Networks and Systems
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Lecture-46
Introduction to Laplace Transform

In the last few lectures, we have discussed the Fourier Transform technique for the analysis of linear and networks systems. You recall that, the Fourier Transform technique is considered very appropriate, in dealing with networks and systems which are characterized by their frequency response function; either because, the frequency response function is deduced experimentally using convenient techniques or because, the specifications in terms of frequency response, comes naturally for such systems or networks. For example: filter networks.

However, for the analysis of general linear networks for the transient performs that is, the Laplace Transform offers a number of definite advantages and for this particular application it is unrivaled and therefore, we would like to spend some time now, in discussing the Laplace Transformation techniques for the analysis of linear networks and systems.

For the first few lectures, we would like to discuss what is meant by Laplace Transform and find the transforms of several important time functions and the properties of the Laplace Transforms. Then, we will take up the question of its application, to various network and systems. In the Laplace Transformation, the type of Laplace Transformation that we talk about is what is called

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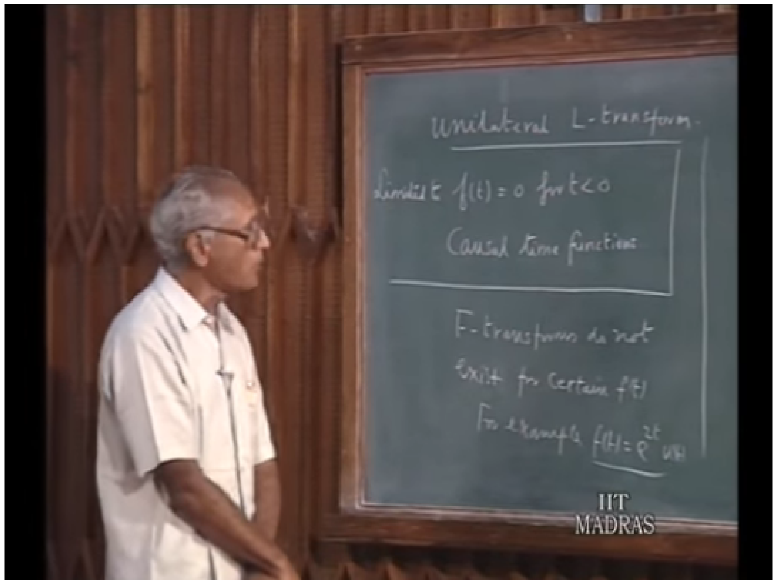

Unilateral Laplace Transform. So this is the type of Laplace Transformation that we are going to talk about. What we mean by that is, we assume that f of t is 0 for t less than 0. So our discussion will be limited to f of t which is 0 for t less than 0 that means: causal time function. If indeed if we have f of t which is fails to be 0 for t less than 0, we simply disregard the value of the function for negative values of time. We take it to be 0 even if it is not originally 0.

So our discussion will be confined to such functions and this is not a great disadvantage because, in transient analysis of networks and systems, some switching takes place at particular point of time and what follows the switching operation what is the important was. And we can always take the switching to take place at t equal to 0. And the past history of circuit, the network and system is summarized in terms of the energy storage, in certain elements for example, in the electrical network in the reactive elements.

So the energy storage in the reactive elements at t equals 0 plus the knowledge of the excitation function, from t equals to 0 onwards for positive values of t. These 2 factors determine the response of the network uniquely for t greater than 0. Therefore, if you know the excitation function only for t greater than 0 and process that; that will not entail any loss of generality because, whatever needed about the past history of the network is summarized by the conditions with the reactive elements.

So this does not lead to any loss of generality, as far as transient performs is concerned. Now the Fourier Transforms of certain time functions is what we have already derived. However, you notice that Fourier Transform does not exist. Fourier Transform does not exist for certain time functions.

For example, if f of t is e to the power of 2t u t then, the Fourier Transform of such a function does not exist because, the defining integral for the Fourier Transform is which is e to the power of 2t e to the power of minus j omega t dt; when you integrate from 0 to infinity that integral does not converge. Therefore, this does not exist.

So the Laplace Transformation what it does is; enlarge the type of functions for which Fourier Transform the type of functions which are handled by the Fourier Transform that means: Laplace Transform enlarge the type of function for which, we can find out the Fourier Transforms that means: certain functions which are not for which Fourier Transforms do not exist, let themselves to Laplace Transformation and therefore, enlarge as the class of networks class of functions for which transforms can be found out. How do we do that?

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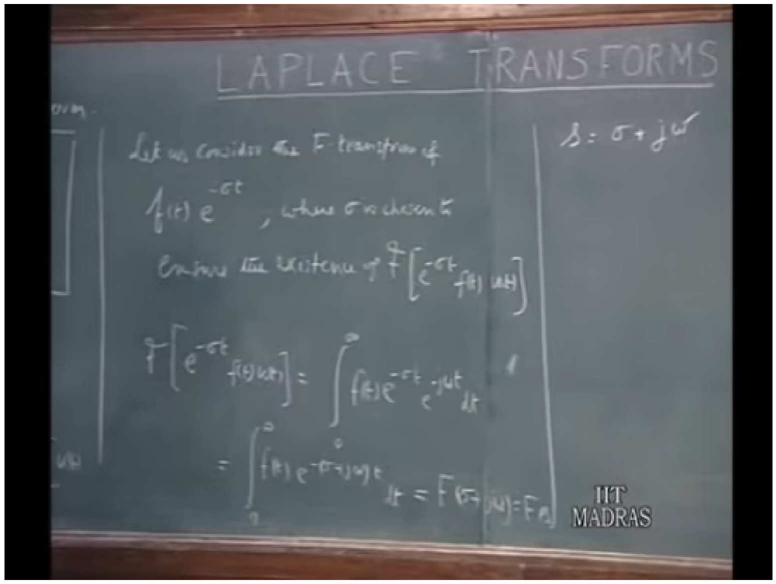

To do that let us consider the Fourier Transform of not f of t but, f of t multiplied by e to the power of minus sigma t. So given a f of t we do not find the Laplace Transform, we will not find the Fourier Transform of that as such. Let us consider f of t e to the power of minus sigma t where, sigma is chosen to ensure the existence of the Fourier Transform of e to the power of minus sigma t f of t.

So, even if f of t does not have a transform, if you multiply f of t by suitable factor e to the power of minus sigma t, it is possible to have a Fourier Transform. For example, if e to the power of 2t is multiplied by e to the power of minus 3t then, it becomes e to the power of minus t then, Fourier Transform exists; so let us see.

So the Fourier Transform of e to the power of minus sigma t f of t by the definition is; Fourier Transform of e to the power of minus sigma t f of t u t because I mentioned, we are assuming this f of t in our Laplace Transformation technique to be those function for which, the value 0 for t less than 0 to make it explicit I am putting f of t u of t.

Therefore, Fourier Transform of f of t u t this is, makes it very clear that this product will have 0 value for negative values of time. This is equal to f of t e to the power of minus sigma t e to the power of minus j omega t dt. And since, you are talking about f of t u t the integrand 0 value for negative values of time.

Therefore, this 0 to infinity instead of minus infinity to plus infinity I am taking 0 to infinity because, f of t u t makes it 0 for negative values of time. So f of t e to the power of minus sigma t e to the power of minus j omega t dt and this will be equal to 0 to infinity of f of t e to the power of minus sigma plus j omega t dt. If the Fourier Transform of f of t u t is f of j omega then, instead of minus j omega t you have minus of sigma plus j omega t.

Therefore, this will be a function f of sigma plus j omega instead of, j omega and this I will call f of s where, s is a complex variable and the dimensions of frequency and it is given by sigma plus j omega. So this f of s now, which is the Fourier Transform of e to the power of sigma t f of t u t is now, expressed is also a Fourier Transform but, instead of being a function of omega we are treating this as a function of s.

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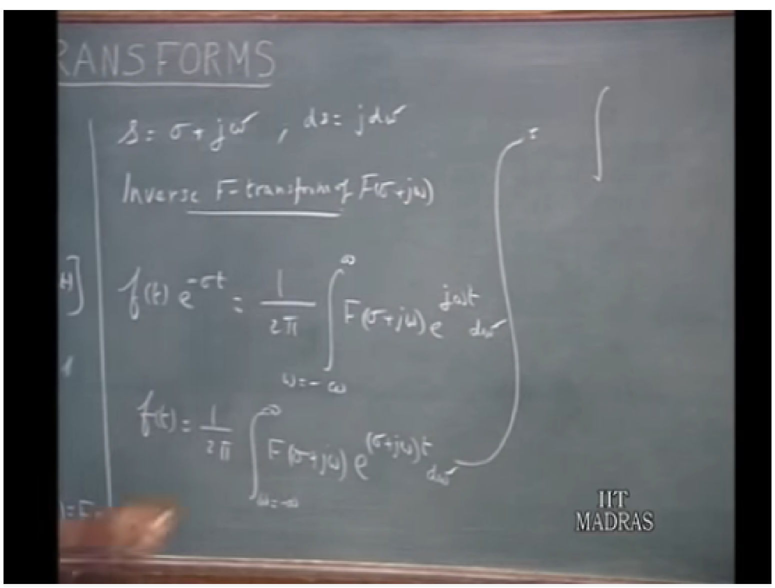

Now, the Inverse Fourier Transform if you want to find out; Inverse Fourier Transform of this f of sigma plus j omega, how do we find this? The Inverse Fourier Transform of this must give us f of t e to the power of minus sigma t. So, f of t e to the power of minus sigma t. That is the Inverse Fourier Transform of this function. So how do we find the Inverse Fourier Transform of usual formula? 1 over 2 pi minus infinity to plus infinity.

Now this is omega of course because, that is the defining Inverse Fourier Transform relation for the integration, is in terms of omega this is f of sigma plus j omega e to the power of j omega t d omega dt. That is the Inverse Fourier Transformation of this f of sigma plus j omega and if you apply the Inverse Fourier Transform you must recover back to your original function f of t e to the power of minus sigma t.

Now from this you can multiply both sides by e to the power of sigma t then, you get this 1 over 2 pi. Now, I am multiplying by this e to the power of sigma t and since, e to the power of sigma t is independent of omega which is the variable of integration, I push inside the integral sign without disturbing any value. So I can write this f of sigma plus j omega e to the power of sigma plus j omega t d omega.

Now that the range of integration is now, is an omega from infinity to plus infinity. But now, I would like to put this in term of the new variable s we have taken. So if s equals to sigma plus j omega and omega is the 1 which is vary, then i can write this as if omega is varying then ds is equal to j d omega because, omega is the variable factor how does vary s vary and omega vary? ds is equal to j d omega. So, I would like to put the entire thing here in terms of s.

So, if I do that then what i would get is the integral now, omega is equal to minus s will be sigma minus j infinity, when omega is equal to plus infinity s will be sigma plus j infinity.

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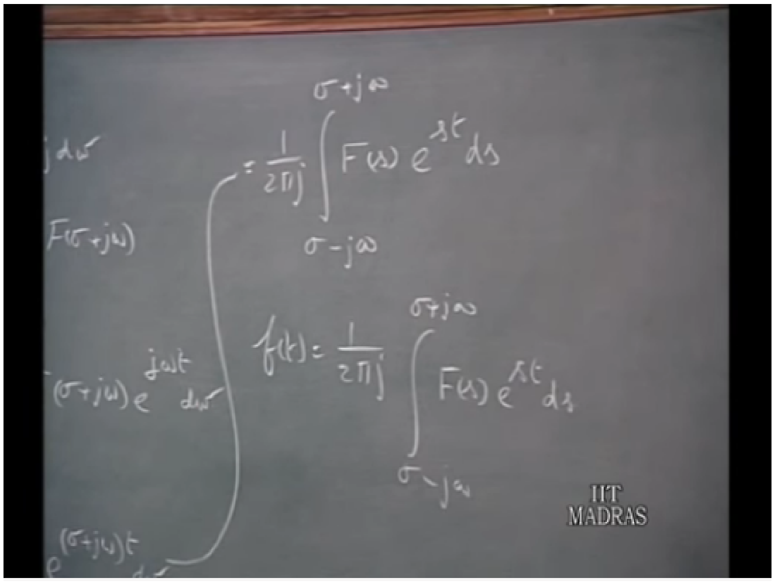

So it will be sigma minus j infinity to sigma plus j infinity this is the variable of integration, f of sigma plus j omega is F of s e to the power of sigma plus j omega t equals e to the power of st and since, d omega equals ds up on j i write this as ds and ds up on j so i write this 1 over 2 pi.

So this means: that we are having f of t as 1 over 2 pi j sigma minus j infinity to sigma plus j infinity of f of s e to the power of s t ds. So we have, let us summarize what we have done so far. We are thinking of finding out the Fourier Transform of f of t but, such function of this type do not let themselves to Fourier Transformation.

So what we can do is, we can try to decrease its growth by multiplying by a function like e to the power of minus sigma t and choosing a suitable value of sigma, we can make sure that this function decreases with increasing values of t such that, the Fourier Transform integral converges.

So, we are associating with f of t a convergence factor e to the power of minus sigma t and this value of sigma is something which depends up on the particular f of t which we choose. Naturally for each value of f of t there is a certain minimum value of sigma which we should have, we will see about that.

So after all borrowed this e to the power of sigma t minus sigma t as a convergence factor, we find the Fourier Transform e to the power of minus sigma t f of t u t because, we are going to talk about functions which are 0 for negative values of time. Therefore, the Fourier Transform integration instead of starting from minus infinity, we can start from 0 itself because; the value of the integrand will be 0 for negative values of time.

So, consequently the Fourier Transform of this will be f of sigma plus j omega where, instead of j omega we have sigma plus j omega because that is, now the variable which we like to treat as the new variable s. So, we have F of s therefore is; f of t e to the power of minus sigma plus j omega t dt which means: f of t e to the power of minus s t dt.

After having find out this f of s which is f of sigma plus j omega, if you like to get back your original function of time, first of we find the Inverse Fourier Transform which f of t e to the power of minus sigma t and multiply that e to the power of sigma t then, you get f of t which goes like this and finally you end up with this.

So, instead of now always talking in terms of Fourier Transforms by using this convergence factor, we must straightaway talk in terms of transformation with reference to the variable s. We can straightaway say: that given a function f of t you have the transformation which is obtained by multiplying f of t by e to the power of minus s t and integrating from 0 to infinity that will give me f of s.

And once you have got f of s, we can get f of t in this manner using this inverse transformation. And these 2 relations constitute the 2 central relations as far Laplace Transformation is concerned. So the origin of Laplace Transformation of as an offshoot of the Fourier Transformation is what we have discussed now. But let us see afterwards straightaway define the Laplace Transformation relations and then study the various properties.

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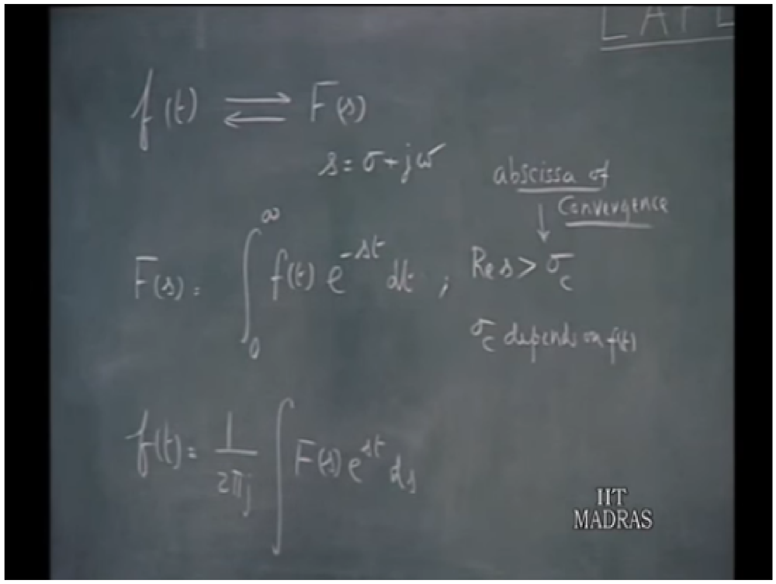

So, given any function f of t, we will indicate its Laplace Transformation as F of s where, s is the complex frequency variable which is real part sigma and imaginary part omega. And so we indicate the a function of time and Laplace Transform pair in this manner. F of s is obtained from a given f of t by this defining integral 0 to infinity of f of t e to the power of minus st dt and this is called the Laplace Transform integral.

The Inverse Laplace Transform is obtained from the Laplace Transformation F of s by this relation 1 over 2 pi j F of s e to the power of st ds. Now, the limits of this integration I will explain in a moment. Now for this integral to exist as I said, there must be a convergent factor e to the power of minus sigma t is the convergence factor which is build into the Laplace Transformation.

So, depending up on the type of function that we are considering f of t there is a certain minimum value of the real part of s that we like to have. So the real part of s here for this integral to exists must be larger than a certain value sigma c which, depends up on the function f of t; sigma c depends on f of t and this sigma c is called abscissa of convergence.

So when deriving the Laplace Transformations of various function, we just briefly have a look at the abscissa convergence that is required, we assume that the real part of s is greater than this. Fortunately, we do not have to keep track of the abscissa convergence in the work related with Laplace Transformations because, we always assume that the s value that we are using, as a real part which exists the abscissa convergence for the particular function or set of functions we are dealing with.

However, when you want to substitute a particular numerical value of s in certain cases then, you have to pay regard the abscissa convergence and make sure that, the numerical value of s that we want put into the expression, it satisfies that its minimum part is greater than sigma c. Normally, we do not want to bother about the abscissa convergence in our routine work.

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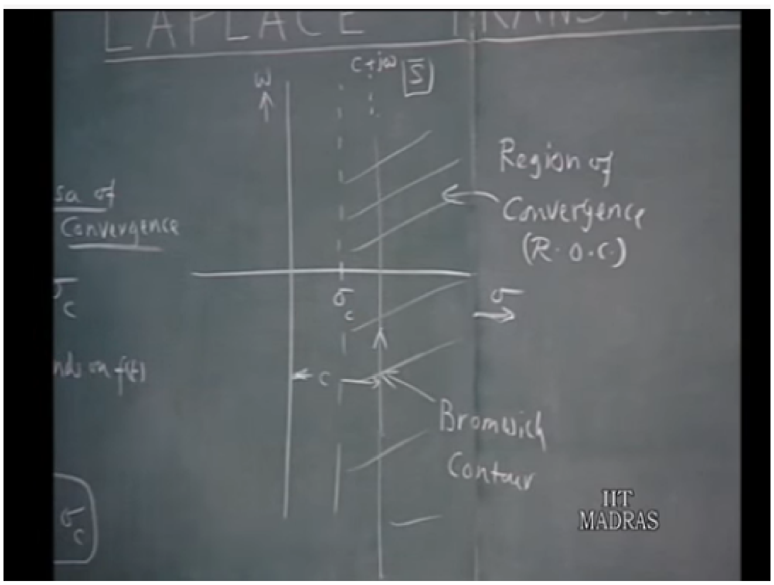

Then, so in this transformation that means if you take this is your complex plane; s plane sigma and omega. So, there is a certain abscissa convergence sigma c. That sigma t depends up on the particular function that we are dealing with as I mention. And in this integration that we are having, the Inverse Laplace Transform integration we are you recall that, when we derive this from the Fourier Transform theory, we said c sigma minus j omega j infinity to sigma plus j infinity.

So, instead of that I simply put c minus j infinity c plus j infinity where, c is the value which is like this. So, we do this integration from c minus j infinity to c plus j infinity where, c is the real part of s. So, instead of sigma I am using value of c just for convenience sake. So, c minus j infinity to c plus j infinity is the contour of integration.

So, we are taking starting from c minus j infinity and integrating up to c plus j omega in this direction. This is what is called Bromwich contour. In literature this is called Bromwich contour and so we are integrating this F of s e to the power of st along a vertical line in the complex frequency plane along which, Bromwich contour c minus j infinity to c plus j infinity where, the value of c is greater than sigma c.

So, what we have therefore is the sigma c defines the region of convergence. This is the region of convergence of the Laplace Transformation integrals of abbreviated as R O C. So, the Laplace Transformation exists provided the value of s is the region of convergence that means: the real part of s must be larger than sigma c which is the abscissa convergence.

So real part of s which is given by sigma, is must be greater than sigma c. As far as the integration in the Inverse Laplace Transformation is concerned, we take a vertical line in the region of convergence that means; the real part of s could be any general value of c but that c should be larger than the abscissa convergence sigma c. So this is the, these 2 are the fundamental relations relating to Laplace Transformation.

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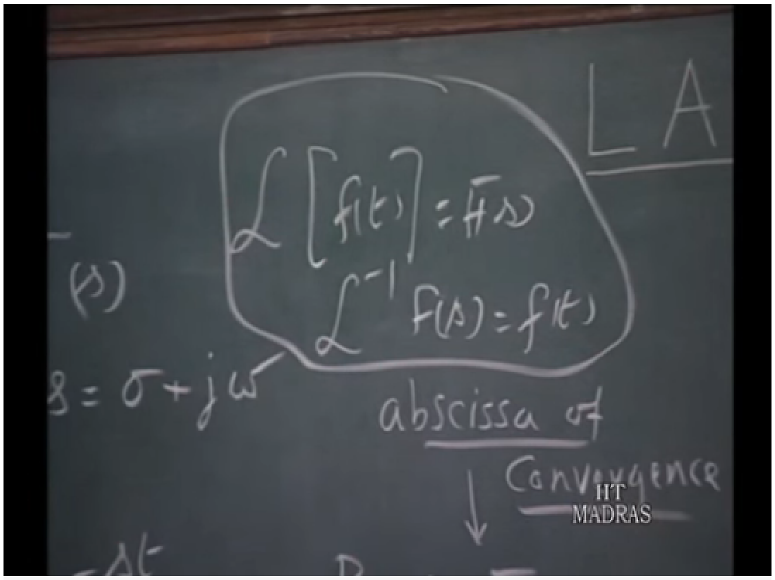

We can also abbreviate this as Laplace Transform of f of t we can write this F of s and we can write Inverse Laplace Transform of F of s equal to f of t. This is the alternate way of writing the forward transformation and transformation in the reverse direction.

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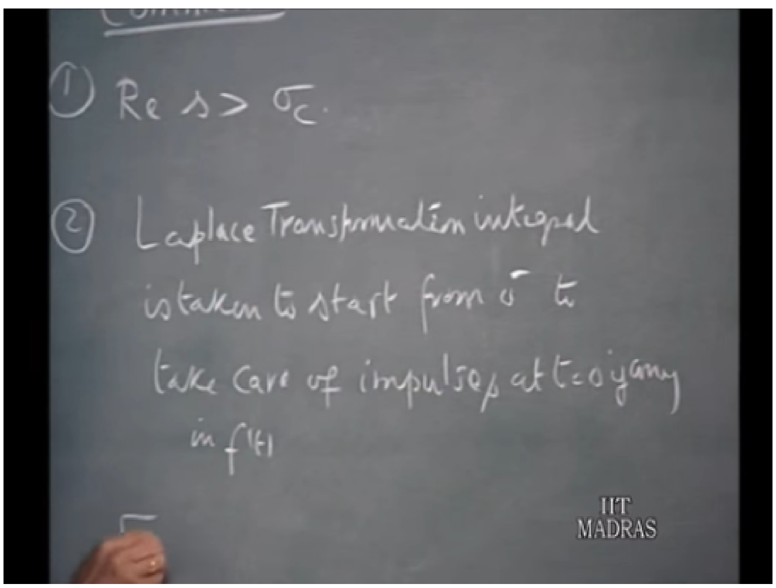

So these, then is the general relation that we need to we have keep in mind and therefore we proceed further, let me make a few commons. One we will first say that the real part of s should be the abscissa convergence and this must be kept note of whenever, you want to substitute numerical values of s as I mentioned earlier. Normally, we do not want to keep track of sigma c in our usual routine work.

Secondly, if f of t has the impulses at the origin then, when you integrate from 0 to infinity the impulses are sitting right at the origin. So, to take of impulses which are present in the origin, we need to integrate through the impulse. Therefore, we must start the integration 0 minus. So, the Laplace Transformation integral is taken, to start from 0 minus, to take care of impulses at the origin, if any in f of t.

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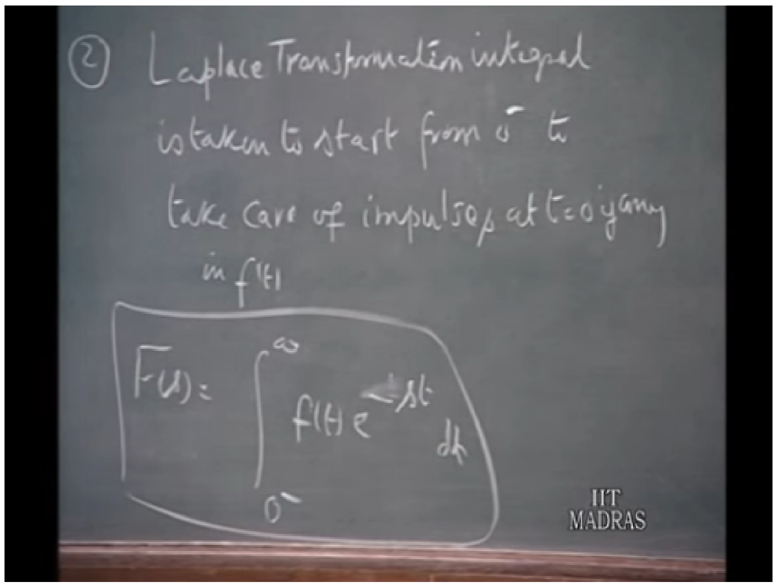

So, F of s 0 minus to infinity f of t e to the power of minus s t dt. If f of t does not have any impulses at the origin, it does not matter whether you take it from 0 or 0 plus or 0 minus. But, if f of t has impulses at the origin and if you want to include the impulses in the origin your transformation, you must perforce start the integration from 0 minus. So normally, when we define the Laplace Transformation integration, we take it starting from 0 minus to infinity you take into account these impulses also.

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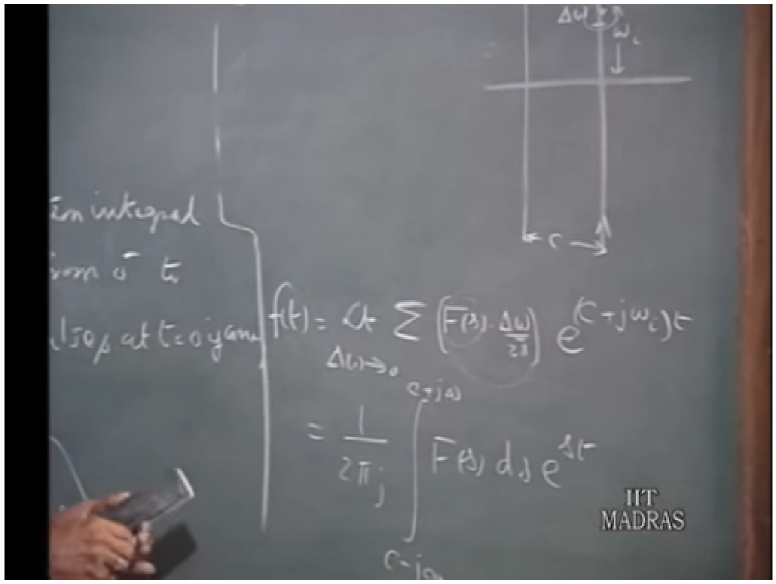

A third point which we like to notice: that in the complex frequency plane this is the Bromwich contour and we are taking the integration along this and so as you move along this line, you are incrementing omega. So you can say, you can divide this entire contour into small intervals of width delta omega and suppose, you have the centre point is omega i then, we can think of f of t as composed of elementary exponential signals of the form; e to the power of sigma.

Suppose, this is c e to the power of c delta omega i t e to the power of c omega i t, this is the exponential signal but the coefficient density given by F of s times delta omega over 2 pi and we take the summation of all such signals. In other words we take this as limit as delta omega tends to zero.

Let me rewrite this more clearly: f of t can be a thought of as the summation delta omega goes to 0 of number of elementary signals; exponential signals of the form e to the power of c plus j omega i t. This is the exponential signal sitting at this point in the complex frequency plane. And its coefficient is F of s delta omega by 2 pi.

This is the coefficient density; we can treat this as the coefficient density just as, we are treating in the case of Fourier Transform. Now in the case of Laplace Transform the coefficient density is F of s because ,the complex frequency signal is e to the power of st, rather than e to the power of j omega a t and the density is defined as; so much coefficients per cycle per second.

Therefore, delta omega pi 2 pi what we have taken. And if you take this limit then, this becomes a integral, so instead of delta omega we are putting delta s. Therefore, this can be taken as 1 over 2 pi j integral F of s ds pi 2 pi j, ds by j becomes d omega and then e to the power of c plus j omega i t is the running variable s. Therefore, e to the power of st and we take the limit from c minus j infinity to c plus j infinity.

So, this is the defining relation Inverse Fourier Transform relation. So, even here just as the case of Fourier Transforms, we can think of f of t as composed of number of exponential signals of this value where, omega i runs from minus infinity to plus infinity along with Bromwich contour and at each particular frequency, spot frequency there is certain coefficient density and the coefficient density is given by F of s multiplied by coefficient density is F of s.

And so the coefficient of this exponential signal which is concentrated in the small elemental width can be thought of, F of s times delta over 2 pi. This is the coefficient density multiplied by the width, this is the coefficient of this particular exponential signal and we take the summation of all such elemental signals along this line, then this becomes this Integral.

So, just as Fourier Transform Fourier Integral split up f of t as the infinite summation of exponential signals e to the power of j omega t type, Laplace Transform also can be thought of as splitting up f of t as a number of elementary signals e to the power of st where, s is the variable along which Bromwich contour and having a coefficient density equals to F of s.

And therefore the particular coefficient of e to the power of st would be F of s ds over 2 pi j that is what we are having. This is the interpretation which would be useful later when, we talk about the system function h of s just we talked about the system function h of j omega when, we are dealing Fourier Transform theory.

So, this is generally the; what we need to know about the introduction to the concept of Laplace Transformation defining Laplace Transform relation and the inverse transform relation. Now, we will take up the question of Laplace Transforms of various important time signal f of t and find out the abscissa convergence of each of these, in a routine fashion.

We will do that, we take it up next after having introduce ourselves the concept of Laplace Transformation. To start with, let us note the condition for the existence of Laplace Transform

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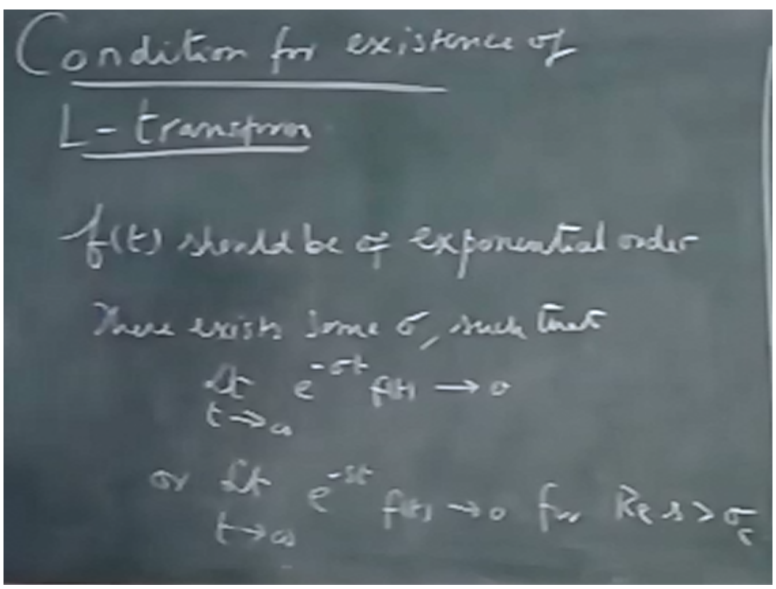

This condition is usually stated as: f of t should be of exponential order. In other words, f of t cannot grow with positive t more than, an exponent of some value or in other words, there exist some sigma real value such that, limit as t goes to infinity of e to the power of minus sigma t f of t goes to 0. So, there must be some real value sigma such that, as t goes to infinity e to the power of minus sigma t pulls down the value of f of t to 0, to negligible proportion as t goes to infinity.

For example, if f of t e to the power of 2t e to the power of minus 3t makes it go down to 0. So, depending on f of t you can choose the values of sigma and the sigma should be larger than the abscissa convergence as we have seen or we can put this as: limit as t tends to infinity of e to the power of minus st f t goes to 0 for some for real value of sigma real value of s some sigma c; so that is what we are having.

This is the abscissa convergence. So, e to the power of minus st times f of t as you put t tends to infinity must go down to 0. It becomes negligibly small. So, the value of real part of s which must be satisfy this condition; sigma c which is the abscissa convergence which, depends up on the particular function which f of t that we have on hand. So this is, in other words, to put this in a very compact fashion we say f of t should be exponential order

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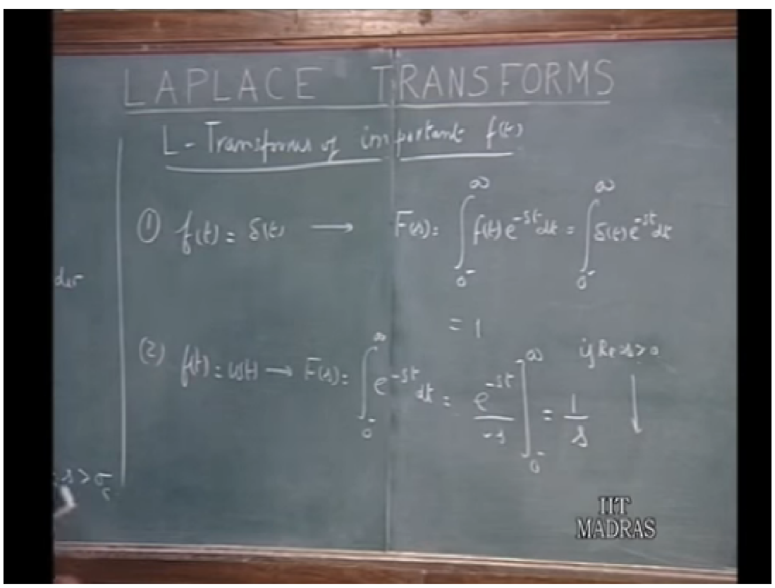

Now, we will take up Laplace Transformation of important time function; important functions of time. Let us start with, let f of t be delta t and impulse at the origin. So F of s equals 0 minus to infinity of f of t e to the power of minus st dt this is the defining integral for the Laplace Transformation. And in our particular case 0 minus to infinity f of t is delta t e to the power of minus st dt.

And what we have any delta e to the power of minus st dt the characteristic of delta e to the power of st that means: this is equivalent to delta t times the value of this function at the value of s equal to 0 that is 1. Therefore, in other words we are integrating delta t dt over the interval 0 minus to infinity which includes t equals to 0. Therefore, the value of this is equal to 1. So, we have delta t has the Laplace Transformation equal to 1, just like in the case of Fourier Transform also delta t equal to 1. This is also the same case F of s equal to 1.

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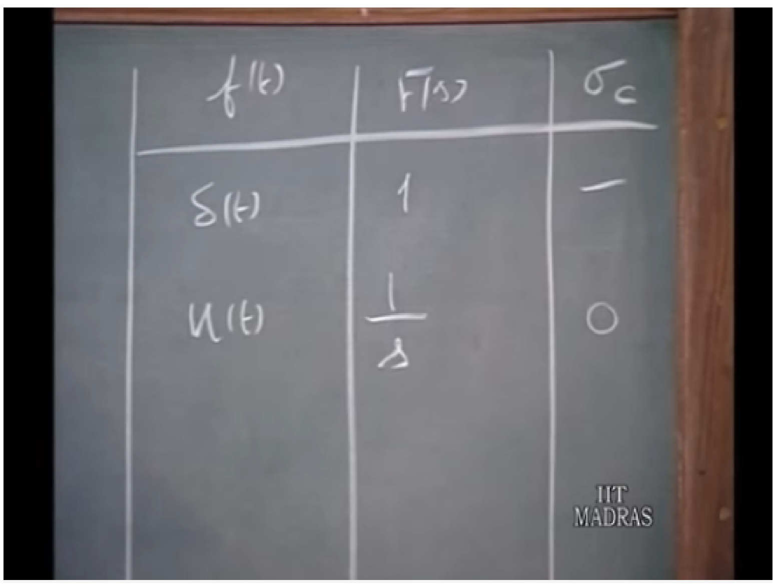

So, I will write here a list of f of t and the corresponding F of s and the corresponding abscissa convergence sigma c. So, as and then we derive the Laplace Transformation we enter them here. So, delta t the Laplace Transform is F of s. As far the abscissa convergence is concerned, it does not matter what value of s, what is the real part of s, it will always be 1. It does not depend up on this and as well put this as nothing. We do not have any special particular restriction on the real value of s.

Now, let us take f of t as u t; unit step function. Then, we find the Laplace Transform F of s as 0 minus to infinity and in this range of integration u of t happens to be equal to 1. Therefore, I can write u of t e to the power of minus st dt and since, u of t is equal to 1, I may as well drop that and write e to the power of minus st dt because, u of t in this range of integration is equal to 1.

Therefore, this will be e to the power of minus st divided by minus s 0 to infinity and now, at upper limit is e to the power of minus infinity times t minus s times infinity. So, if real values of s is greater than 0 then, you have say minus a small real value of s times t, therefore this will be e to the power of minus sigma plus j omega t where, sigma is the real part of s.

So, as long as sigma is positive; real part of s is greater than 0 that means, sigma is positive when t goes to infinity the magnitude this which is governed by e to the power of minus sigma t. After all this is equal to e to the power of minus sigma t times e to the power of minus j omega t.

The magnitude of this is 1 irrespective the value of t but, the magnitude of this depends up on sigma and t as long as sigma is positive and t goes to infinity this goes to 0. Therefore, at the upper limit we make sure that this goes to 0, by taking the real part of s to be greater than 0. Therefore that is we have assumed and the so at upper limit this is 0 and lower limit when t equal to 0 this is equal to 1. Therefore, this becomes 1 over s.

And therefore, F of s the Laplace Transform of f of t which is ut will be 1 over s provided; we take the real part of s to be greater than 0. And that means, the integral will converge only if you take that particular condition and that means: the abscissa convergence for this sigma c happens to be 0, which is the real part of s; minimum part of real part of s that we should have.

So, we have the relation now that u of t 1 over s and this Laplace Transformation is valid as long as real part of s is greater than 0 that means: sigma c is 0 that means, the real part of s must be some positive value which, is larger than 0 of course. So, these are the 2 important functions for which we have found out the Laplace Transformation. Let us move on.