Networks and Systems
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Lecture-42
Integration in Time Domain

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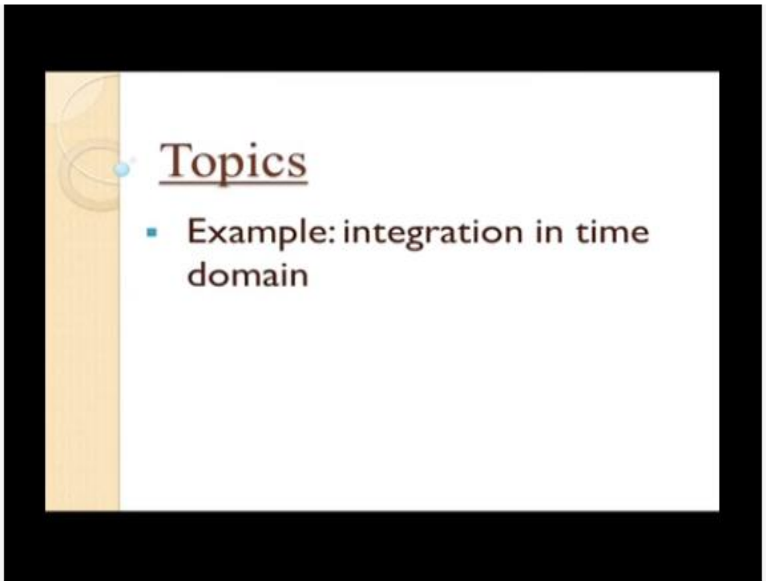

In the lecture, we saw how a complicated integral operation like the convolution in time domain is simplified in the transform domain by the pure multiplication of the pertinent transforms. Let me illustrate this, first with an example, before we go on to discussion another properties.

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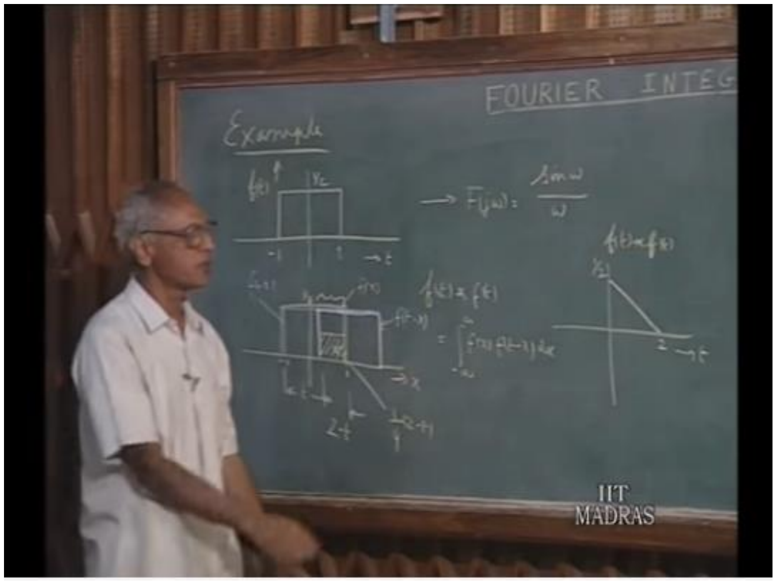

Let us consider, here f of t which is a pulse of height half and lasting from minus 1 second to plus 1 second, I will call this f of t it is Fourier Transform f of j omega in our earlier discussion, will be, if the amplitude is a and the pulse duration is d ad that is: half times 2 1 sin omega d up on 2.

So, if is 2 d up on 2 is 1 therefore, sin omega divided by omega d up on 2 it is omega. So in this standard terminology, if we had a height a and duration d it would be a d sin omega d up on 2 by d omega d up on 2 and ad happens to be 1 and d happens to be 2. This is the Fourier Transform of this pulse.

Now what I would like to know is: what is the Fourier Transform of f of t star f of t that is: f of t convolve with itself. So, if I want to find out the f of t convolve by with itself then first of all I write this is; let us say, f of x as the function of x. So, in order to find f t convolved with f t I should take the integral of f of x times t minus x dx that is: the integral that, I should workout.

So, this is f of x this also happens to be f of minus x that means; if I take another curve like this, this I can consider to be f of minus x, but I have to multiply f of x by f of t minus x that means; the minus f of x curve I must push forward by an amount equal to t.

So that would be obtained by let us say, this curve which I would like to say is f of t minus x.

So, originally this was minus 1 this is 1 and I push this forward by an amount t therefore, this intervals equals to t. So, the overlap interval between f of x and f of t minus x is this is 2 this is t therefore, this is 2 minus t so that is the overlap interval between these 2. And when you multiply these 2 curves each of amplitude half you get the area under the curve after all this is the area under this curve and this is the area that, you get amplitude is the height is 1 fourth and the duration is 2 minus t.

Therefore, the area of this will be 1 fourth times 2 minus t that would be that, would be the value as you push forward the curve in the positive direction. If you push it on the negative direction you get a similar expression therefore, the variation of f t convolve with f t would be for positive t.

When t equal to 0 we have to take these 2 the f of minus x and f of x multiplied together that, will be 1 fourth lasting for 2 seconds therefore, it will be half but, then as the t progress this reduces in this manner. So, it starts with half and become 0 when t equals to 2.

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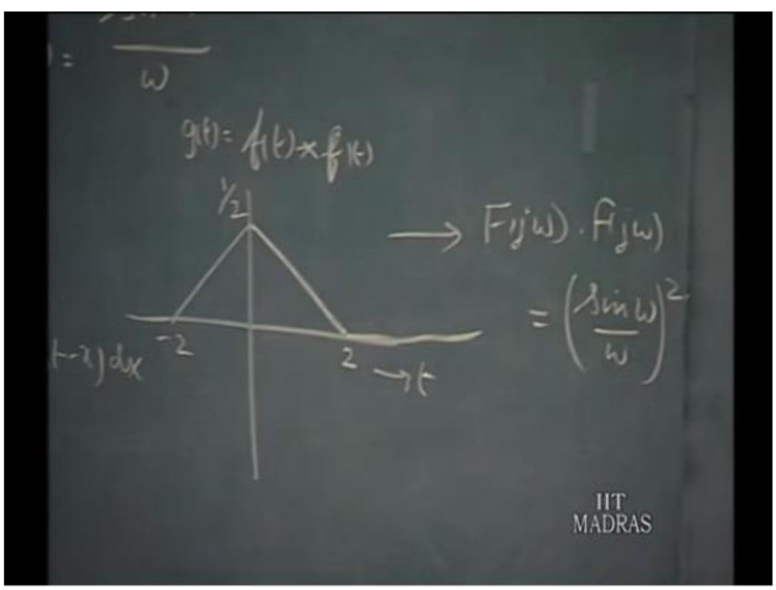

And beyond t equals to 2 this white curves comes away from this therefore, there will be no overlap area therefore, it become 0 from that point onwards and if you proceed in the negative direction you get a symmetrical result it can be easily seen. So f t convolved with f of t will be having, if you call that, g of t this will be the variation of this.

So, to find the Fourier Transform for this, you do not want to spend any additional effort because we know the Fourier Transform of f of t so g of j omega. The Fourier Transform of this is obtained by multiplying f of j omega multiplied by, f j omega which is indeed sin omega by omega whole squared.

So, you can find the Fourier Transform of this; as the Fourier Transform of the convolution of a rectangular pulse by itself. So, you can independently calculate the Fourier Transform of this and we can show that this is equal to this.

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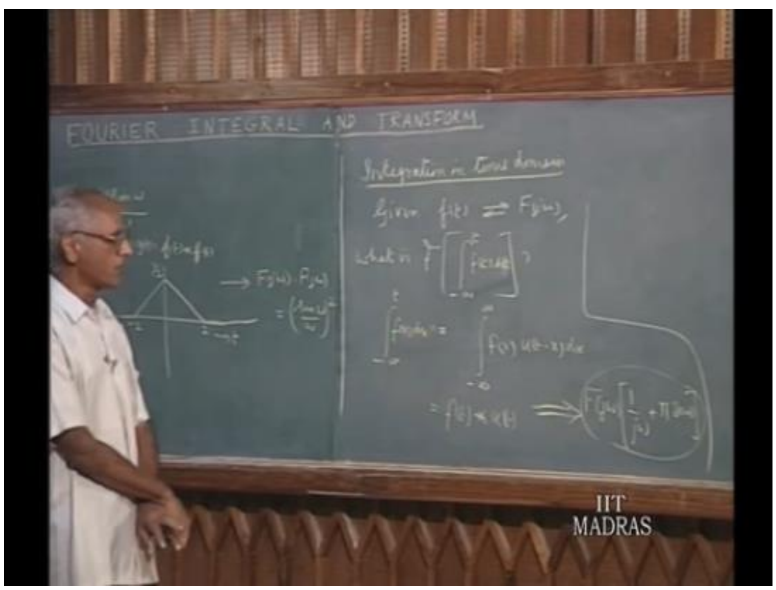

We next study the rule corresponding to integration in time domain, if you recollect that, we said when a function is differentiated in the time domain, in the transform domain it gets multiplied by j omega. And that time I observed that, if you want to integrate a function in the time domain a transform domain it is not nearly division by j omega it may end at something else

Let us see, how that works out now, what we particular, we like to workout is given f of t and f of j omega has the transform pair. What is the Fourier Transform of integral of f of t dt from minus infinity to plus infinity minus infinity to t. So, this is the function gt which is obtained by integrating f of t from minus infinity to t you can see that; minus infinity to t of f of t dt can be written as minus infinity to infinity of f of x.

I can write this also as f of x no problem this can be written as after all this is dummy variable, we can write this as f of x dx. We can write this as f of x times u of t minus x dx because what happens when x goes beyond t when x is more than t t minus is the negative quantity therefore, this become 0 u of t minus is 0 for x greater than t.

Therefore, in this range of integration whenever, x exceeds t it is the integral is 0 therefore, instead of confining our integration from minus infinity to t, we are normally taking the upper limit of integration plus infinity. But we know the value of this integral integrand from x equal to t from x equals to infinity, it is going to 0 so these 2 are equivalent that means; to this, I am adding something which is equal to 0.

So, both are equal and you can see that this is: immediately equal to f of t convolve with u of t after all that is: the formula for a convolution 2 time functions f of t convolve with u of t will give me this. Therefore we observe that, integral of ft from minus infinity to t is equivalent to convolving f of t with u of t and we know from the convolution rule that the Fourier Transform of this is obtained by multiplication of these 2 Fourier Transforms.

The Fourier Transform of this will be f of j omega 1 over j omega pi delta omega that is being the Fourier Transform of unit step function. So, the Fourier Transform of this is given by this you observe now, that f of j omega is not merely divided by j omega but, you have an additional component f of j omega times pi delta omega.

So this is the important result that, we have when you want to integrate a function of time from minus infinity to infinity you have to multiply by 1 over j omega plus pi delta omega.

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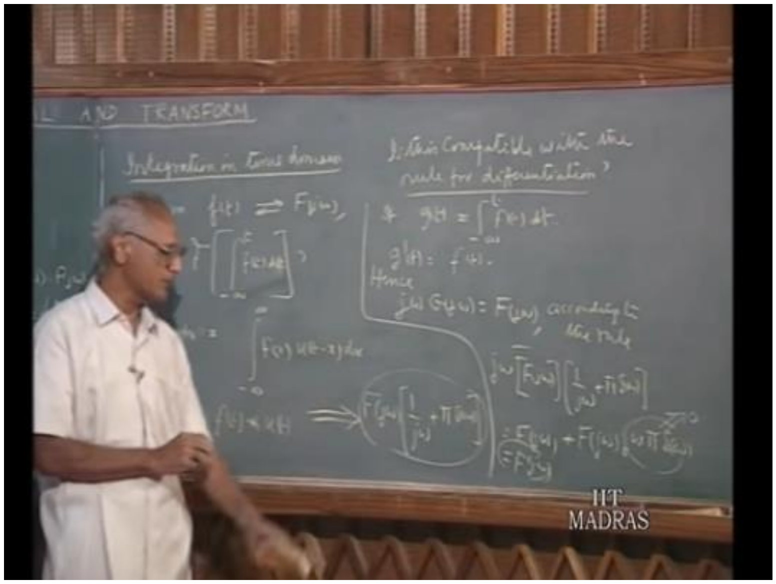

Let us see, the implications of this is this compatible with the rule for differentiation, which we have already talked about that; question we like to settle first. What we are saying is if g of j omega is from minus infinity to t I am sorry g of t suppose, I write g of t I define this as g of t f of t dt then we know the, derivative of g t with respect to t is f of t.

Therefore, if the Fourier Transform of g of t is g of j omega then j omega is multiplied by that; must be the Fourier Transform of f of t hence, j omega times g of j omega must be equal to f of j omega according to the derivative rule according to the rule for differentiation. Now, it is indeed so, we know the g of j omega has got the Fourier Transform therefore.

Let us illustrate this; J omega is multiplied by f of j omega multiplied by 1 over j omega plus pi delta omega, if you take that the product of f of j omega 1 over j omega and 1 over j omega gives me f of j omega plus I have additional term which is f of j omega j omega pi delta omega.

Now what we would expect is this should result in f of j omega but, we have an additional terms but, we observe that omega times delta omega is 0 because any function delta omega multiplying delta omega will be equal to that function evaluated at omega is equal to 0 multiplied by delta omega.

Therefore since, this quantity goes to 0 this is indeed to f of j omega therefore, there is no incompatibility this is quite alright, if it is consisted with the rule for differentiation that we have already derived. So, in the summary the integration in time domain from minus infinity to t of f of t dt is translated into the transform domain as multiplication of f of j omega by 1 over j omega plus pi delta omega.

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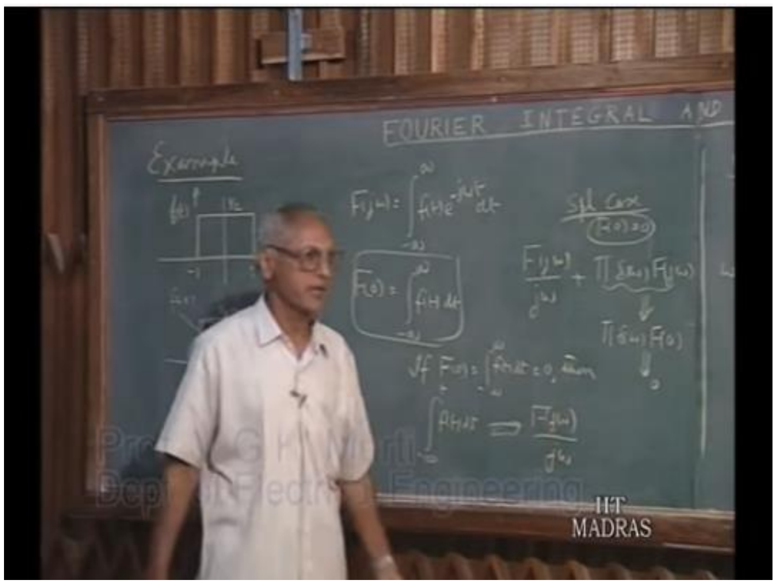

However, we observe that, for a special case, if this g of j omega which is the Fourier Transform of this quantity is f of j omega divided by j omega plus pi delta omega f of j omega. So special case of f 0 is being equal to 0 suppose, we have a special case where Fourier Transform evaluated at omega is equal to 0 is equal to 0 then this will be 0 because this is equal to pi delta omega f of 0 and in this case it equal to 0.

Therefore f of 0, if it is equal to 0 then integration will result in f of j omega you do not have the extra term and when does f of 0 become 0 then you recall once again F of j omega equals f of t e to the power of minus j omega t dt minus infinity to plus infinity. So, f of 0 minus infinity to plus infinity f of t and omega equals 0 this is 1 f t dt.

So, the summary of this is. If f of 0 which is equal to minus infinity to plus infinity of f t dt is 0 then the Fourier Transform of f t dt is simply f j omega over j omega the second term drops out. So, summarize in general whenever, you have an integral minus infinity to t f t dt it is Fourier Transform is obtained by multiplying f of j omega by 1 over j omega plus pi delta omega on the other hand for the special case where the integral of f t dt t from minus infinity to plus infinity is 0 which is f 0.

If that happens to be 0 then the Fourier Transform of this integral will be multiplied by f will be obtained by multiplying f of j omega by 1 over j omega.

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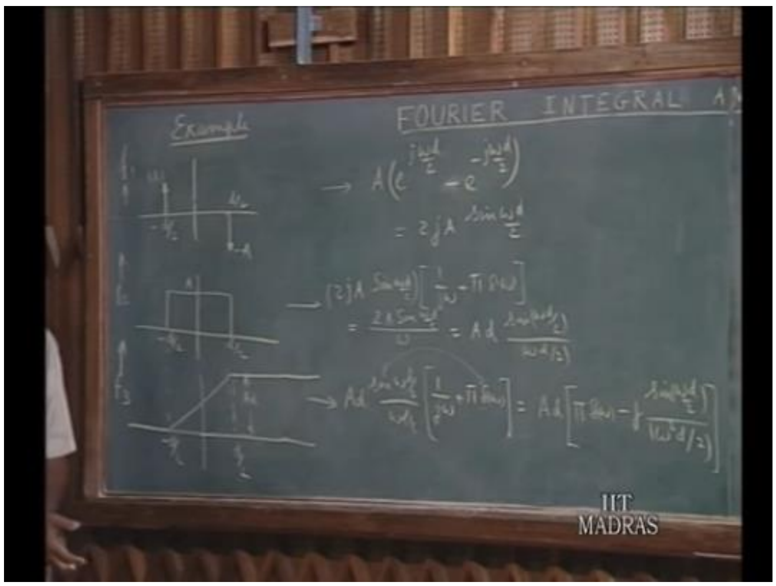

Let us work out an example, illustrating the integration. Let us, consider an example to illustrate this, I will start with a f1 of t which consists of a pair of impulses of strength a and minus a at minus d up on 2 and plus d up on 2 this is f1. I obtain from f1 a function f2, which is the integral of f1 from minus infinity to t as, I integrate from minus infinity to t become 0 area.

At this point I have a jump and the jump is equal to a and that continues up to this point and beyond that, you have another jump of minus a because you are integrating with an impulse and therefore, this becomes 0. So, this will be f2 is obtained by integrating f1 from minus infinity to t therefore, what we have is this kind of f2 derivative of f2 of course, f1.

Now, if I integrate this once more I get a function f3 which is minus d up on up to minus d up on 2 this is 0 and then gradually increases. Because you are picking more and more area under this pulse therefore, it increases like this up to d up on 2 and beyond that it; flat because there is no further area is being added to this curve that will be the nature of f3. And the total area picked up from this point to this point is a times d base area therefore, this is equal to ad.

Now, our interest is to find out the Fourier Transform of such a curve like this f3 now, we can put the problem in this way suppose, you want to find the Fourier Transform of such a curve like this then you know that this is the integral of this which in turn the integral of this. So, if you can find the Fourier Transform of that. By integration rule you can find the Fourier Transform of this and by the integration rule again find.

The Fourier Transform of this, this is what we want to do our motivation for doing this is the Fourier Transform for this is very simple because, it consists of pair of impulses we know the Fourier Transform of this is a 2 pi a the strength is I am sorry the impulse of strength is therefore, e to the power of j omega d up on 2 representing this pulse and this is; minus a e to the power of j minus omega d by 2. Because it is shifted in time by d up on 2 seconds and this is negative therefore, minus omega

So, this is the Fourier Transform of this corresponding to this impulse of strength a you have a Fourier Transform of a since this is advanced in time you have to multiply j e to the power of minus j omega d up on 2. Since. this is delayed in time you have to multiply e to the power of minus j omega d up on 2 and since, the negative impulse you have minus sign here and this of course, equal to 2 a sin omega d up on 2.

Now when you come to this, this curve is obtained by integrating this therefore, you have to multiply this by 2 j a sin omega d up on 2 multiplied by 1 over j omega times pi delta omega, by the integration rule that, we have just now observed and here is a case where the integral from minus infinity to plus infinity of f 1 t is 0 in other words f 0 the value of this evaluated at omega equals to 0 is therefore, this term drops out.

We can also verify this from substituting omega equal to 0 this quantity is 0 that multiplied by pi delta omega leads to 0 therefore, you get contribution only from the product of this term and 1 over j omega. Therefore, that will be 2 a sin omega d up on 2 by omega 2 a sin omega d up on 2 by omega, which I can write this further as ad sin omega d up on 2 divided by omega d up on 2 that is: the form which we are familiar just wanted to put it in that forms.

So, that once we have this pulse function we know immediately it is Fourier Transform of this of course, the same thing is obtained through finding out the Fourier Transform at the derivative to start with. Now, the Fourier Transform of this is once again obtained by multiplying a d sin omega d up on 2 divided by mega d up 2 by 1 over omega plus pi delta omega and now, f of 0 of f2 of 0 capital f2 of 0 that means Fourier Transform of this evaluated at omega equal to 0 is not 0.

Therefore, the second term of also has it place, it is not going to vanish therefore, if you write this, we can have first of all the product of these 2 delta omega multiplied by a certain function is equal to the function evaluated omega equals to 0 multiplied by delta omega equals 0 this becomes ad times delta omega.

Because sin omega d up on 2 by d omega up on 2 evaluated at omega equal to 0 equals 1 and then you have 1 over j put this as minus j ad is already there sin omega d up on 2 divided by omega because this omega is there another d omega up on 2 there omega squared d up on 2. So, that will be the Fourier Transform of this so, this example illustrates how 1 can repeatedly differentiate a function or integrate a function.

And then get 1 transform of the other using the rule for differentiation and integration of the case may be earlier we had workout some examples, where the function was continuously differentiated. Now, we have taken a reverse order we started with a simple function whose Fourier Transform, can be easily be obtained and integrate this at 2 successive steps and therefore, from the Fourier Transform of this we are able to find the Fourier Transform of this.