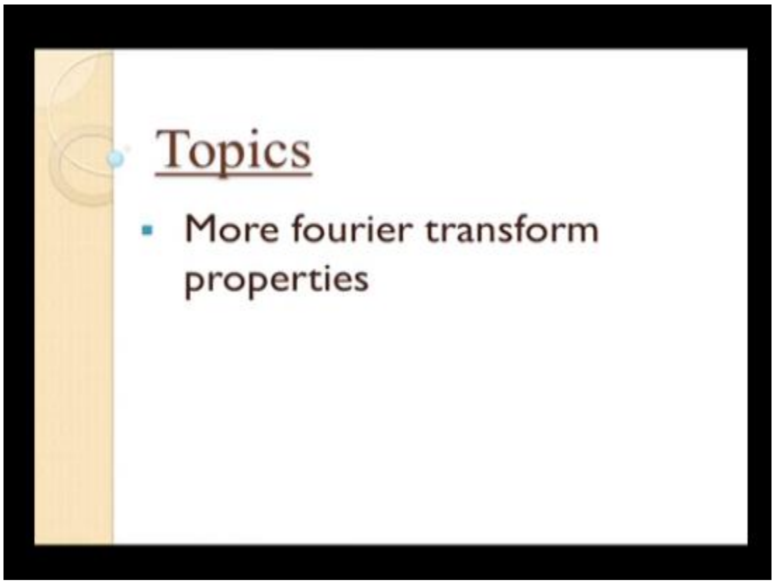
Networks and Systems
Prof. V.G.K. Murti
Department of Electrical Engineering
Indian Institute of Technology – Madras

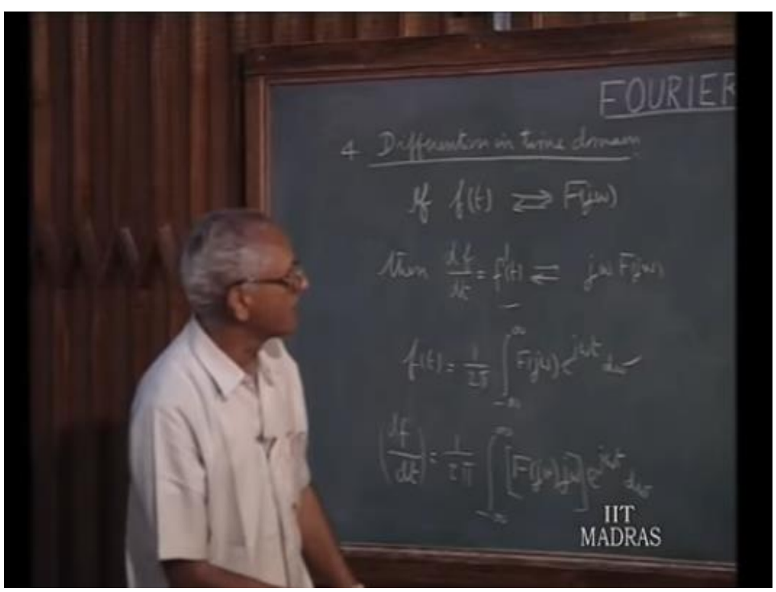
Lecture-35
More Fourier Transform Properties

(Refer Slide Time: 00:11)



Let us now enquire into what happens when f(t) is differentiated in time domain? What is it effect in the transform domain?

(Refer Slide Time: 00:32)



So, we will say differentiation in time in time domain. What is it effect in the transform domain? If f(t) and f of j omega form a transform pair then, this particular property tells us df by dt which may like to call f prime t has its Fourier transform which is obtained from the original Fourier transform by multiplying by j omega.

So, the differentiation time domain corresponds to multiplication in transform domain by j omega and this is 1 of the very useful prosperities of a transform of a particular time function. Because, the calculus that is involved in the differentiation is replaced by an algebra algebraic multiplication j omega and this is a trend which we noticed as we go along not only in Fourier transform but Laplace transform also.

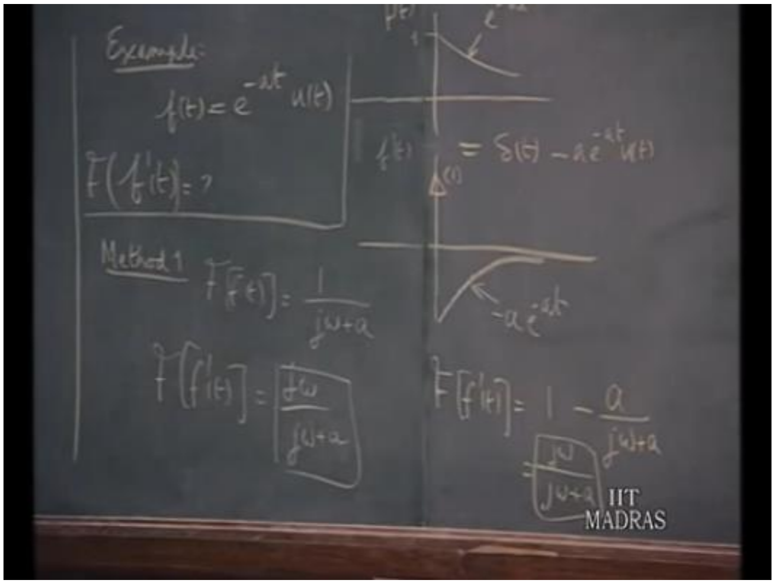
Operation involving in calculus integration and differentiation or converted into multiplication or division by j omega and makes for the great convenience as far as the working is concerned. Will talk more about later but let us see why this is so, f(t) can be obtained from f of j omega to the inverse transformation.

That is f(t) can be written as 1 over 2 pi from minus infinity to plus infinity of f of j omega e to the power of j omega t d omega. Now, let us differentiate this signal, so d f by dt the derivative of this with respect to time df by dt. And assuming that we can differentiate under the integral sign here, if we carry that out here this will become 1 over 2 pi from minus infinity to plus infinity f of j omega j omega e to the power of j omega t d omega.

Because, this is the only function of time inside the integral sign. So, what we have here is a quantity f j omega j omega multiplied by the e to the power of j omega t d omega. So obviously, whatever we are having here is obtained is 1 over 2 pi times minus infinity to plus infinity plus some quantity function of omega into e to the power of j omega t d omega.

This is the defining relationship of a inverse Fourier transform. Therefore, d f by dt must be the inverse Fourier transform of f of j omega j omega, so that is what we have here. If we have j omega f of omega its inverse Fourier transform is df by dt or in other words d f by dt f j omega j omega f j omega is the transform of d f by dt. So, this is the very useful result and to illustrate this let us work out an example once again.

(Refer Slide Time: 04:00)



Let us take a simple signal f t is e to the power of minus a t u t. Let us work this the Fourier transform of f dash t in 2 ways 1 by straight forward method by differentiating this and then trying to find out its Fourier transform. Or alternately finding out the Fourier transform of this and then find out Fourier transforms the differentiated signal.

So what we look to find out is Fourier transform of f prime t is what. So the method 1: Fourier transform of f(t) this is the function which we have been talking about all the time is 1 over j omega plus a. so the Fourier transform of the derivative function f prime t is j omega over j omega plus a. That is all it illustrates, it is quite simple and straight forward. Now, we like to check on this result by actually carrying out the differentiation in time domain and finding out its Fourier transform.

So, let us see what it is. This is the function f(t) this is 1 this is e to the power of minus 1. If you plot the f prime of t its derivative e to the power of minus a t its derivative will be minus a e to the power of minus a t. So, you have minus a e to the power of minus a that is now that is not all, why? The Fourier transform can considers the time function extending from minus infinity to plus infinity therefore; we must take the derivative at every point.

Now, you observe this f(t) all 0 all along and suddenly it jumps to the value 1 therefore at this point it jumps from 0 to 1. Therefore, there is the derivative must be infinitely large and that quantity when integrate at must be raised to 1 therefore obviously what we are having here is the delta of magnitude 1. So, f prime t really is not simply given by this alone but also you must have a delta function.

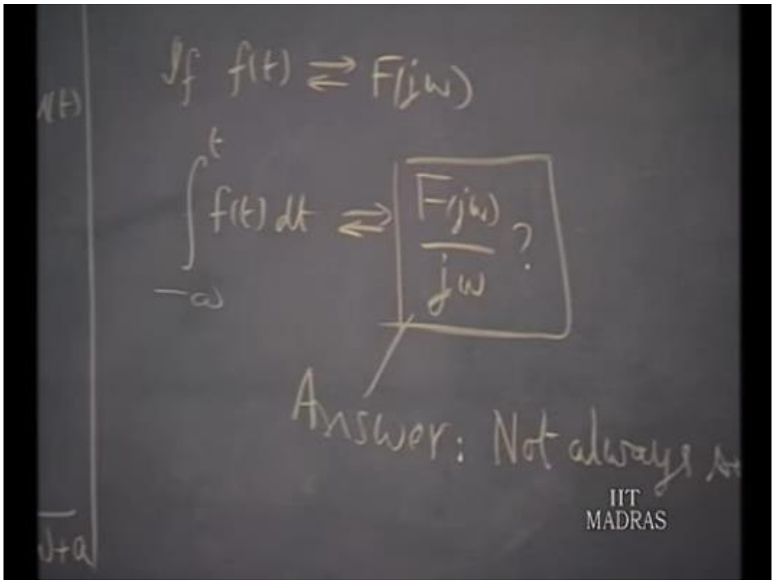
So, in other words f prime t is equal to delta t plus this quantity which is minus a e to the power of minus a t because, this function starts only from t equal to 0 onwards. So, f of j omega for this the Fourier transform of f prime t would be the Fourier transform of this plus the Fourier transform of this. The Fourier transform of delta t is obviously 1 and e to the power of minus a t u t as 1 over j omega plus a except we have 1 more a here and negative sign minus a over j omega plus a.

This is in the j omega over plus a result which is an record with. So, the important factor fact that example brings up is for 1 thing when we are taking the differentiation you must also consider the derivatives associated in the jumps as shown here. And secondly, to find out the Fourier transform of differentiated signal we do not have to carry out the differentiation time domain. If you know the Fourier transform of this signal all you have to do is multiply by j omega.

So this makes as a pointed out for great convenience in manipulation of differentiated signals. Now, I would just introduce a topic we will not carry this out. What 1 might argue is if the differentiation is equivalent to multiplication by j omega in time domain would it mean that integration will be equivalent to division by j omega.

After all this signal is obtained by integrating this signal therefore, if you have some Fourier transform for that. The Fourier transform the original signal must be obtained by dividing j omega. This is true up to a point but this is not quite true. In other words what you like to point out postponed the discussion to later stage.

(Refer Slide Time: 09:02)

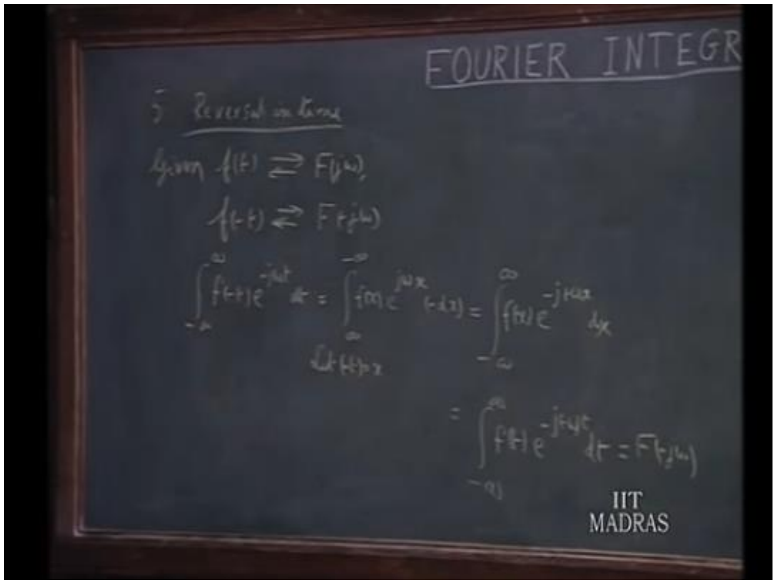


Integration if f(t) has the Fourier transform f j omega then integration then also say minus infinity to t f(t) dt. What its Fourier transform? Is this f j omega by j omega? This is the question which we like to ask, the answer is not always so. So, you must be careful try to integrate the signal in finding out the Fourier transform. We will discuss this at later point of time after we find out the Fourier transform of the step function.

So, we will postponed the discussion of this what I would like to caution you at this stage is you should not jump to the conclusion that because, differentiation involves multiplication in j omega integration means division by j omega in the transform domain that is not so. We have so far seen how certain operations in time domain how they reflected in transform domain for example; we thought we saw how the function of time shifted in time what it affects in the transform domain.

When it differentiated in time what is the effective transformation domain these property we have studied. Let us now continue this trend take up now the case where it is reversed in time.

(Refer Slide Time: 10:39)

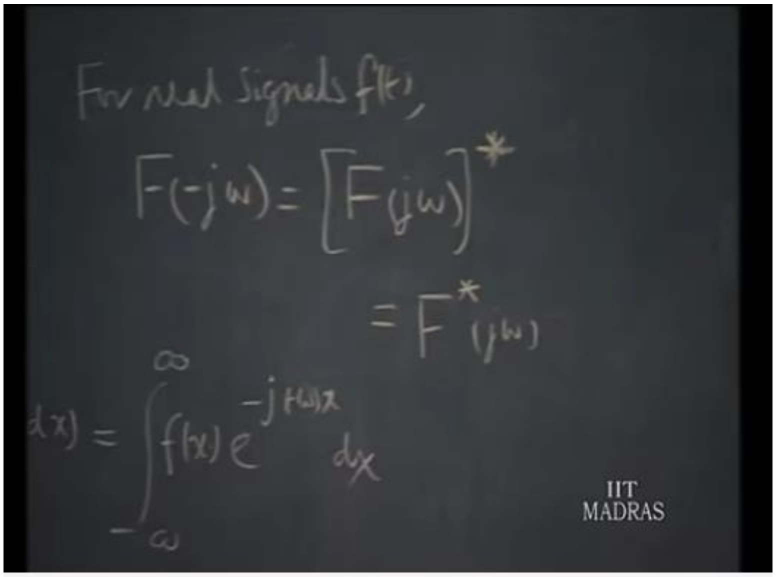


So, if f(t) and f of j omega form a transform pair we want to know what the corresponding f of j omega for f of is minus. In the sequence of event takes place in the reverse direction what is the corresponding f of j omega? The answer simply is f of minus j omega. It is straight forward to show this because after all you have now minus infinity to plus infinity of f of minus t e to the power of minus j omega t dt.

You can replace, let minus t to be x then I can write this as f of x e to the power of j omega x minus dx dt I will put minus dx. And when t equal to minus infinity x will be plus infinity and when t equal to plus infinity x equal to minus infinity. And this can further be written as this minus sign and that integral limits can be taken care of by putting this plus sign and reversing the integral limits.

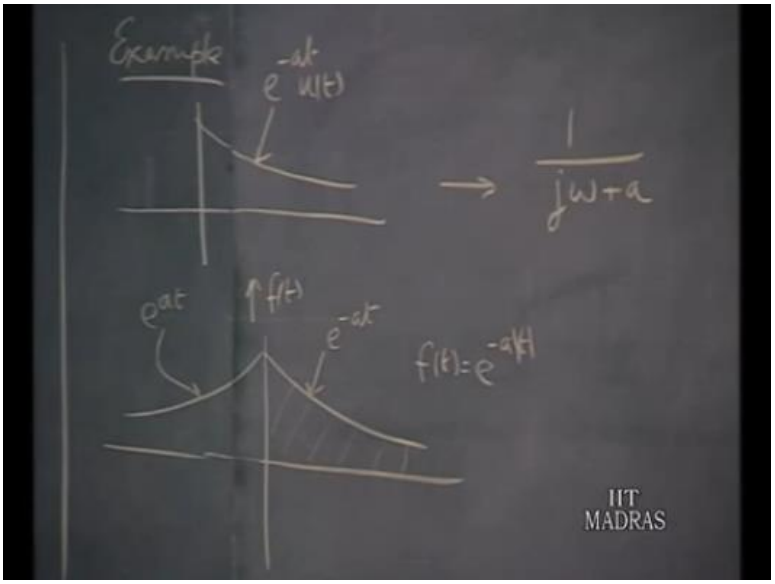
Therefore, I can write this minus infinity to plus infinity of f of x e to the power of minus j minus j omega x. So instead of plus omega x I put minus omega x therefore I introduce the minus sign here this of course written as minus infinity to plus infinity after all x is the dummy index this is f(t) e to the power of minus j minus omega t. So, the difference between the relationship of the Fourier transform of f(t) and this is instead of plus omega I have got minus omega. Therefore, this is f of minus omega right.

(Refer Slide Time: 13:08)



And what is the further significance of f of j omega? For real signals f(t) that means, when ever substitute real value of time the f(t) is also turns out to be real then it turns out that f of minus j omega equals f of j omega conjugate. That is the angle will be reversed the magnitude will remain the same you can write this also f star there that is in other words conjugate f of j omega.

(Refer Slide Time: 13:55)

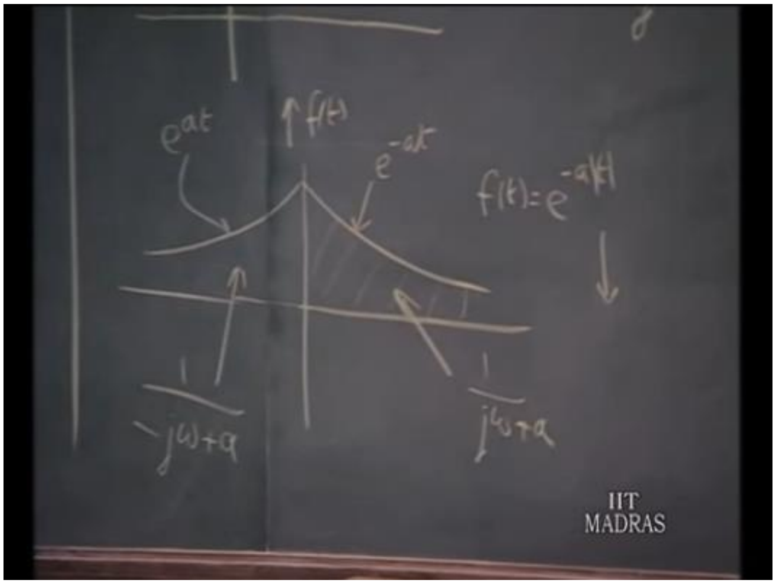


Again according to our practice let us work out an example. This time function e to the power of minus a t u t we know the Fourier transform 1 over j omega plus a. Now, I would like to enquire about the Fourier transform of a time function which is given by; e to the power of minus a t for positive t and e to the power of plus a t for negative t. That means, this portion is e to the power of minus a t. It is a reflection on the negative side this will be e to the power of a because, for negative t the exponent will become negative and again decrease.

So, this f(t) that we have now can be described as f(t) is e to the power of minus a t magnitude. So, the composite curve can be described analytically in this fashion. Now, what is the Fourier transform of this it is quite straight forward because, we know the Fourier transform of this section. The Fourier transform of this section is 1 over j omega plus a we know that. And what is this time function? How it is related to this?

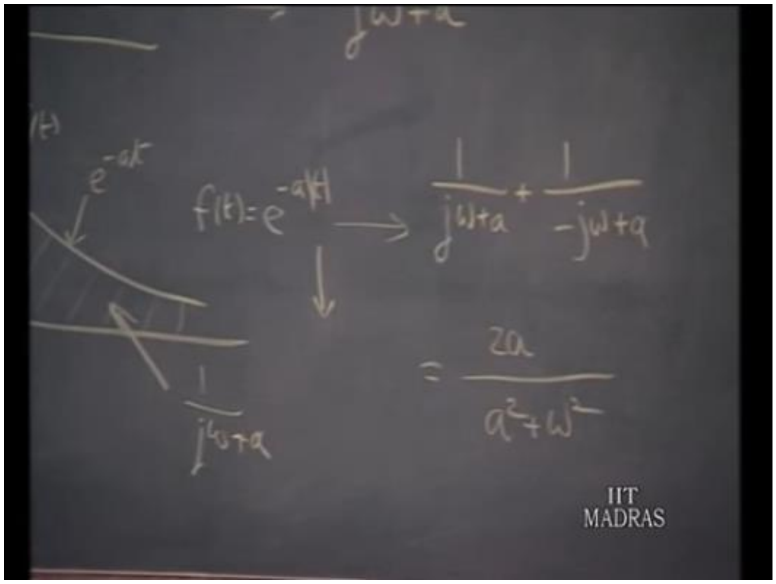
If this is f(t) this is f of minus t the whole sequence of even that takes place in the positive time axis will now be taken in the negative time axis in the same order. Therefore, the Fourier transform of this if it is f of j omega the Fourier transform of this is f of minus j omega.

(Refer Slide Time: 15:50)



And after all, what we have here is u f(t) is the summation of these 2 curves. Therefore, we can say its Fourier transform is the Fourier transform of this is 1 over j omega plus a and the Fourier transform of this portion is 1 over minus j omega plus a.

(Refer Slide Time: 16:07)

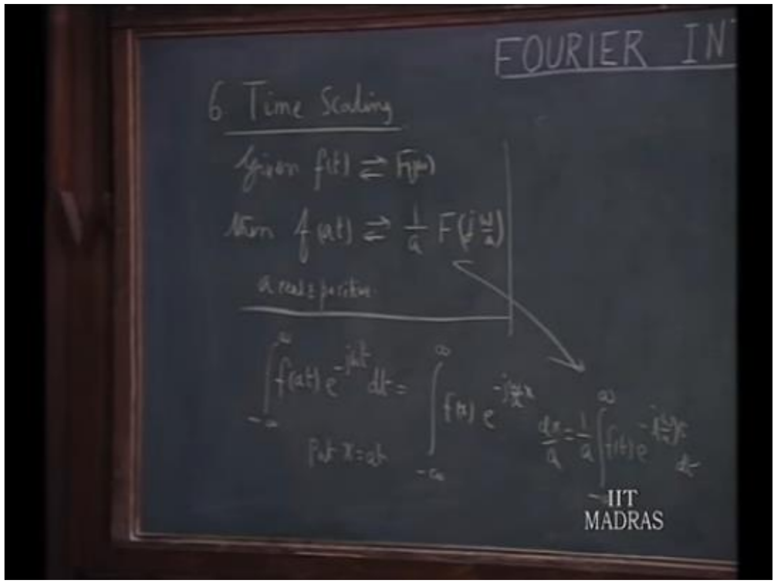


So, this particular f(t) which we have given e to the power of minus k magnitude t we have the Fourier transform minus k magnitude t we have the Fourier transform which is 1 over j omega plus a plus 1 over minus j omega plus a. And combine these 2 you will get denominator a squared plus omega squared and in the numerator we have 2 a.

Now, the essential motivation for our study of the various properties of Fourier transform is to find shortcut to find the Fourier transform in this manner. You do not have to try to do integrate this in both direction and get its result. Once we know the property we easily deduce the Fourier transform of this in this manner.

So, this is the real motivation for our study of the various properties of the Fourier transform. So, that we can intelligently obtain the Fourier transform for such a functions and also find the inverse Fourier transforms of certain functions, once we know the various properties. Let us now, continue our discussion additional property where, we are thinking of scaling in time domain. So far, we talked about translation time domain, reversal in time domain, differentiation time domain.

(Refer Slide Time: 17:50)



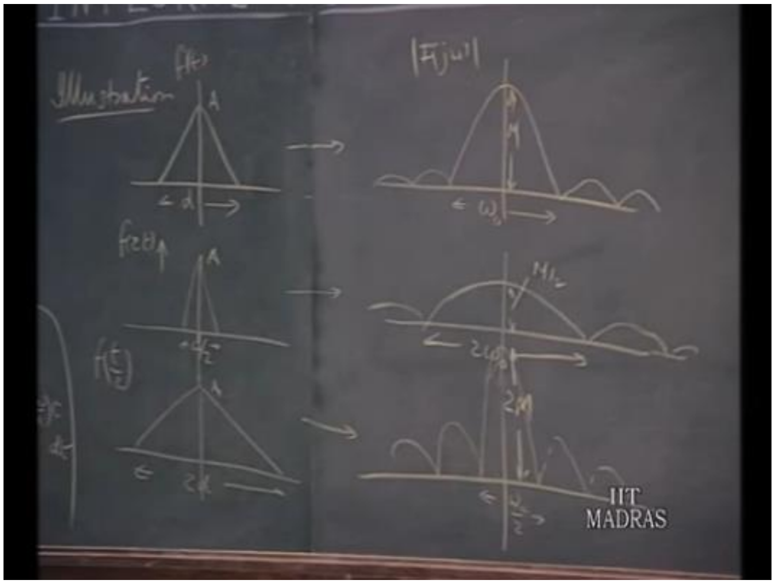
Now, let us consider what happens if you scale the function in time domain. Again, we start with basic f(t) f of j omega pair. Then, I would like to know what is the Fourier transform of f of a t where a is real and positive. Then it turns out to be this 1 over a f of j omega over a. Now, how do establish this? Proof is again straight forward we take the Fourier transform of f of a t which is minus infinity to plus infinity of f of a t e to the power of minus j omega t dt.

Now, by this time you must have got an idea how to go about this we must substitute some x for a t. So that we try to bring this into a form which we can compare with the Fourier transform of f(t). Therefore, if you substitute x for a t then this becomes when t becomes minus infinity x also becomes minus infinity.

When t becomes plus infinity x also becomes plus infinity f of x and e to the power of minus j omega t equals to x up on a and d t is d x up on a. So, this is obviously 1 over a from minus infinity to plus infinity of f of x e to the power of minus j omega by x d x. And since, the dummy variable x in the integration I can as well replace it by t.

This will be f(t) e to the power of minus j omega by a times t dt and this clearly is 1 over a times f of j. Instead of omega you have got omega by a therefore in the defining relationship of f of j omega t you would had f of minus j omega t. But, instead of omega you have got omega by a so instead of being f of j omega it becomes f of j omega by a and this a is stitching outside.

(Refer Slide Time: 20:47)



So, this again has got some important aspects in relation to scaling of time functions. Let me illustrate this; illustration of this suppose we had a time which is given like this a and stitching from over an interval d. And let we imagine that its Fourier transform would be like this magnitude spectrum I am plotting and this is equal to let us say some omega I will say some omega 0 whatever it is.

And then, this is a this is the Fourier spectrum for this magnitude spectrum. Now, let us see what would happens if I compress the in the time scale that means, the events now come closer together. So, I have the same a but the events occur faster so this is d up on 2. So, the same information sequence of information events now compressed in times they are carrying over a small interval of time t up on 2.

So this is f(t) this will be f of 2 t because, at a smaller value of t you get the same original values. What is the consequence of the spectrum? From what we have seen here first of all f of a t 1 over a, the amplitude must go down by factor half and then omega by a. So, it means j omega by 2 it becomes that means it get spread out.

So that means the amplitude like this so, this will become m up on 2 and this will become 2 omega. On the other hand if I allow the things to take place more. So, let us say this is 2 d and this is a this will be f(t) up on 2. That means, it takes the longer time for a particular value to reach compared to this, the t must be doubled the same values what it had in the case of f(t).

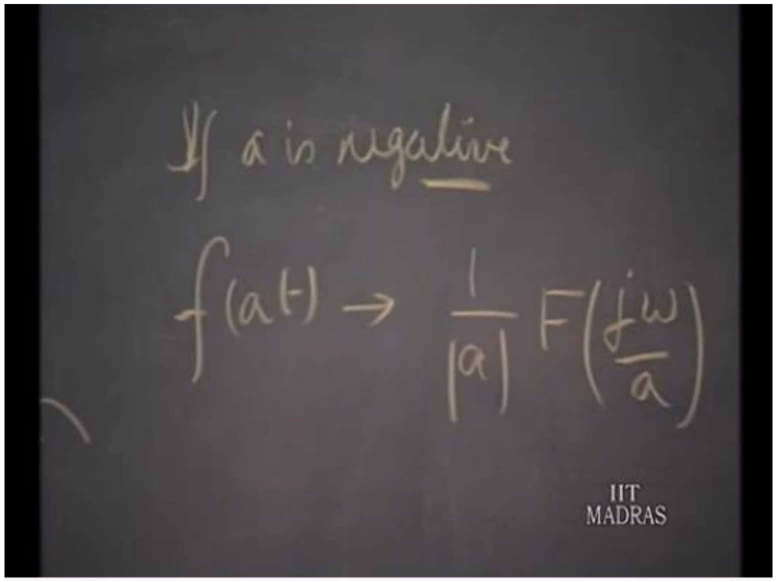
So, in this case a is equal to half in the original formula that we have here a is equal to half now therefore, it means as far this spectrum is concerned it will be 2 m. This will become larger but 2 f of 2 j omega therefore, in the frequency spectrum things becomes crowded now. So it means, it will be very large compare to this and this become omega up on 2 omega 0 up on 2 and this will be 2 m.

So, what it really means is the significance of this is, that if you allow certain events to come faster become faster. Then it requires as the larger frequency components; larger frequency component compare with this you have got certain frequency component. And then this spectrum is spread out that means you have more high frequency components.

So, if you have a time signal in which events occur faster then it requires higher frequency components. On the other hand the same sequence of event take place more lesser fashion that means, they get more of lower frequency terms coming to the picture and their amplitude also gone. That is what it is things which occur faster require high involve high frequency component and thing which occur slow fashion which has a greater concentration of low frequency components.

So, in other words the slower the rate of change then you need to have a smaller band information transmittance is concerned. But, the other hand you like thing to happen fast if you have got faster changes then you require a larger band between higher frequency components both. So, this is the important property in the information transmission which we have we can say information centered a faster rate needs to have as higher frequency component.

And therefore, the channel must also have the corresponding bandwidth to handle such signals. If we have seen now that f(t) it goes f of j omega f of a t is 1 over j omega by a. And we assume a is positive real and positive so that, we are talking about a b is greater than 1 and less than 1 and we talked about this. It will be of interest for us to know what happens when a is negative.
(Refer Slide Time: 26:13)



Then it can be shown that if a is negative f of a t as far as its Fourier transform 1 over a magnitude f of j omega over a. So, all it means is the magnitude of a comes into the picture rather than the absolute value actual value of a. So, this is a result which can be derived on the same basis. So, all it means is the scale factor is now depends of the magnitude of a even if a is negative.

So far, we have studied several properties of the Fourier transform you recall it talked about the linearity of Fourier transforms. Then various operations in the time domain like translation in time domain, then multiplication by exponential factor e to the power of j omega c t. Then we talked about differentiation in time domain, then we talked about time scaling and we talked about reversal in time domain. And saw, what all these operations in time domain how they are reflected in the transform domain.

The study of all these properties will help us to find out the Fourier transform of certain functions in much more in simpler way than if you have desire to direct integration. And also when sometimes we want to find the inverse Fourier transform that means, f of j omega is given and we want to find the corresponding f(t). If we keep these properties in the back of our mind sometimes, we get the solutions in more straight forward way.