Networks and Systems
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Lecture-31
Fourier Transform Examples

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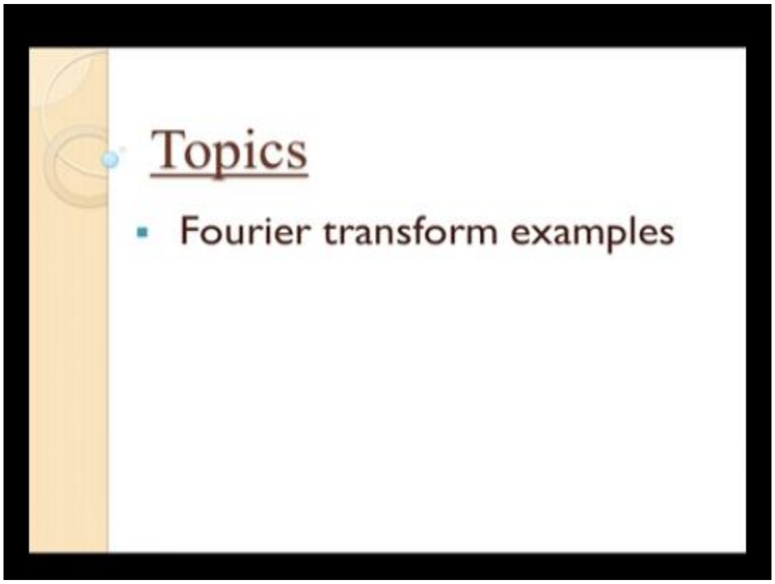

In the last lecture we introduced ourselves to the concepts of the Fourier Integral and the Fourier transform. Let us, quickly recall what we did? We observed that, a periodic function of time can be considered to be a periodic function with period going to infinity and we found that is a consequence the function will have frequencies of all values extending from minus infinity to plus infinity.

But then, the amplitude of the various frequency components will vanishingly small in fact, tend to 0. So, we will not have any meaningful data if we take up continue to take up the approach of evaluating the coefficient Fourier coefficient as we did in the case of a periodic function. So, we took up an alternative approach we talked in terms of the coefficient density which is the coefficient divided by the base frequency f 0.

And the coefficient density turns out to be a meaningful concept in the sense that it does not vanish and it gives the relative idea of the different frequency component magnitudes.

Coefficient density we also called the Fourier transform F j omega and Fj omega is in general is a complex number therefore, it has got both magnitude and phase.

So, corresponding to each f of t you have the Fourier transform F of j omega and both f of t and F of j omega can be thought of has 2 different windows through which we can look at a function: 1 is the time description and other description in terms of frequency. Both give equivalent information about the physical phenomenon we are used to observe a function as a sequence of values with respect time.

So, a function of time comes more naturally for us to visualize a physical situation of this sort. But imagine that you have the instrument or creature which have got senses receptive different frequencies, different frequency bands. Then, that particular instrument will observe the phenomenon as in terms of the relative different frequencies that will be in terms of the for example: Fourier transform F of j omega.

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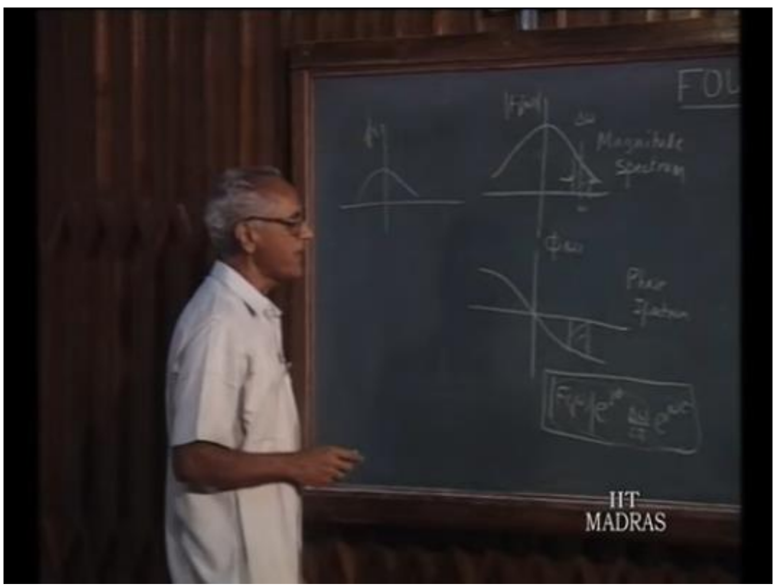

So, let us now look at once again you have a function of time f of t and correspondingly its Fourier transform F of j omega will have magnitude F of j omega magnitude and a phase function phi of omega which is an odd function. Therefore, this is the magnitude spectrum and this is the phase spectrum.

We also observe that is, this both of the magnitude and phase are essentially functions of which are called Fourier coefficient densities if you take a small band of frequencies, delta omega centered around a particular omega this section of the spectrum both the magnitude and phase together will tell us, about idea of the strength of the signal at this frequency omega.

In fact, these 2 sections represents at function of time which can be written as F of j omega magnitude e to the power of j phi e to the power of j omega t. So, this is the time function which is identified by these 2 sections of the spectrum. In other words, what we are saying is even though there is a small there is small difference in the frequencies in this band.

If you assume that, entire spectrum represents frequency component at this point omega at the center of this band. The time function corresponding to that is e to the power of j omega t and its magnitude is the Fourier coefficient times of course, delta omega you must also have delta omega. Because, this is f of j omega i write this again F of j omega e to the power of j phi this is the coefficient density.

But since the coefficient density we are talking over a band delta omega is: delta omega over 2 pi because, the density is in terms of the frequency e to the power of j omega t. So, this is the signal that is represented by these 2 sections this is the strength of the signal this is the coefficient and this is the time function.

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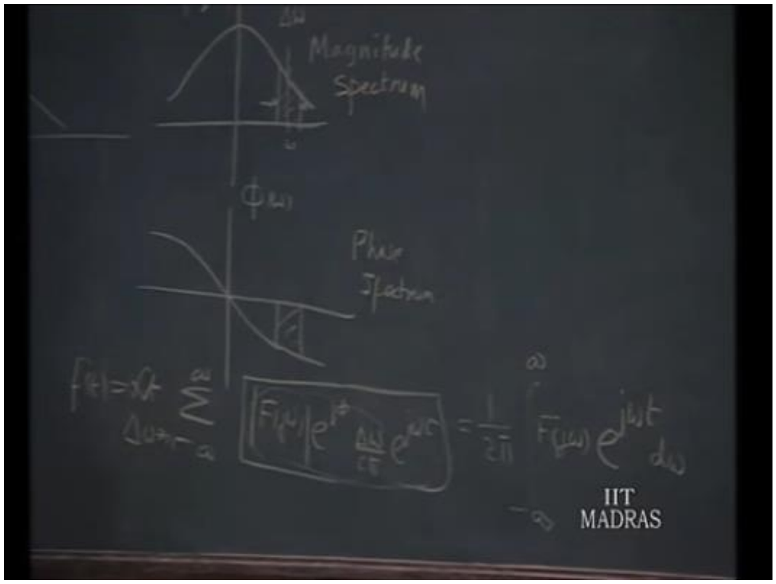

And if you take the limit of all such individual components, over the frequency band extending from minus infinity to plus infinity. This will add up to that is from minus infinity to plus infinity limit as delta omega tends to 0. if you take, such all such function this will be 1 over 2 pi minus infinity to plus infinity F of j omega.

Where F of j omega now, talking about combining F of j omega the magnitude and e to the power of j phi together is a complex number F of j omega e to the power of j omega t that will be your f of t. So, f of t can be thought of as 1 over 2 pi F of j omega e to the power of j omega t d omega.

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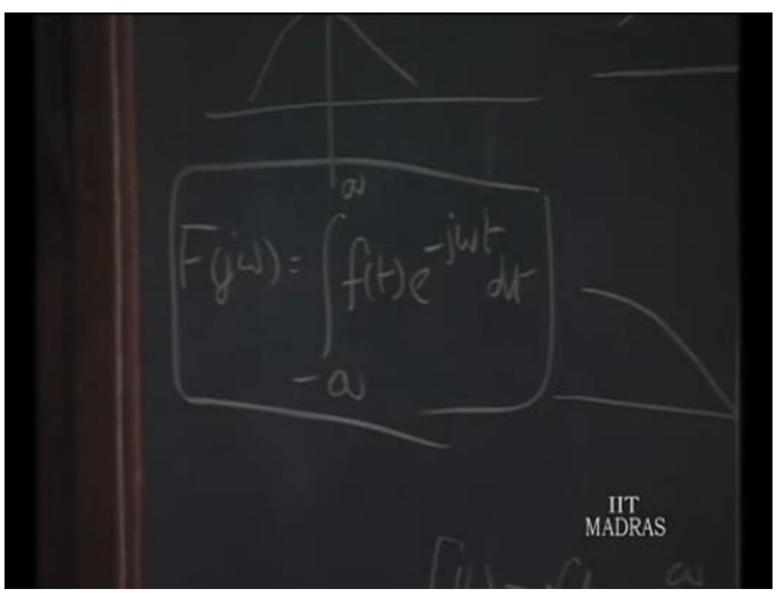

So, this is the Fourier integral and to get F of j omega from f of t this is minus infinity to plus infinity f of t e to the power of minus j omega t dt. So, these are the 2 relations which are important in the Fourier transform theory. You can get F of j omega from f of t and f of t from F of j omega.

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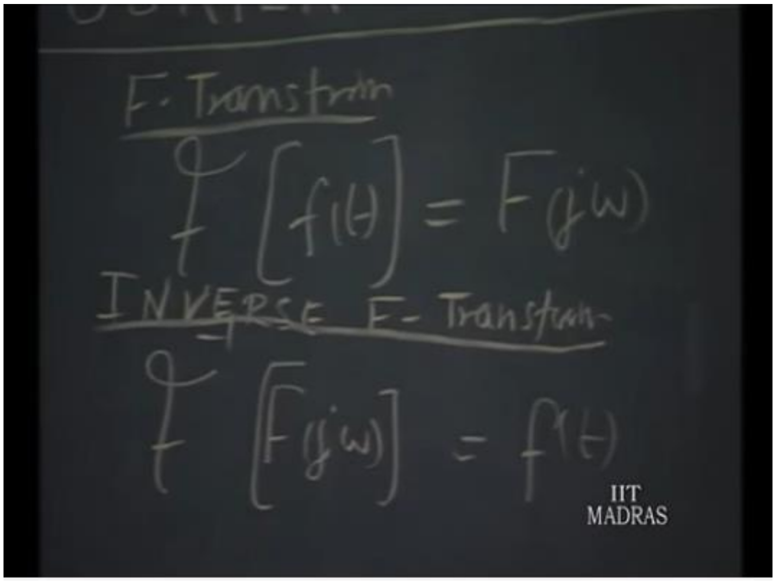

We write this relation in more compact fashion in this manner Fourier transform t is indicated in this manner a script f as the function of f of t this will be the Fourier transform this will be F of j omega. So, to recover f t from F of j omega we write this f of minus 1 the inverse Fourier transform this will give me f of t. So, this is called the Fourier transform that is your transform function of f of t function of ft to get the Fourier transform F j omega and what you have here is called the inverse Fourier transform

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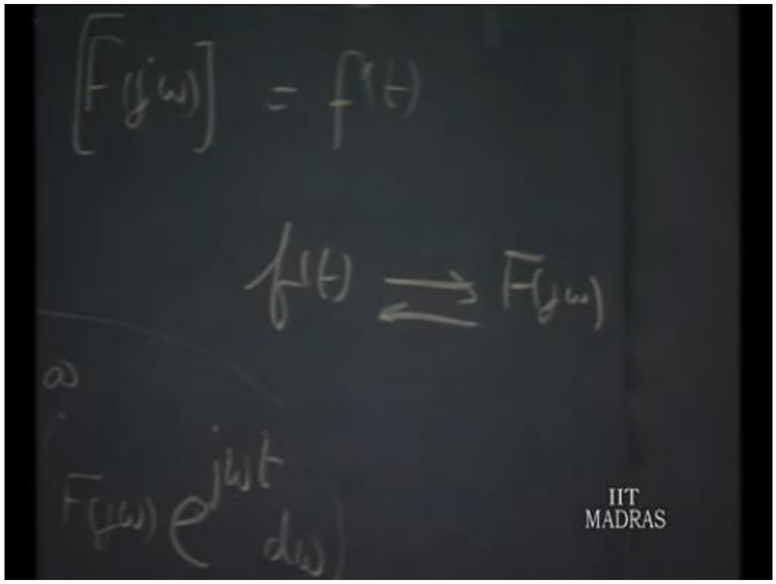

You would also like to indicate this transform relations occasionally, in this fashion f of t and f of j omega from a transform pair. So, we can indicate that functional relationship in this manner f of t arrow F of j omega is there in the forward direction you are doing the Fourier transform and in the reverse direction you are doing the Inverse Fourier transformation.

So, as i mentioned f of t and F of j omega are the 2 alternatives descriptions of the same phenomenon and what you like to do now, is to find out the Fourier transforms for a few representative functions of time before we go on to study of the properties of Fourier transform.

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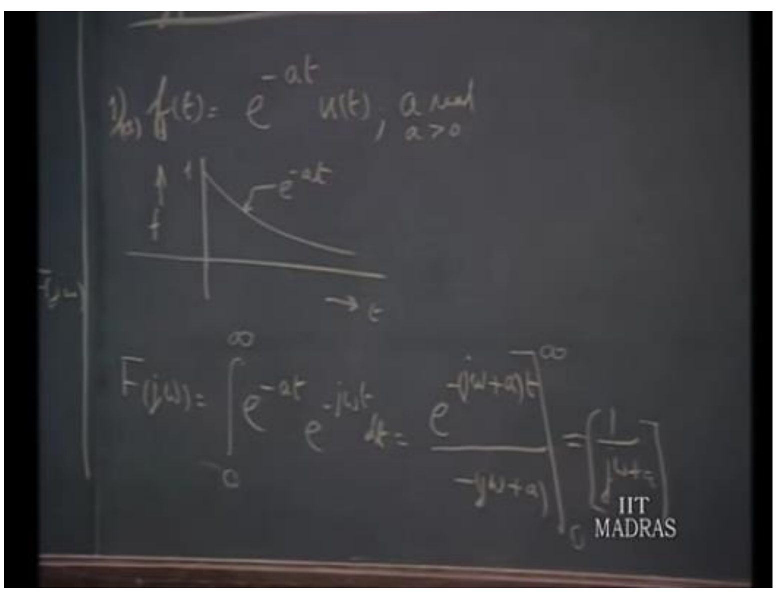

So, to get some physical idea of how this F of j omega comes about common time signals let us, work out a few examples typical Fourier transforms of a few representative time signal: 1 suppose, we have f of t as e to the power of minus at u t i call that 1 a let u, say a is real quantity a is real and a is greater than 0. That is i have taken a exponential starting time t equal to 0 t is the unit step function.

So, it will be 0 up to time t equal to 0 for negative values of t f of t is 0 it starts at 1 and then, decrease exponentially. So, this is e to the power of minus at, but the Fourier transform for this using the formula that we had here will be F of j omega you integrate this from minus infinity to plus infinity of f of t e to the power of minus at u t dt.

But since we know f of t is 0 identically from t minus infinity to 0 i can start the integration 0 go up to infinity strictly speaking we should start from minus infinity to plus infinity, but minus infinity to 0 f is 0. Therefore, i am starting the integration from 0 from 0 to infinity the value this function e to the power of minus at and i have e to the power of minus j omega t dt.

So, this will be e to the power of minus j omega plus at that is what it is to be integrated. So, you have in the denominator minus j omega plus a this should be evaluated between the limit 0 and infinity. Since, we have taken a to be a real number and minus k is the negative real number.

When, t goes to infinity e to the power of minus at become 0 and e to the power of j omega t is something which oscillates between 0 and 1 in this magnitude at least therefore, e to the power of minus at become 0 at t equal to infinity. Therefore, the upper limit is 0 and the lower limit it is 1 because, when t is equal to 0 the exponential become 1 therefore, the result is this will be 1 over j omega plus a. So, e to the power of minus at u t has a Fourier transform which is 1 over j omega plus a.

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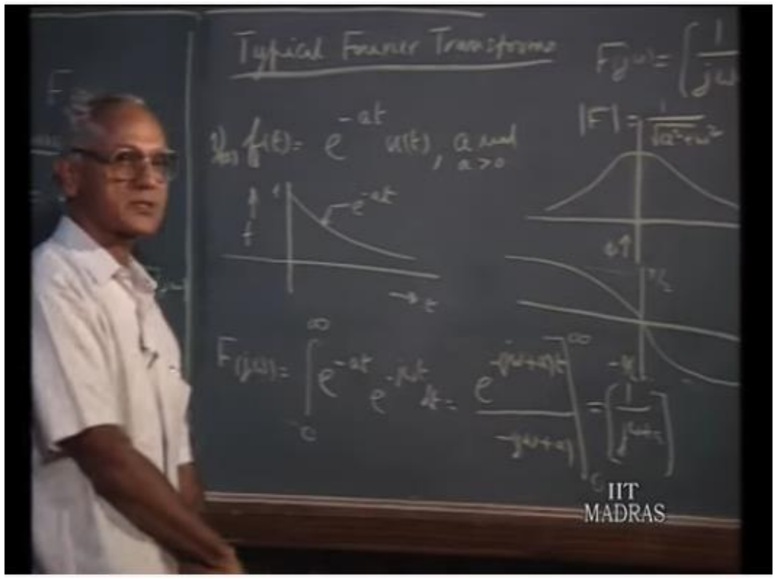

So, how does the spectrum look like? F j omega equals 1 over j omega plus a. So, the magnitude spectrum will be F magnitude 1 over square root of a square plus omega square right. Therefore, it will be something like this and the phase spectrum as a function of this all of the function of frequency when, omega goes to very large positive value the phase of this the angle of this complex number becomes minus 90 degrees.

That means, it goes to asymptotically minus pi upon 2 and because, of the phase spectrum is the r’th function of omega. So, it reaches plus pi up on 2 into positive axis that is so, this is the angle spectrum or phase spectrum, this is the magnitude spectrum. Now, even though we said that in a periodic function has 0 amplitude signals at all frequencies we still from the coefficient density we can see that, the component of the signals at dc are stronger than the component of the signals at some other frequencies.

So, this spectrum gives us an idea of the frequencies at which the densities concentrated in this signal. The energy density is concentrated in the signal because, the components are more done here than here. So, you can see the relative proportions of the different frequency component that go to build up the signal in terms of the Fourier transform magnitude which is really the coefficient density.

So, even though coefficients are all 0 the coefficient density gives us as the measure of the strength of the signal strength of the signal at different frequencies. So, let us now continue this now i purposely put this 1 a e to the power of minus at a is real and a is greater than 0. Because, i wanted to extend this idea and say that this particular formula that we had f of t as a Fourier transform 1 over j omega plus a will be valid even if a is complex number.

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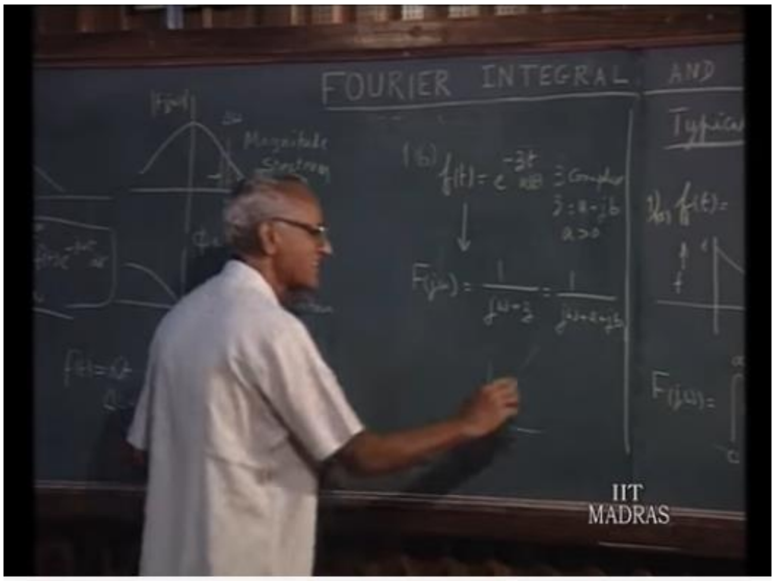

So, f of t suppose is e to the power of minus z t where z is complex z. Let us, say a plus i b and a is the real number greater than 0. Then, the Fourier transform for that can be shown to be 1 over j omega which is 1 over j omega plus a plus i b. So, this formula that e to the power of minus at ut here also, you must have ut as this Fourier transform 1 over j omega plus a will be valid even if instead of a you have complex number.

The only requirement is that, the real part of the complex number must be negative because minus z that is minus z is the coefficient of t. Then, the real part of the minus z must be greater than real part of minus must be negative or the real part of the z must be greater than 0.

The reason is if you had exponentially rising like this. For example: if you have instead of this being negative suppose it is exponentially rising signal then, this integral will not converge. This integral when, t becomes larger will not converge therefore, you must have an exponential decaying signal then only it converges at t equals to that is the reason why, we had this restriction.

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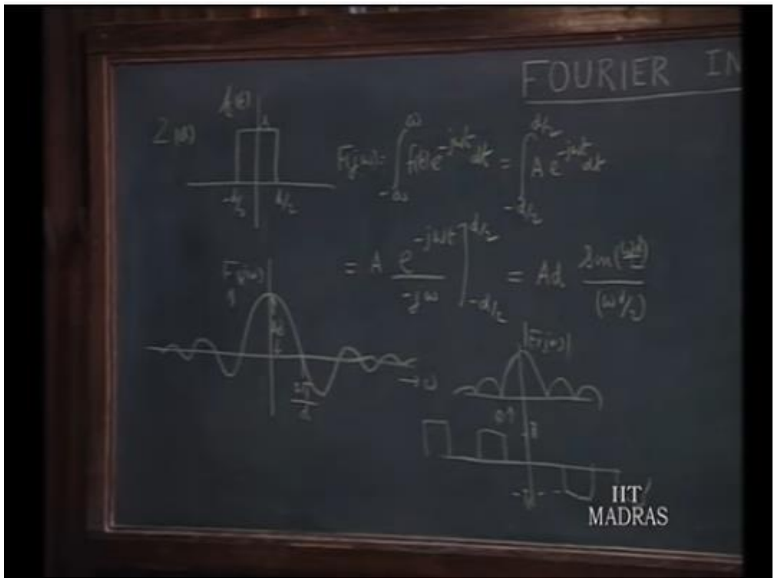

Now, let us work out another example now let us, take a second function of time a pulse function which is quite common and appears quite frequently in Fourier transform theory. So, we have a pulse of width half d units amplitude a. So, this is f of t so, f j omega will be from minus infinity to plus infinity f of t e to the power of minus j omega t dt this is the standard form in our case, this function of time only from minus d upon 2 to plus d up on 2.

Therefore, this can be written as minus d up on 2 to plus d up on 2 and in this interval f of t equals a. So, a e to the power of j omega t dt. And that will be a e to the power of minus j omega t divided by j omega evaluated between the 2 limits minus d up on 2 to plus d up on 2. And this can be shown you can work this out and this can be shown to be Ad sin omega d up on 2 divided by omega d up on 2 that is the Fourier transform for that.

So, this is of the form sin theta by theta type of variation. So, the spectrum for that F of j omega can be plotted in this fashion like this, where at the dc the Fourier transform will be having the value Ad. Because, you recall sin theta by theta will have a value equal to 1 when theta equal to 0. So, this will be Ad And then, it oscillates, but with diminishing amplitude.

Now, when does the first 0 occur first 0 occurs when sin omega d up on 2 is 0 that omega d up on 2 equals pi. Therefore, the first 0 occurs when omega equals 2 pi up on d. In fact, this is the spectrum which we plotted in the last class when, we talked about the evaluation of the Fourier series coefficient density starting from periodic pulse strain this is the exactly the type of spectrum that we plotted.

Now, since F of j omega happens to be real i do not plot the magnitude spectrum and phase spectrum separately. Because, F of j omega is real it is either positive or negative. So, i can combine both the phase and magnitude information in 1 plot like this.

However, if you wish you can plot the magnitude spectrum separately like this, this, is F of j omega magnitude and the phase spectrum will be whenever, this is negative you can say that is minus pi up on 2 minus pi this minus phi and then, you can write this as plus pi.

So, this can be considered to be the phase and then could be the magnitude. So, this is the alternative way of this repeating the spectrum was placed in magnitude. But when the Fourier transform is real there is no point doing this separately as well exhibit the entire f of j omega being the real function of time real function of omega in 1 plot like this.

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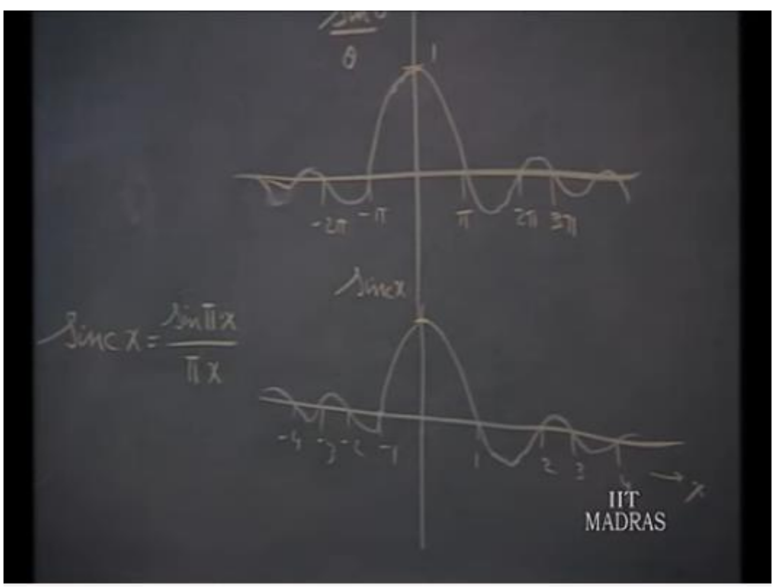

Now let us, have little bit of diversion here we know the sin theta by theta curve which occurs frequently in Fourier transform theory will be like this, this is 1 this is pi, 2pi, 3pi this is minus pi, minus 2 pi etc. Now, there is another function which is called Sinc theta or Sinc x equals defined as sin pi x over pi x. This is in the literature Sinc x is the term the function that is defined as sin pi x over pi x.

Therefore, if you plot Sinc x as versus x it will be similar to this because, when x is 0 both the numerator and denominator is 0 sin 0 by 0 type of thing. So, it will be 1 and it will have oscillations, but the 0 occurs whenever sin pi x is 0 that means, for integral values of when x is equals to 1 2 3 4 the sin pi x become 0 that means, the curve will be something like this.

Similar to that, but what we have now here is this will be 1, this will be 2, this will be 3, this will be 4, this is the minus 1, minus 2 ,minus 3 like that. So, occasionally people prefer to use the Sinc function instead of sin theta over theta compact at special Sinc x, which is will be the value of the function of x will be vary in this fashion.

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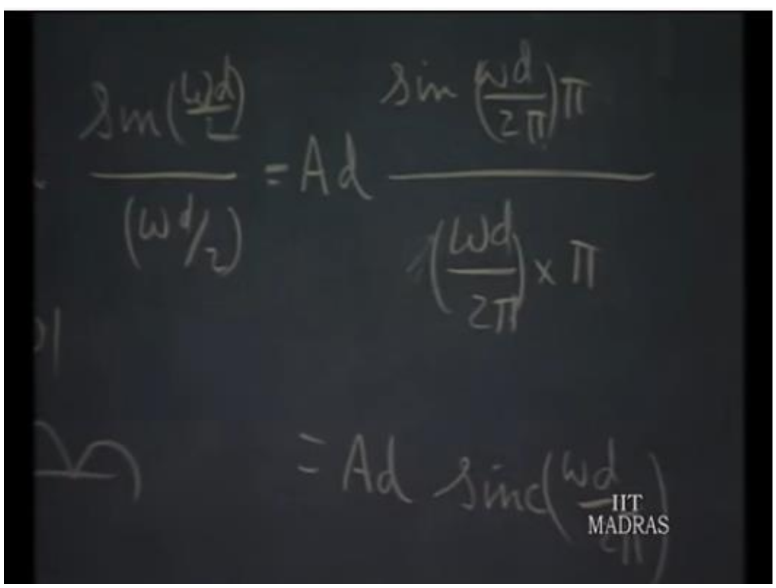

So, we can write this if you like as Ad sin omega d over 2pi times 2pi over i am sorry sin omega d up on 2 pi times pi right divided by omega d up on 2 pi times pi. So, we can write this as Ad times Sinc omega d up on 2 pi. Ad sin omega d up on 2 by omega d up on 2 can be written as Ad Sinc omega d up on 2 pi and the Sinc function vanishes at integral values of the argument.

When, x equal to 1 2 3 4 so, what values of omega will become 0 2pi upon d, 4pi upon d and so on and so forth. 2pi upon d, 4pi upon d, 6pi upon d. So, this is an alternative way of writing this So, this function this pulse function sometimes called gate functions because it has the non-zero value only between these two limits rectangular pulse function as the Fourier transform Ad Sinc omega d upon 2pi. The Sinc function tries to normalize the values in a nice way because, you are now having integral values of x and this will become evident.

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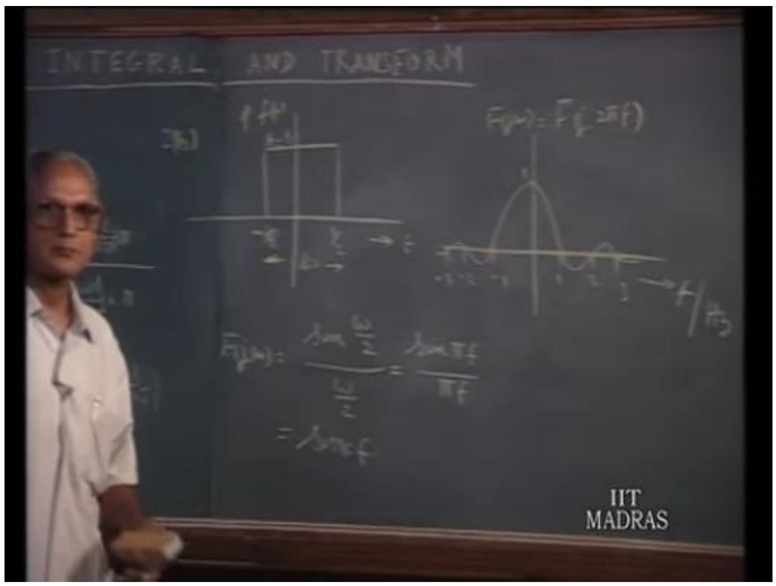

Now, if i take the second example we normalize the pulse so that you have unit amplitude and unit duration that means, this is minus half, this is half, this is t and A equals 1. Example where you normalize the pulse so, that you have unit amplitude and unit. This is f of t. So, the Fourier transform for that if you go back to our old formula, sin omega d upon 2 by omega d upon 2 A happens to be 1 d happens to be 1. So, Ad is 1 sin omega d is 1 this is A is 1. So, omega d upon 2 d is 1.

Therefore, this is omega upon 2 divide by omega upon 2 which if you like to put this in terms of frequency, this is sin pi f omega being 2 pi f this is sin pi f. Therefore, and this is indeed Sinc f so, consequently the pulse normalize to a unit amplitude and unit width we have a Fourier spectrum which is given by Sinc f that means, if you plot the frequency F of j omega, but it turns now F of the frequency you calculate the x axis in terms of frequency you will have this is 1 f is 1 2 3 4 minus 1.

So, what does Sinc function only tries to normalize the things if you have normalize pulse unit amplitude unit width then, the Fourier spectrum will have will be like this unit height and going to 0 1 2 3 cycles per second. So, this is just a especial case of this because, once we normalize we have the Sinc function which surprisingly very simple function. Let us, now work out the third example.