we have a transform diagram of a whole circuit we have similarly generalized impedance for inductance capacitors so on.

A whole assembly is for the transform diagram this is the transform diagram, but 1 element which is an inductor. So, after having consider this who will now go on to the study of the third property of the Laplace transformation multiplication by t.

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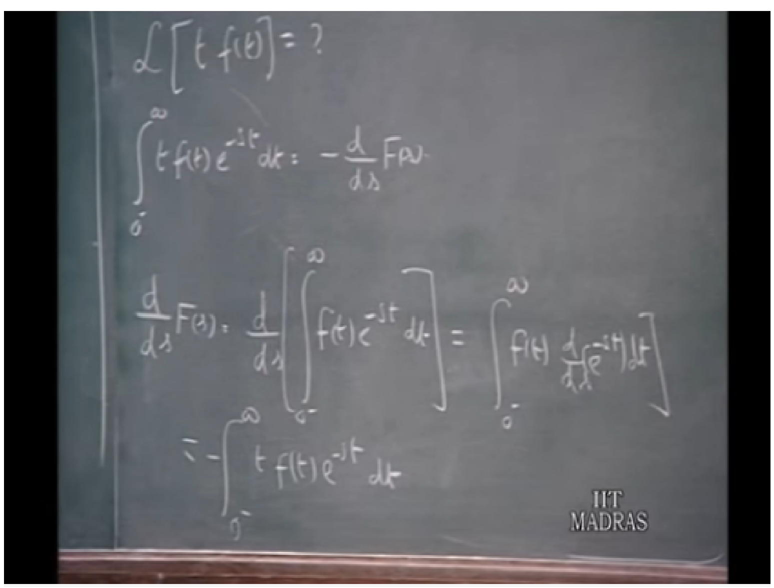

So, we will like ti ask the question what the Laplace transforms of t ft. This is the question that we like to ask. So, the Laplace transform of t ft 0 minus to infinite t ft e to the power of minus st dt, this is what we like to ask. Now, to do this let me take d by disease of f of s. Suppose, I want to take the derivative of f of s which is Laplace transform of f of t this is d by ds of 0 minus to infinite of f of t e to the power of minus st dt this is f.

The integration is to the reference to t and we are taking the derivative reference to another variable. So, we can introduce this inside the integration time and this can be written as 0 minus to infinite of f of t d by ds of e to the power of minus st dt. And this when you take the derivative of this with reference to s that is minus t is sin outside. So, 0 minus t infinite of a input a minus sign because of that, t ft e to the power of minus st dt.

And that is exactly what we are having here you wrote minus infinite t ft e to the power of minus st dt. And that integral is equal to minus d by ds of f of s as we can see. So, this integral is equal to minus d by ds of f of t because is a negative sign; therefore, this will be minus d by ds of f of s. So, when you multiply a function of time by t as for the transform is concerned. It is equivalent to the negative of the derivative of the f of s to s.

So, you can see that there is a nice arrangement here, you multiply by t you get minus d by ds this. On the other hand, when you take the derivative of f of t df by dt you are multiplying that s f of s. In the derivative formula you get d by dt of f of t has the Laplace transform s times of s. On the other hand, when you multiply t of t is d by ds of f of s except for the negative sign is somewhat a dual relationship exists.

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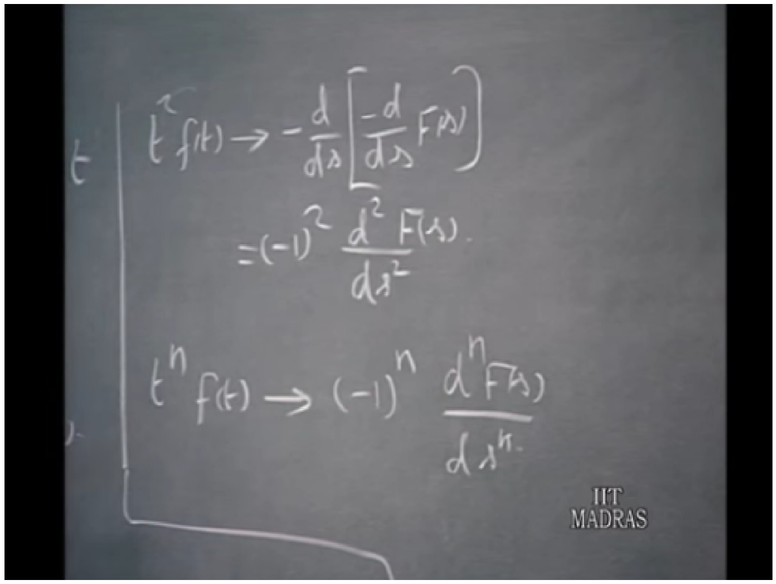

Now, based upon this let me extent this result. Suppose, I take t square f of t then t f of t is minus d by ds of f of s; therefore, I am straight minus of d by ds of the Laplace transform of t ft which is minus d by ds of f of s. So, this will be minus 1 square d squared by ds square f of s that is all. So, you have tn f of t the terms to be if you take tn you have minus n to the power of n n'th derivative of f of s. All the time we are assuming that the causal functions 1 a verity an f of t we assume that tn f of t will be 0 for negative values of n this is understood.

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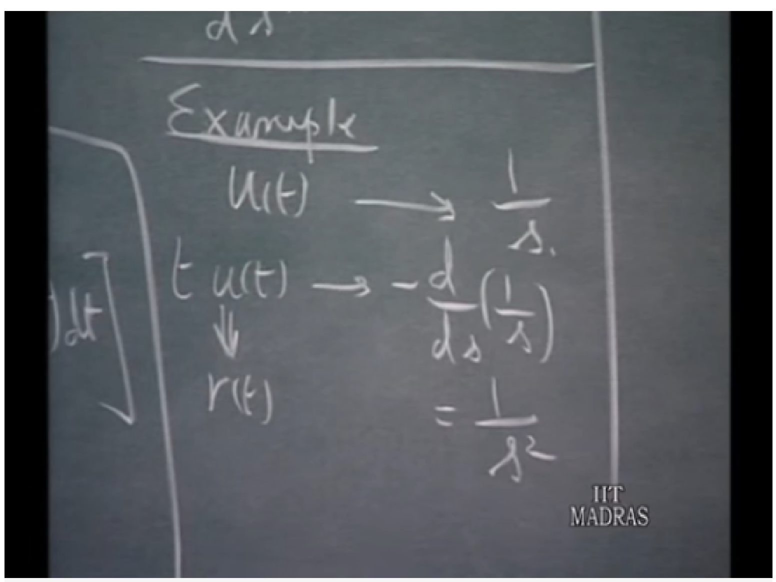

So, let me take simple example. We know that, u of t as the Laplace transform 1 over s. Now what happens, if I take the Laplace transform of t ut. I multiply this by t, then you must take the negative of the derivative of this with reference to s minus d by ds of 1 over s that is 1 over s square and t ut is indeed rt. So, that Laplace transform of a ram function is 1 over s square.

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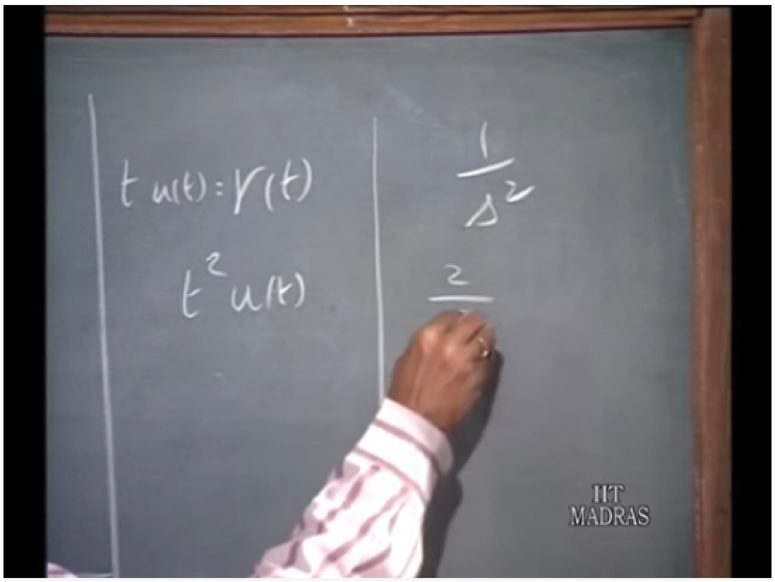

So, we can write the extend over Laplace transform implies the table, we have already consider u of t delta t sin omega not t cos omega not t we can also say Rt Laplace transform is 1 over s square. Now, suppose i take t square ut this is t ut of course, t square ut; that means, state this derivative of these ones again and data's are negative sign. So, if you do that it is minus 2 over s cube will be derivative and you attach the negative sign it is a 2 over s cube.

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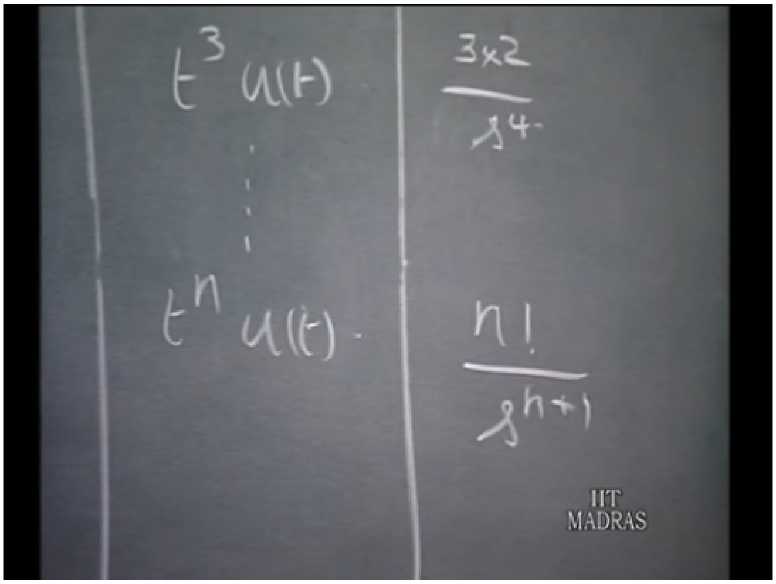

Continuing in this fashion suppose i have 2 cube ut then it will be again you take the derivative of this is 3 times 2 divided by s to the power of 4. And if you proceed further in this fashion, if you have tn ut then you have n factorial over s to the power of n plus 1. That's how it goes as we take higher and higher powers of t the Laplace transform goes in this manner. Let us now consider another example.

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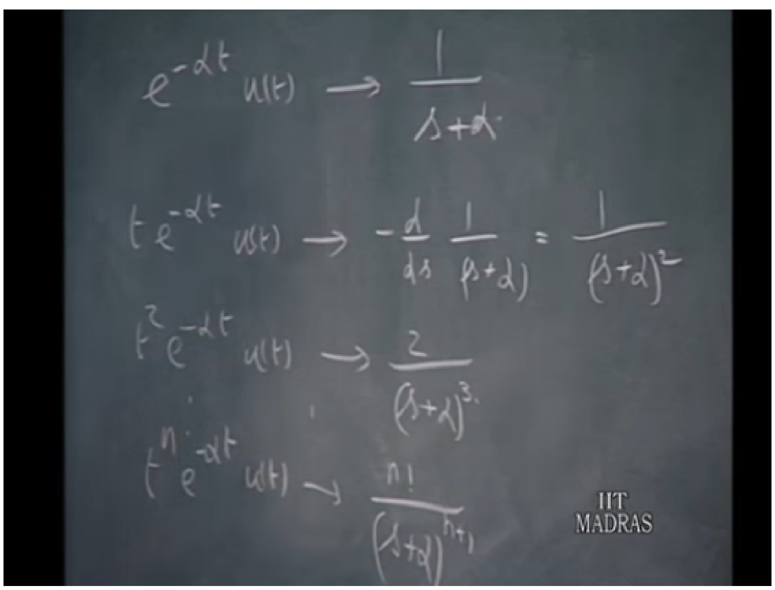

Suppose I have e to the power of minus alpha t ut, then we know the Laplace transform is 1 over s plus alpha. Now, I suppose multiply this like t e to the power of minus alpha t ut then its Laplace transform is minus d by ds of 1 over s plus alpha. And that is indeed 1 over s plus alpha Whole Square 1 over s plus alpha Whole Square. And suppose I have t square e to the power of minus alpha t ut than it can be shown this is indeed 2 over s plus alpha to the power of 3.

And in fact, as before you have t to the power of n e to the power of minus alpha t ut. You continue in the same fashion you have n factorial over s plus alpha to the power of n plus 1. So in other words, in we need not remember the Laplace transforms for t multiplied answers like this either e to the power of minus alpha t or in terms of u of t. It’s enough you know, the basic formula for ut then when ever is multiplied by t to the power p square t cube extra than you can easily derive them without any difficult.
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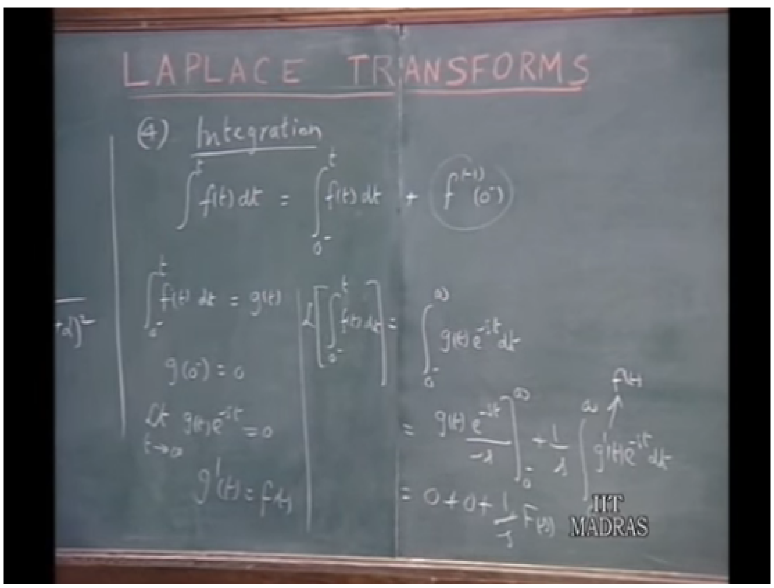

Let me now take the next property which is the integration which is again very important and which commonly occurs. Let me consider, f of t dt in infinite form which we can always regard this as 0 minus to t of f of t dt plus. Suppose, this f of t is a current let us say then what we are doing this by integrating only go for the charge. So, we find it convenient to consider the charge at a function of point at a time t.

As the initial charge plus the additional charge that is carried by the current the point 0 minus t the interval 0 minus to t. So, this is the value of integral to limit 0 minus to t to this the initial value of the integral of this write this as f inverse minus 1 with in bacl at 0 minus. That is, this is the initial value of this integral plus the addition to the integral in the value in the interval from 0 minus to t.

So, this is how we can pit the interfered integral in the form of definite integral because they initialize. Now, you not to find out the Laplace transform of that. Let us find out the Laplace transform of the 2 individual quantities. So, 0 minus to t of f of t dt suppose I called the g of t. Then, we know it straight away that g 0 minus equal to 0 because like in the integral between the limits 0 minus to 0 minus that must specify.

And further limit as t has going to infinite of g of t e to the power of minus st equal 0. Because, we are taking the value of g of t to be Laplace transformable and you are taking the value of real part of s is that this tends to 0. That is value of st is the region of convergence of the Laplace transform of g of t.

With these conditions and further we also know that g prime t the derivative of g of t g prime t is f of t because this the integral of this the fundamental role of the integration tells as that the derivative of this is going to be f of t .Using this the Laplace transform ft dt Laplace transform of that equals 0 to infinite of this quantity which is gt e to the power of minus st dt.

And that would be obtained as, let us do the integration by the parts. I will take g of t e to the power of st take the integral of this divided by minus s between 0 minus t infinite plus 1 over s 0 minus to infinite of e to the power of minus st g prime t the derivative of g of t times e to the power of minus st dt.

And this will be at the upper limit this will be 0 because limit as t tends to infinite of g to the power of minus st this is 0 at the lower limit this is also 0 g or g minus is 0 therefore, this is 0 and you have 1 over s of g prime t is the same as f of t. Therefore, 1 over s times f of s. So, we have that the Laplace transform of this is a f of s over s and this is a constant f.

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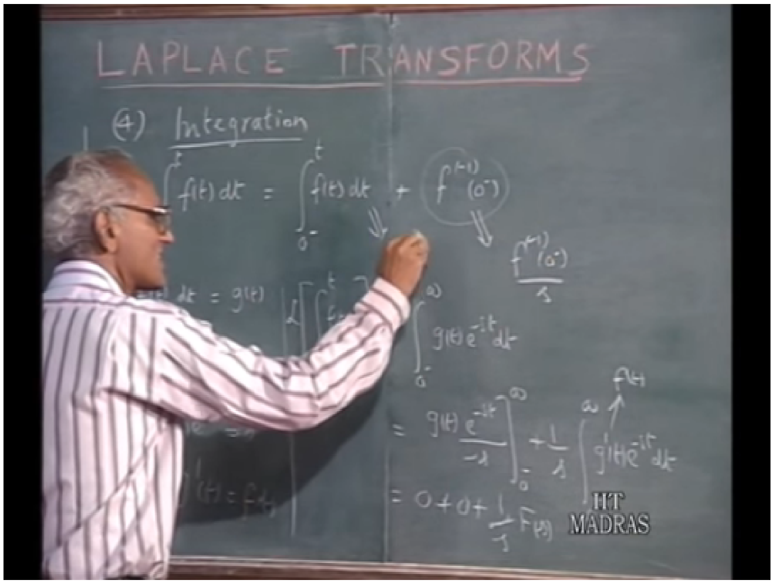

The initial value of the integral is going to a constant that will be the Laplace transform of this is f initial value divided by s and the Laplace transform of this is f of s power s.

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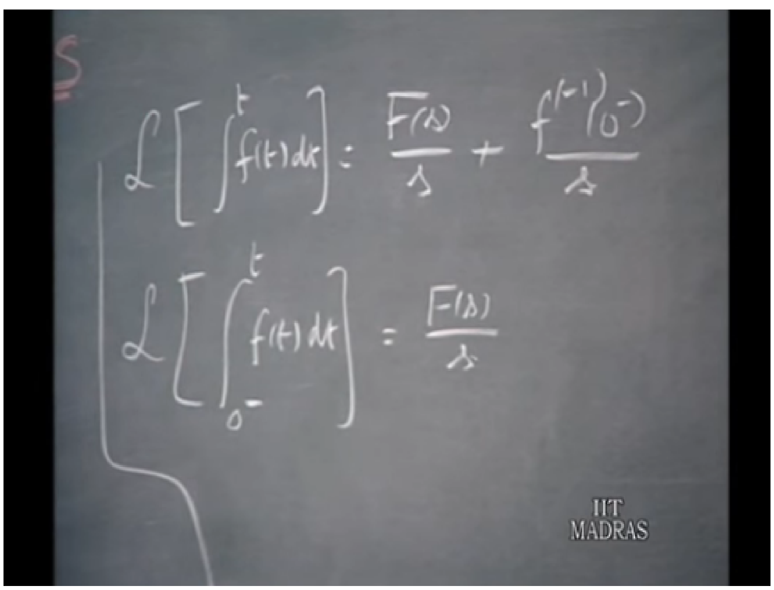

So, we have Laplace transform of the integrate integral f t dt as f of s over s plus the initial value of the integral at 0 minus over s. On the other hand, if you want to take the Laplace transform of the definite integral 0 minus to t of f of t dt; that means, the initial value of the integral is ignored. Where this is simply f of s over s, that this are the 2 important results relating to integration.

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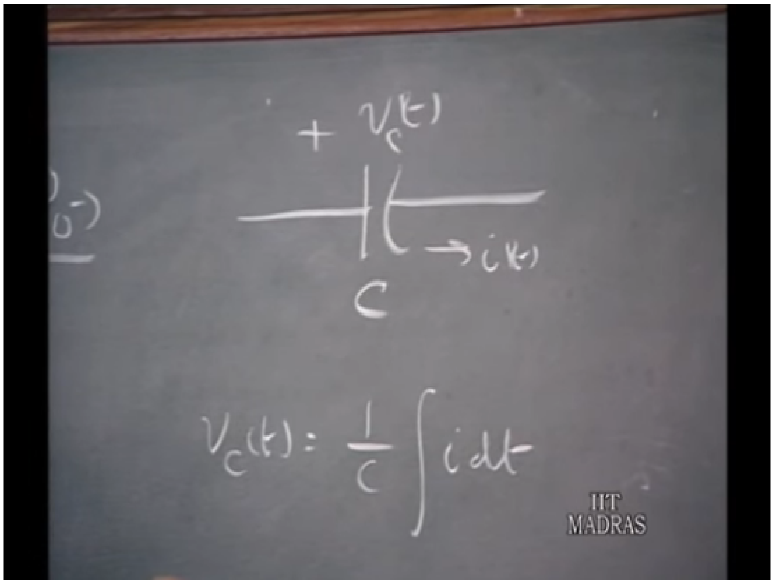

And we can make use of this, when suppose have a capacitor value c and this is the voltage across the capacitor vct and the current the capacitor is it then we can write vct is 1 over c the integral of the current which is the charge or the capacitor.

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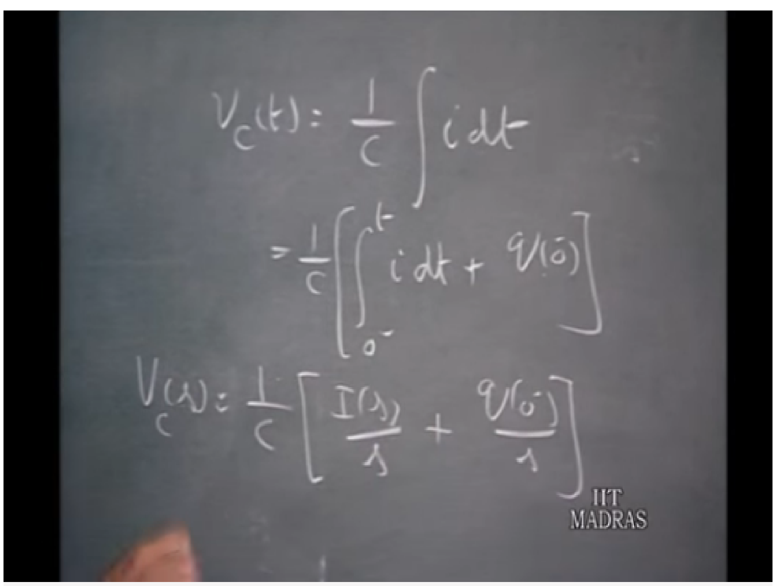

I can write this further as 1 over c 0 minus to t of i dt plus the initial charge on the capacitor 0 minus. So, if i take the Laplace transform of the various quantities. I have Vc of s the Laplace transform of the capacitor voltage is 1 over c this is I of s over s plus a constant this is the constant q 0 minus over s.

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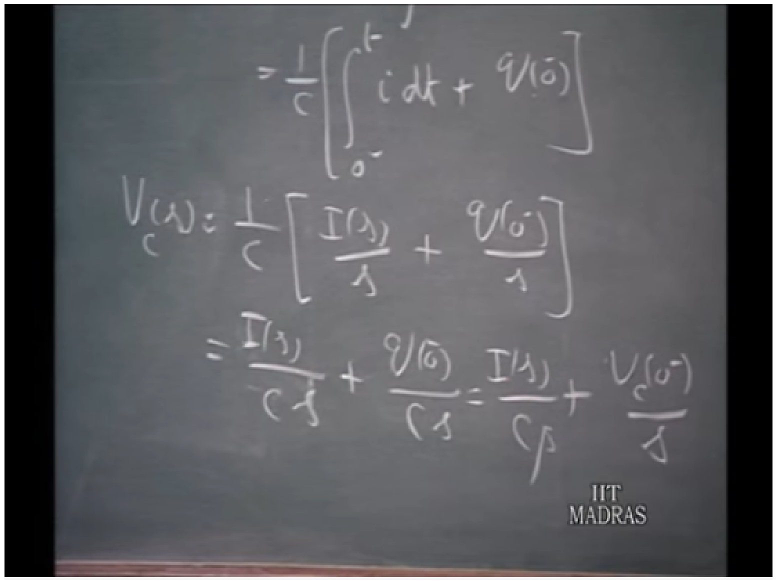

So, i can write this as I s over cs plus q 0 over cs which is I s over cs plus q 0 minus over c is the initial value voltage across the capacitor; therefore, I can write this as Vc 0 minus over s.

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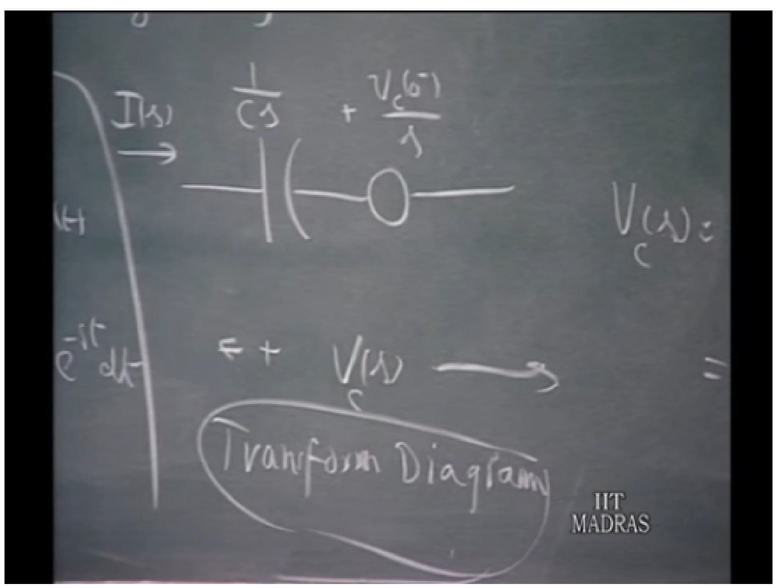

So, this equation can be represented nicely in these fashions. This is the Laplace transform the current this is the Laplace transform of the voltage across the capacitor and this is the generalized impedance 1 over cs I s passing through 1 over cs create this voltage in addition, I have the initial value of the capacitor voltage divided by s a voltage source. So, this is then the transform diagram of the capacitor in the same way as we derived it for the inductor.

An equivalent in the term in the parallel form the capacitor parallel with the current source can also be derived, I will not do that it leave it the exercise for you that it can be drawn up in similar fashion as we did in the case of an inductor is very important. Now, if you see that you must distinguish between that 2 formulas and when you have the initial charge in the capacitors we have that is equivalent to a source which is the represents the initial charge in the capacitor or the initial voltage of the capacitor.

So, will stop this discussion of the properties of the Laplace transform of this point. So, in this lecture what we have covered is we started of with revive of the definition of the Laplace transformation and type of functions which are Laplace transformable then we went to discuss the various properties of the Laplace transform in particular what we talked about with linearity property then we talked about the derivative how the derivative time function carry over in the transform domain.

Then we discussed how multiplication of f of t by various powers of t how such functions are transformed in the Laplace transform domain. In particular we said t ft becomes as the Laplace transform minus d by ds of f of s then we went to the integration of the time functions and we distinguish between the definite integral indefinite integral and in both the derivative case and the integral case.

We saw how they can be applied to the situation of the induct than the capacitor how the relation between the transformed variables of currents and voltages across inductor and capacitors can be represented by means of such a diagrams which we called transform diagrams. We will continue the discussion of the properties of the Laplace transforms in the next lecture.