Networks and Systems
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Lecture-24
Examples

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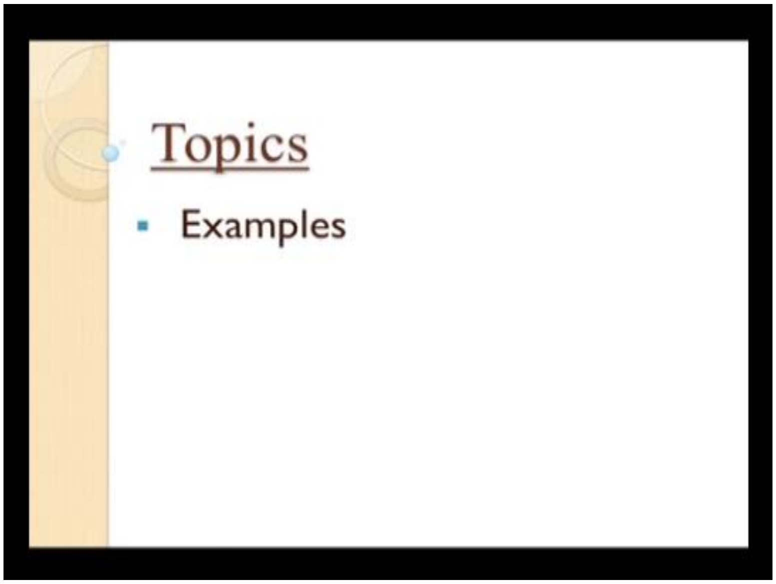

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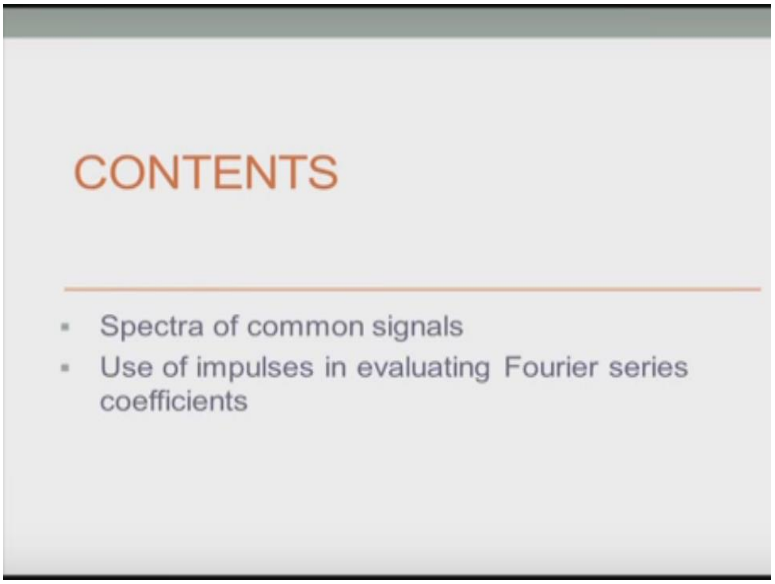

Now, let us use this background, let us plot the spectra of a few representative waveforms. I would like to keep this waveform and this is the spectra.

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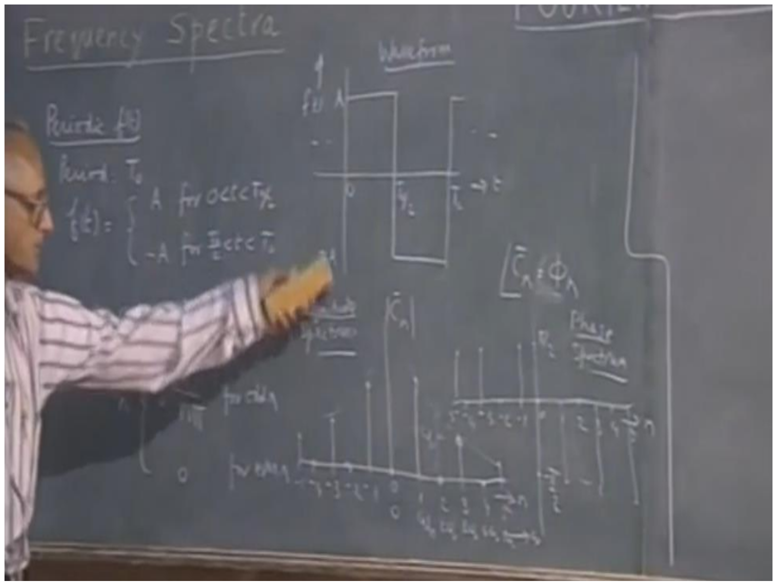

So, let me see.

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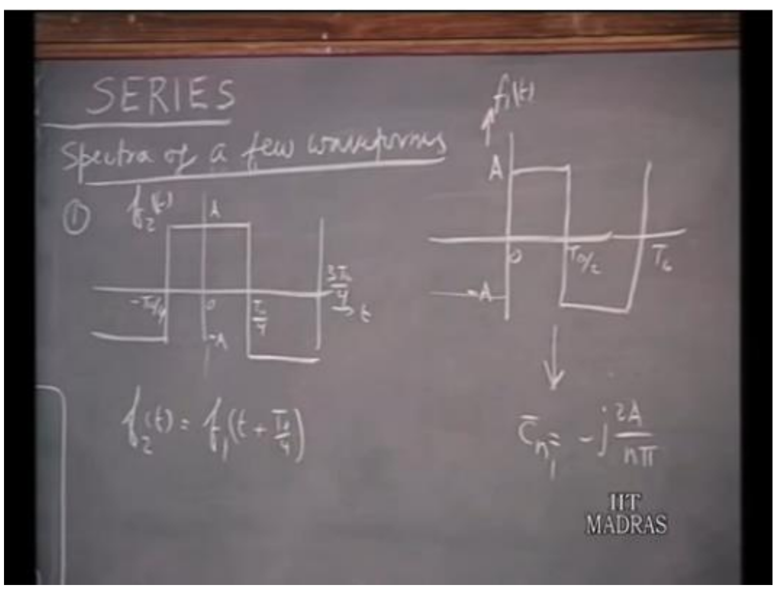

Suppose, I take a waveform, which is related to this, in the sense that it is a square wave all right, but shifted from the square that we have already dealt with. This will be 3 T not upon 4. This is minus T not upon 4. Let me call this f 2 of t. If the original square wave that we have is f1 of t that means, for f1 of t, I am considering this to be f1 of t.

We know that, this needs to a Fourier coefficient let us say, C n 1 because, I am calling this f 1 minus j 2 A by n pi, minus j 2 A by n pi. Now, what would be the Fourier coefficient for this? You recall you can see from this, these 2 are essentially the same waveform except for a shift in the origin. In fact, f of 2 of t is.

Obtained by advancing this by a quarter period. Suppose, similar events are made to occur t not by 4 seconds earlier then, this waveform is obtained from that. So, a rise from a negative a to a plus a occurs at 0. Now, that occurs at t not by 4 seconds earlier. Therefore, f 2 of t can be written as f 1 of t plus t not upon 4. So, the whole thing is advanced by T not by 4 seconds.

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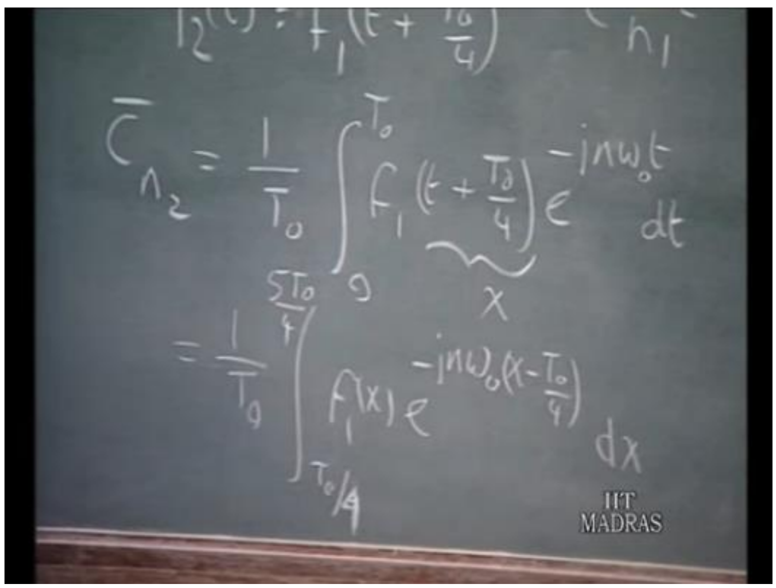

C n 2 the Fourier coefficient for f 2 of t will therefore, be, 1 over T not 0 to T not f 2 of t which is f1 t plus T not upon 4 e power minus j n omega not t dt. Now, in order to relate C n 2 with C n 1 let us, make the identification that t plus T not upon 4 equals X in which case, we can express this as 1 over T not.

Now, substitute the variable t by the variable X. Dt will be dX and this will be f 1 x of course, and e power minus j n omega not. T will be X minus T not upon 4 and the limits of integration when t equals 0 X will be T not upon 4 and when X is t not this will be 5 T not upon 4. X will be 5 T not upon 4. And therefore, this will now we equal to 1 over T not.

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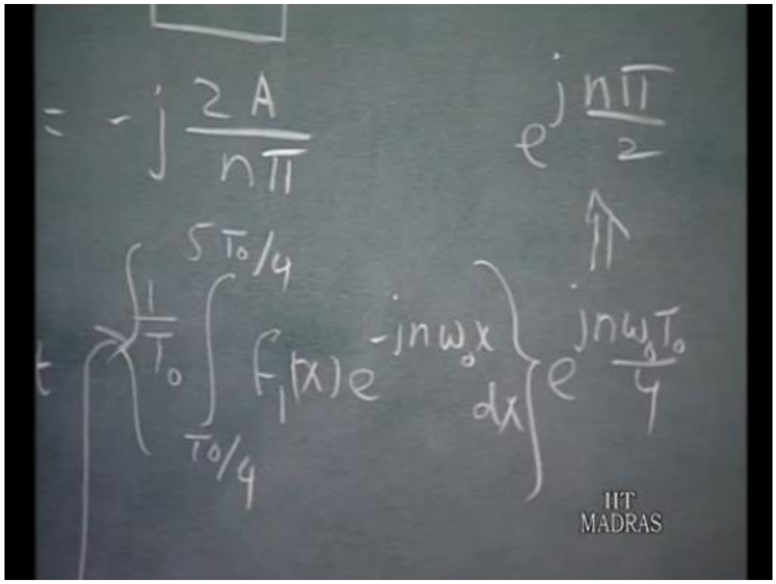

T not upon 4 to 5 T not upon 4 f 1 of x. I can write this as minus j n omega not X dX and the whole thing multiplied by a constant e j n omega not T not upon 4. E j n omega not T not upon 4 is of course, e power j n pi upon 2 because, omega not T not equals n pi upon 2. So, this will now become this multiplied by e power j n pi by 2. After all, X is a dummy variable here.

You can go back to t. This f 1 of t e power minus j n omega not t d t is exactly the formula for evaluating C n 1 because; we are integrating this for a complete period. Therefore this will become.

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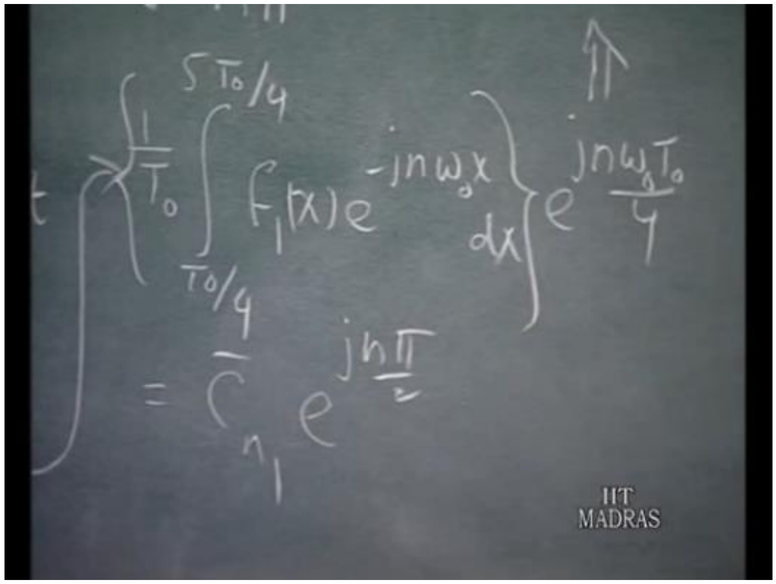

C n 1 multiplied by e power j n pi upon 2. So, e power j n pi by 2 has a magnitude 1 and its angle of course, depends upon the value of n. So, C n 2 happens to be equal to C n 1 e power j n pi by 2. The Fourier coefficient for this; C n 2 C n 1 e to the power j n pi by 2.

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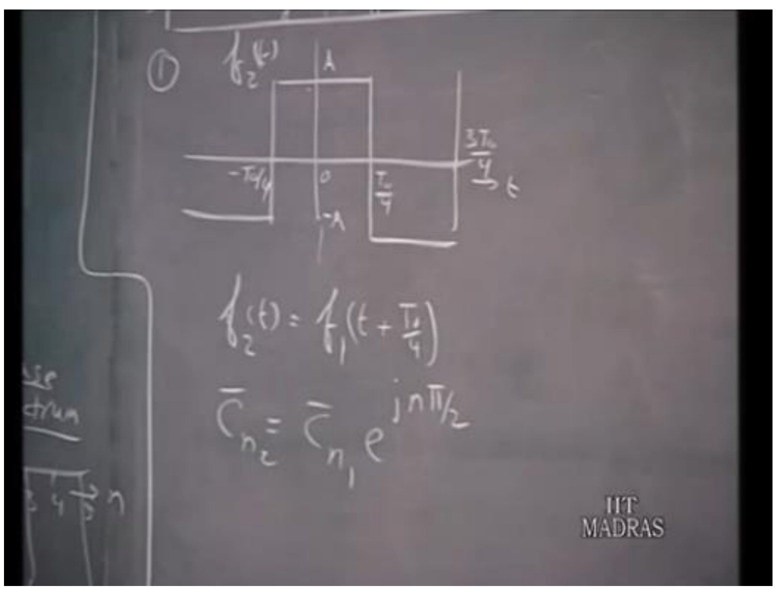

So, what is the spectrum of this? How does it look like? The magnitude spectrum will be the same as that of C n 1 why, because the magnitude of C n 2 is the same as the magnitude of C n 1 because the magnitude of this is 1.

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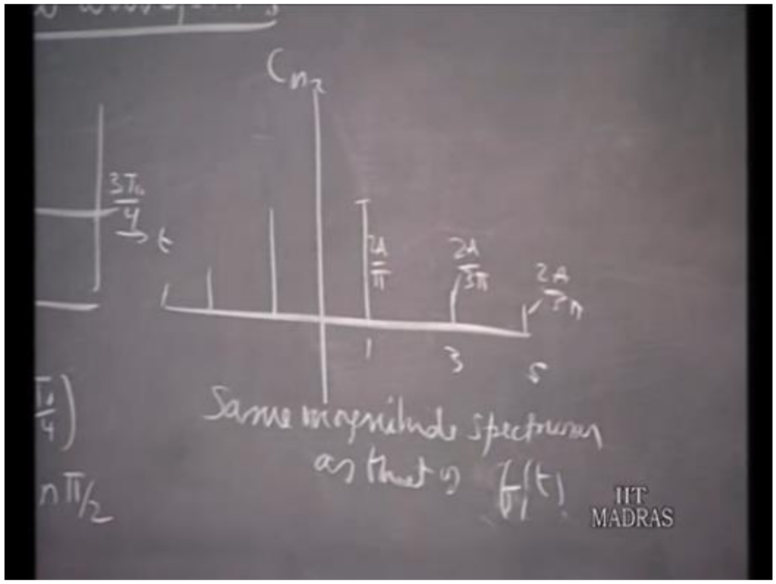

So, you have 1 3 5. This will be 2 A by pi 2 A by 3 pi 2 A by 5 pi etcetera. So, the magnitude spectrum remains unchanged. Same magnitude spectrum as that of f 1 of t. So, this magnitude spectrum and that will be identical. Phase; however, get modified. So, instead of this phase you have at each component, you have an addition of n times pi upon 2.

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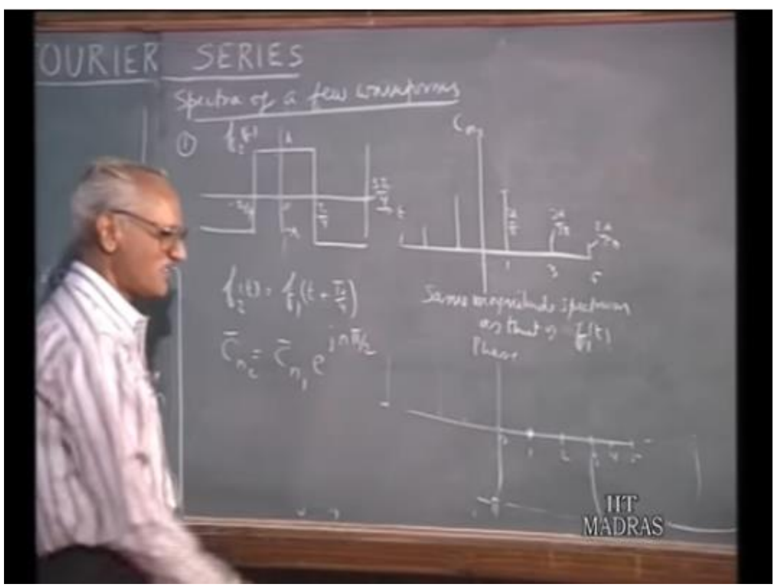

This of course, is nothing. At 1 it is minus pi upon 2. So, add to that another pi upon 2. So, this becomes 0. At 2 there is no component to deal with. At 3 it is minus pi upon 2. To that we add 3 pi upon 2. So, that becomes 180 degrees. You can as well say it is minus pi. Plus pi or minus pi are 1 and the same and at 5 you add 5 pi upon 2. It is already pi upon pi upon 2 minus pi upon 2 you add 5 pi upon 2. It becomes 2 pi.

Once again it is 0. So, you have a whole series of angles 0 or pi alternately and this is the phase spectrum. So, the conclusion we draw from this is: if you have a square wave, whether you put the origin here or origin somewhere here, the magnitude spectrum will remain the same.

So, if you have a periodic function, the amplitudes are of different harmonics or the ratio of the amplitudes of different harmonics will remain the same irrespective of where you pitch the origin, because that is the fundamental property of the waveform; how different harmonics can go together to compose the given wave.

So, the magnitude spectrum remains invariant under translation in respect to time, but the phase spectrum undergoes a change depending upon, where you put the origin. Usually, we are interested only in the magnitude spectrum because, we would like to know the relative proportions of the various harmonic components and therefore, we can say that the magnitude spectrum is invariant under the operation of shift in the time axis. Let us take a second example.

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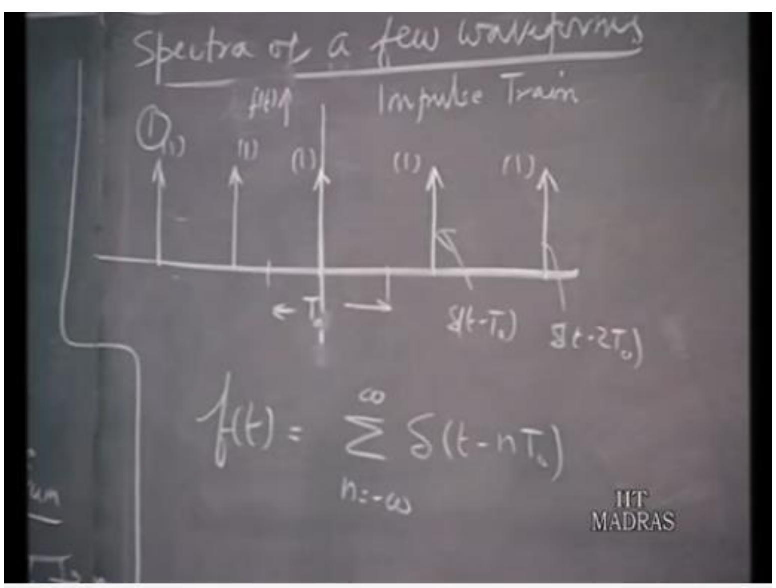

Suppose, the periodic function that we are talking about, is 1 which is a periodic impulse train. So, each is of impulse delta function of unit magnitude, unit strength. So, f of t is a periodic waveform with one impulse sitting in each period at the centre. So, we can write formally that, f of t consists of a number of impulses delta t minus n T not, n going from minus infinity to plus infinity.

So, for each value of n there is an impulse; at 0 this is the impulse. At n equals 1 this is after all delta t minus T not and this is delta t minus 2T not and so on. So, all these impulses go together to constitute this f of t.

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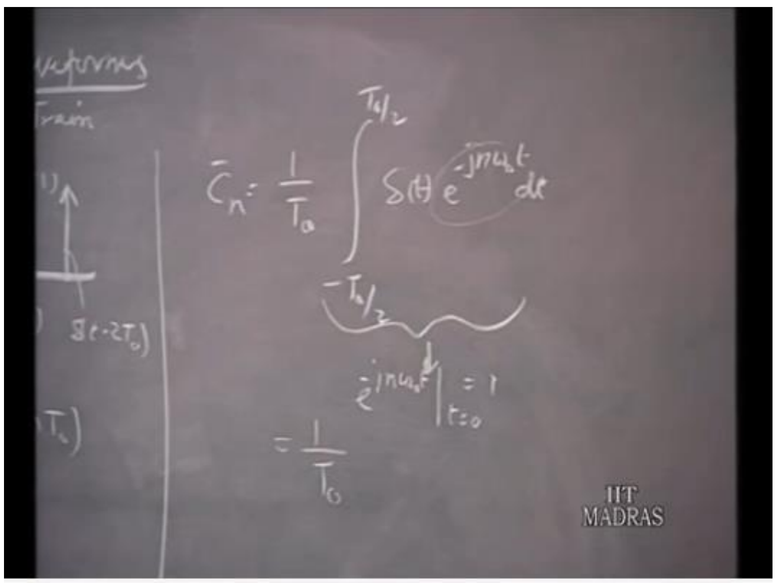

Now, what is the Fourier Spectrum for this? So, you have to find out the Fourier coefficient to start with, and this can be done with surprising ease because, whenever we have impulses inside the integration sign then, the integration becomes quite simple as we had already seen. So, C n here will be 1 over T not, say minus T not upon 2 to plus T not upon 2

So, inside this range, function that we have got is only delta t. E to the power of minus j n omega not t dt. Now, what is the result of this integration? Whenever a delta t multiplies a certain function of time and you integrate over an interval which contain that impulse the value is simply, the value of this function at t equals 0 because, the impulse it sitting at t equals 0.

Therefore, the value of this function is 1. Value of this is simply this quantity evaluated at t equals 0 and that of course, is 1. Therefore C n not is 1 upon T not. Normally, when we evaluate the Fourier coefficients it would be advisable for us, to verify if the general expression for C n is also valid for n equals 0. So, if you want to do it independently this is 1 over t T not.

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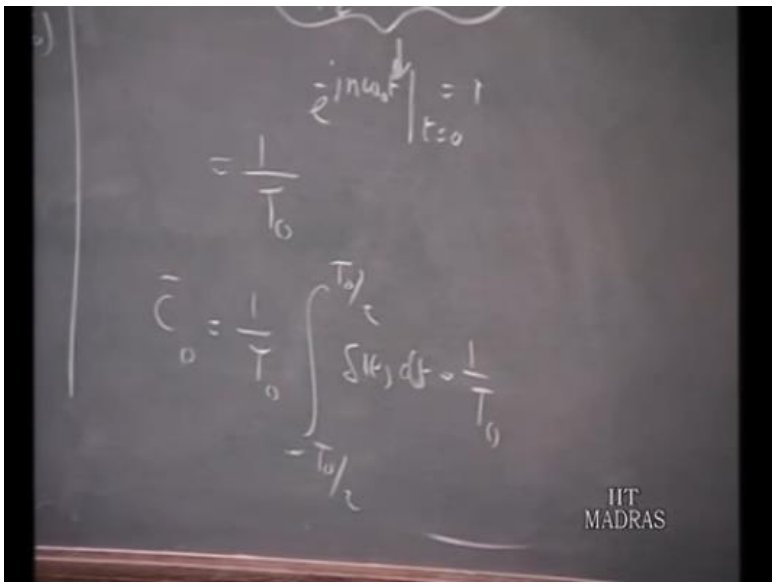

Say minus T not upon 2 to T not upon 2 delta t dt. Certainly, this is equal to 1. Therefore, this is 1 over T not. So, what do we have for the spectrum?

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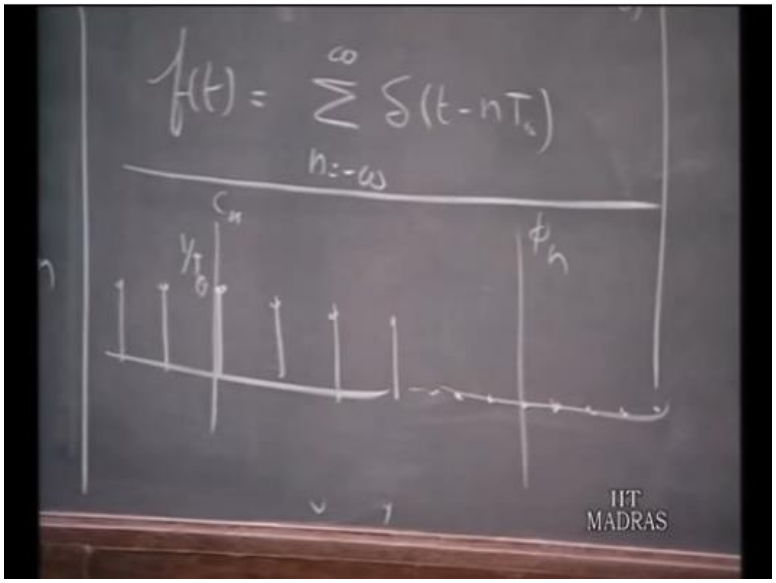

We have C n magnitude spectrum flat all components of the same value 1 over T not. Phase is 0 because C not is 0 that means, phase spectrum. Very interesting result that we got now. You have a train of impulses of unit strength, occurring regularly at intervals of T not.

Then, the spectrum for that is flat here in the sense that, all harmonic components have the same magnitude. They do not do down as n goes up. They retain; that means, all frequencies; that means, we can say a train of impulses is. So, to say, impartial to the order of the harmonic all harmonics of equal strength, go together to compose this impulse time.

This is quite an important fact which, will occur this particular property occurs, again and again in our future discussions. So, we will observe that if f of t equals this train of impulses, the Fourier Series expansion for this is 1 over T not e to the power of j n omega not t n ranging from minus infinity to plus infinity.

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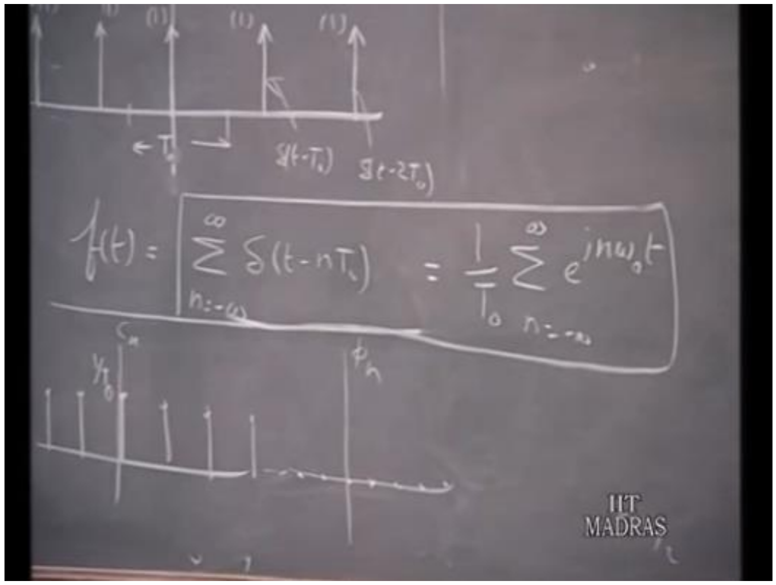

So, this is the result which is very compact and has a very nice look about it. You have summation from minus infinity to plus infinity of the impulses. On the other side, you have summation from minus infinity to plus infinity of exponential terms. So, this spectra gives us an indication, how the various coefficients go remain the same.

You have taken 1 extreme example of a situation where the spectra magnitude remains constant right through. So, with this background let us go further and construct another spectrum.

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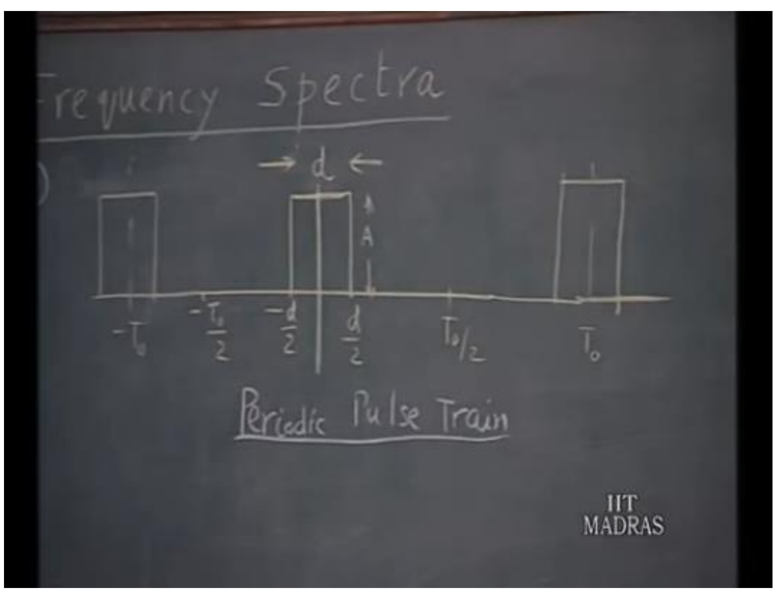

As a third example of finding out the spectra let us consider a periodic pulse train with a waveform depicted as here, where a rectangular pulse repeats itself for every T not seconds. The pulse has a width d and an amplitude a and it is centered symmetrically with respect to the origin. So, this is a periodic pulse train and the duty cycle is d upon t. That is for T not it will last only for d seconds.

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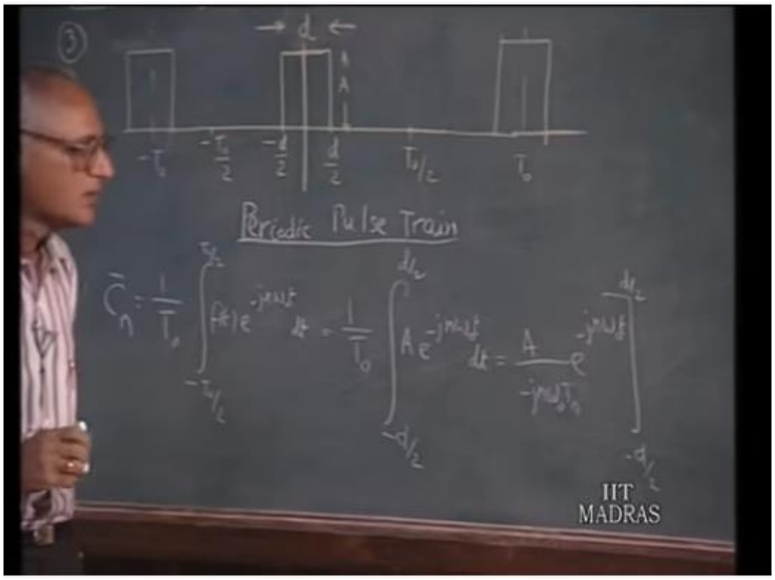

So, C n for this will be 1 over T not minus T not upon 2 to plus T not upon 2 f of t e to the power of minus j n omega not t dt. This is the standard formula. But in our particular case, this f of t lasts only from minus d upon 2 to plus d upon 2 and the rest of the period it is 0.

Therefore, we can write this as minus d upon 2 to plus d upon 2 and in this interval its value is A. A times e to the power of minus j n omega not t d t and this can be written as A minus j n omega not T not e to the power of minus j n omega not t. That is evaluated between the limits minus d upon 2 to plus d upon 2.

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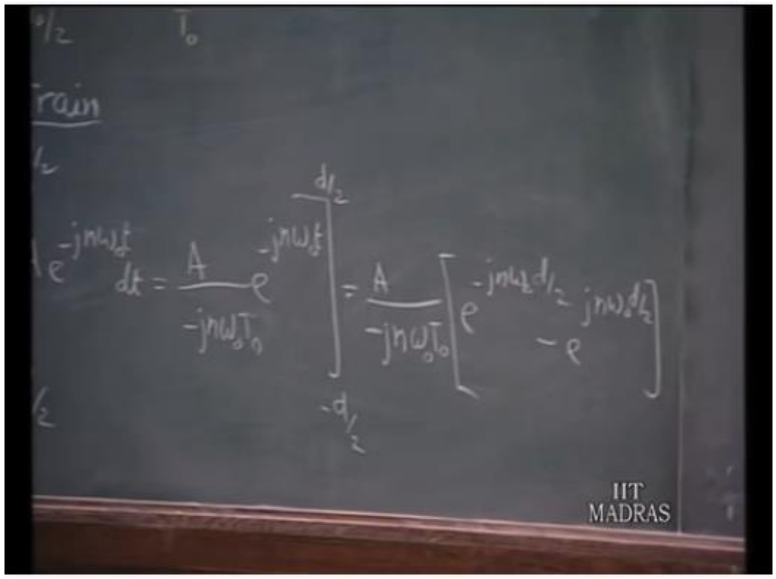

A minus j n omega not T not e to the power of minus j n omega not T not. e to the power of minus j n omega not d upon 2 minus e to the power of j n omega not d upon 2, substituting the lower limit minus d upon 2. So, if you pick up from here and go here.

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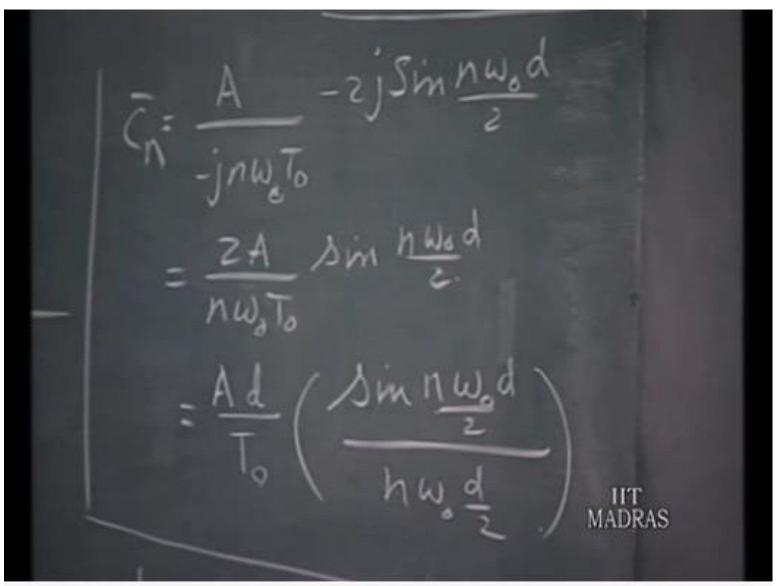

C n will now be A divided by minus j n omega not T not. This you notice, if we have reversed the side e to the power of j n omega not d upon 2 minus e to the power of minus j n omega not d upon 2 would be 2 j sin n omega not d upon 2. The negative sign you reverse this thing. I can write this as minus 2 j sin n omega not d upon 2.

So, canceling this minus j on both sides you have 2 A n omega not T not sin n omega not d upon 2. I will write this in a slightly different fashion. Suppose, I write this sin n omega not d upon 2 I establish a denominator n omega not d upon 2. So, this 2 is taken over here. So, A remains and since, I introduce a d in the denominator i put a d in the numerator.

N omega not are already carried over here. So, all that remains is this. I put this in this particular form because, this function we have is of the form sin theta by theta which is, a function which is well known in mathematical literature and which comes up quite frequently in waveforms of this sort.

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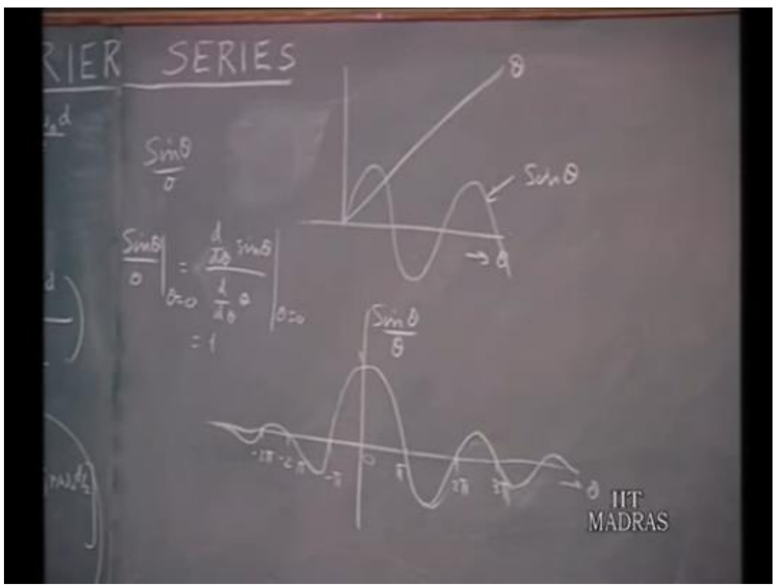

So, to plot this spectrum corresponding to this, let us fill ourselves upon some background information of the variation of sin theta by theta. So, if you are having a function of sin theta by theta, take theta. What we are interested in is to find out the variation of sin theta by theta.

So, let us plot this. This is sin theta and theta will be like this. So, we are dividing this waveform by this. So, we can see that as the theta increases the amplitude of the oscillations decreases. On the other hand, at theta equals 0 this is 0 by 0. So, if you take sin theta by theta, evaluating at theta equals 0 we have to take the limit at theta goes to 0.

That means; you have to take the derivative d by d theta sin theta in the numerator, d by d theta of theta in the denominator and evaluate at theta equals 0 using the l' hospital's rule and that turns out to be 1. So, when you divide this quantity by this you get, sin theta by theta curve.

It starts with 1 and will now exhibit oscillations with decreasing amplitudes and since this is an even function of theta, both numerator and denominator are odd functions. Therefore the ratio is even function. So, you get something like this. This is the variation of sin theta by theta. This function occurs again and again in Fourier theory. So, we would like to see the properties of this.

So, at theta equals 0 it is 1 and it becomes 0 again at pi 2 pi 3 pi and so on and so forth. Again on the negative side minus pi minus 2 pi minus 3 pi and so on so, this is the type of function that we have got. However, we do not have continuous variation of theta. We are interested in finding out the values of this, at specific values of theta which are given by integral values of n.

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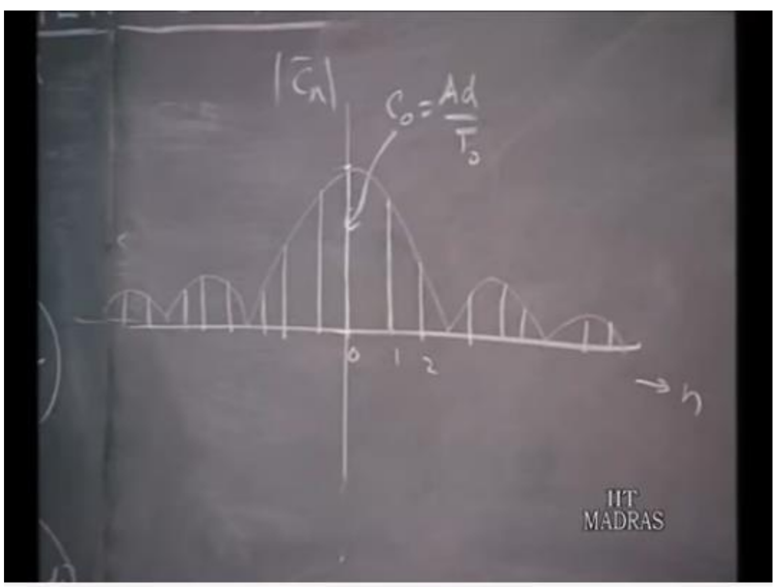

So, consequently the spectrum of this would be magnitude. We have an envelope like this, because we are considering only the magnitude. So, the negative loops are flipped over. This will be the envelope of your C n magnitude, but the lines, you will have only components like this and specific values of integral values of n 0 1 2 etcetera.

In particular, C 0 will be A d upon T not. As you can see, the average over a complete period is A d upon T not. So, that is the C not A d upon T not and like this it will go. So, the spectrum of a periodic pulse time will have this type of character and since, this is a real quantity, as far as the phase is concerned we can it is either 0 or pi and sometimes, when the C n quantity turns out to be real.

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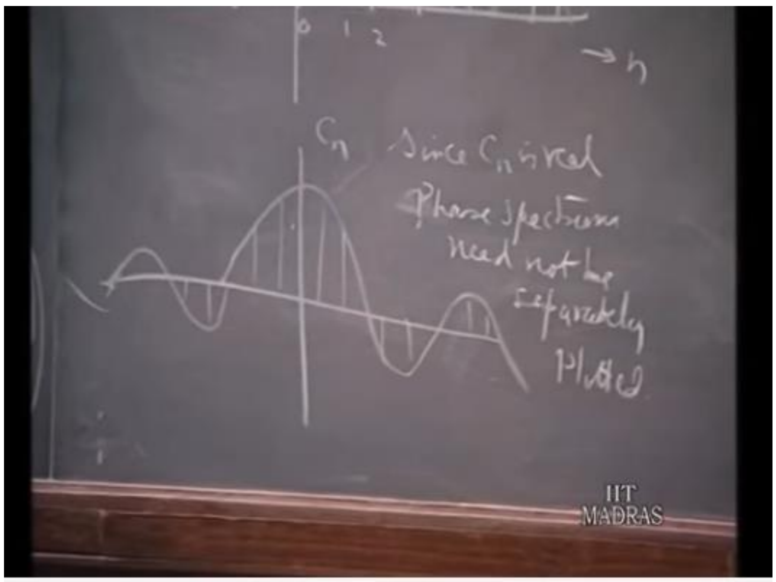

we can plot C n simply as like this. So, we do not have to plot the magnitude and phase spectra separately. So, C n is either real. It is either positive or negative and. So, that information can be given. Since C n is real, phase spectrum is not necessary. Phase spectrum need not be separately plotted. So, the amplitudes go down in this fashion.

We have seen in one of the earlier examples that, when impulse functions the evaluation of Fourier coefficient becomes quite simple, because integration with impulse is easy. So, would like to take an example, where we make use of the impulses in the evaluation of Fourier coefficients.

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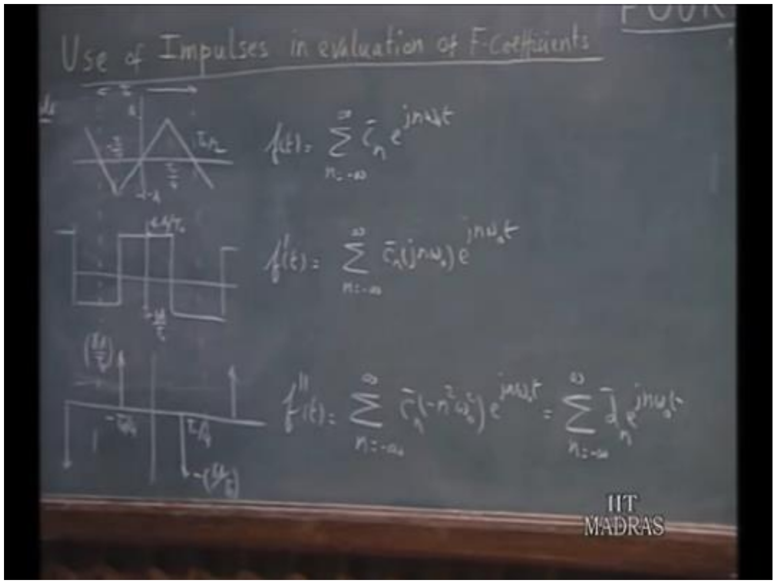

Let us take a triangular wave like this and suppose; I call this f of t and let its Fourier expansion be j n omega not t n from minus infinity to plus infinity. To evaluate this C n in this for this waveform requires, good amount of computational effort. But, let us see, what happens if you take the derivative of this, which is this.

If you take the derivative of this, the slope here is constant. The slope here is constant, but negative. Therefore it is like this. So, if you take the derivative of f of t, it results in a square wave like this f prime t, but we would not stop that. We will take the second derivative of this. The derivative of this now as you can see, it is 0 everywhere except where there is discontinuity.

Therefore, there is an impulse here and an impulse here and what are the magnitudes of these impulses? The slope here is, it raises by an amount 2 A in T not by 2. Therefore, it is 4 A by T not. That is the magnitude here and this minus 4 A by t not. Therefore, the jump that is involved in going from minus 4 A upon T not to plus 4 A upon T not is 8 A upon T not.

Therefore, there are 2 impulses here. One positive and the other negative which tends equals to 8 A upon T not and suppose we find the Fourier coefficients for this periodic waveform, then can we related those 4 coefficients to the Fourier coefficients of the parent waveform. That is what we want to do.

So, if f t has this Fourier series f prime t assuming that, we can differentiate under the summation sign. This will be C n j n omega not e to the power of j n omega not t. And you differentiate again, it will be C n again derivative of this. So, again you multiply with another j n omega not.

So, it will become n square omega not square e to the power of j n omega not t. Let us call this d n. Let us call the Fourier coefficient for this to be d n, n from minus infinity to plus infinity of d n e to the power of j n omega not t. So, if you calculate d n for this periodic function having impulses then, we know d n and C n are related by this C n square C n times minus n square omega not square is d n. So, that is the plan of action. We would find out the Fourier coefficient for d n.

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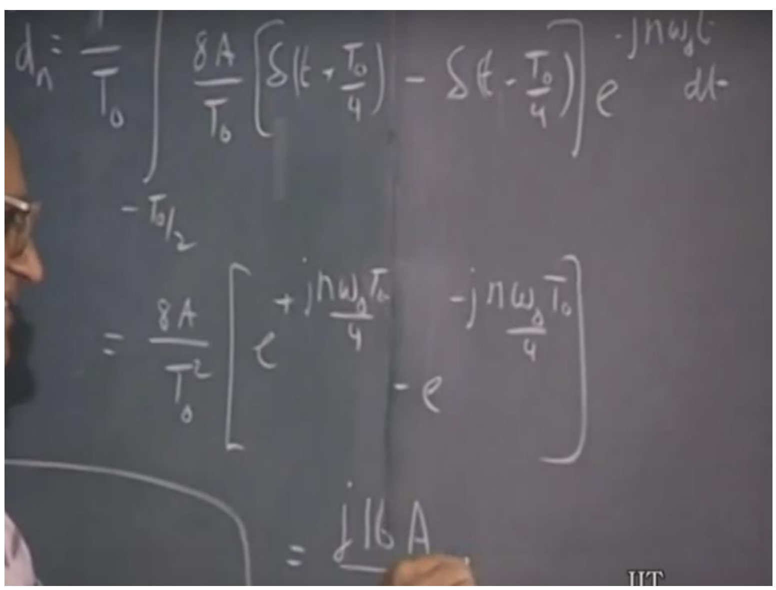

And then find out form that C n. So, d n is 1 upon T not from minus T not upon 2 to plus T not upon 2. In this range of integration, we have 2 delta functions. So, one delta function at both magnitudes: 8 A upon T not delta t plus T not upon 4. That is this delta here at negative values of time and another delta function at t not upon 4 with a negative sign.

Therefore, minus delta t minus T not upon 4 e to the power of minus j n omega not t dt. So, you have got 8 A upon T not square. Now, this delta multiplied by this is being integrated. Therefore, this will be e to the power of minus j n omega not minus t not upon 4 evaluated at t equals minus t not upon 4.

That means, plus t not upon 4 minus e to the power of minus j n omega not t not upon 4 and you can carry this out. Finally, you can show that is j 16 A upon T not square sin n pi by 2. This is your d n. Therefore C n will be obtained by dividing d n by minus n square omega not square.

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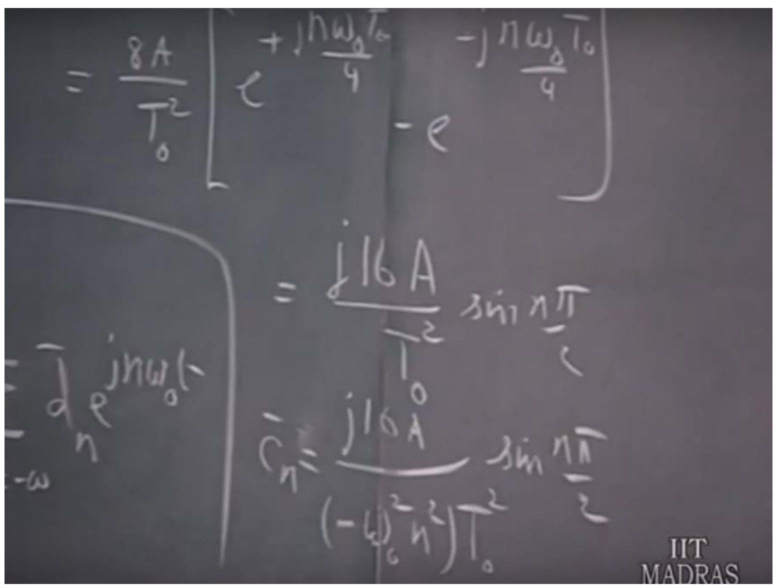

So, j 16 A minus omega not square n square T not square sin n pi by 2. And you can complete the work and you can show that this will turn out to be minus j 4 A by pi square n square minus 1 raised to the power of n minus 1 by 2 for odd n and 0 for even n.

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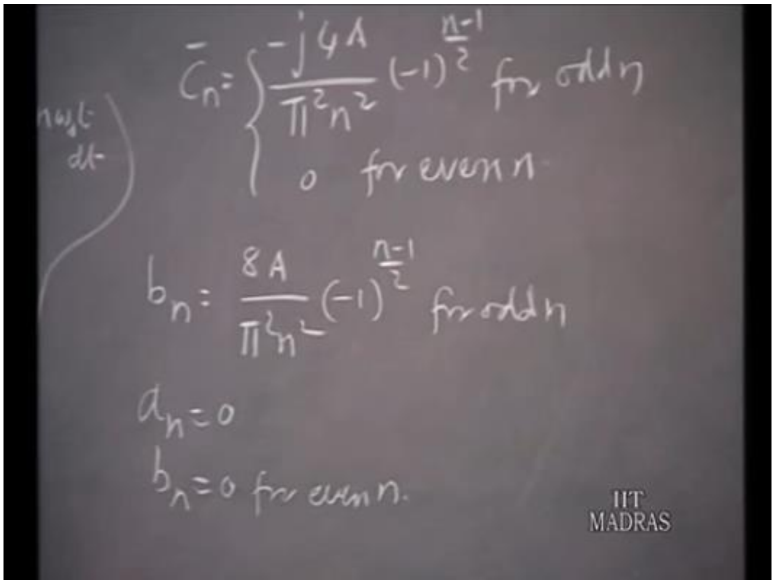

So, that is the Fourier coefficient for the original triangular waveform. Recognizing that C n equals a n minus j b n by 2 you can write, this contains b n is 8 A by pi square n square minus 1 n upon 2 for odd n and we know a n is 0 b n is 0 for even n and what about C not?

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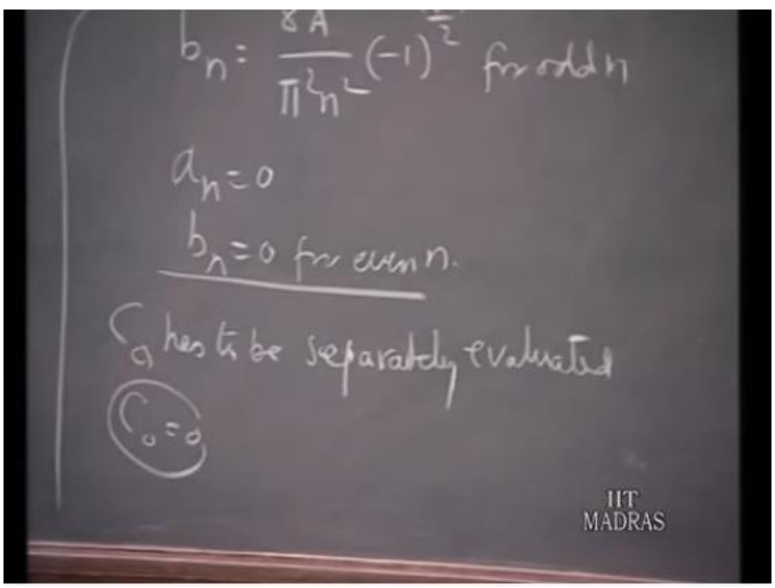

C not has to be separately evaluated. I will explain in a moment. C not for this waveform because it is 0 average the original waveform C not is 0. Other C n had already been obtained.

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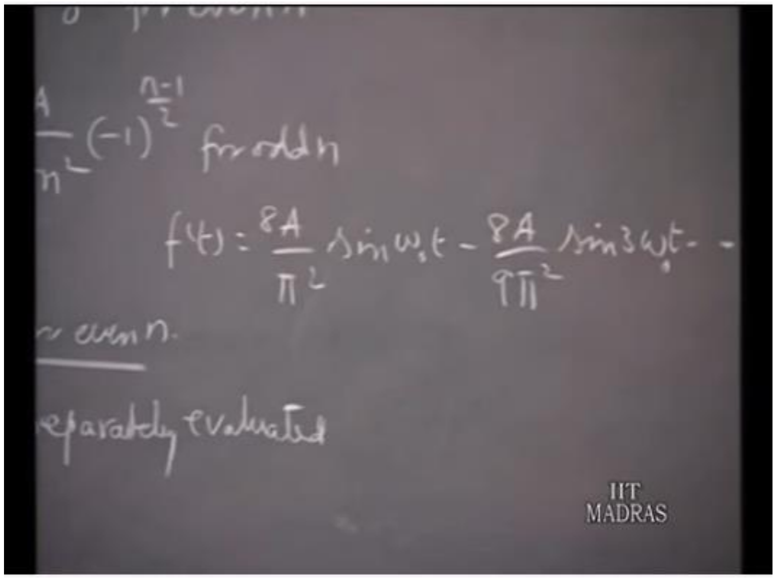

So, finally, f of t will be 8 A by pi square sin omega not t minus 8 A by 9 pi square sin 3 omega not t etcetera. Now, let us see what happens. If you have a constant term here, when you take the derivative, the constant term disappears. Therefore, a d c term gets lost here and therefore you cannot expect to that term to be present here.

So, the dc term in the original function has to be separately evaluated because, that information is lost when you go to this. So, C not in this case has to be separately evaluated, you cannot obtain from this information. Now, what other terms can possibly get lost? If there is a constant term here, that can get lost when you take the derivative, but a constant term here means; that you have a term like k t here.

Unless there is a term like k t you, cannot get a constant term in the derivative And if you have a term like k t in the periodic function that, completely spoils the periodicity of the function. Therefore, if you have periodic function you cannot have a k t term here. So, you cannot have a constant term here, as a derivative of this.

Therefore, there is no trouble on that score. The only trouble which can come because this, C not that is here can get lost and that has to be separately evaluated and in this case the constant term is 0. So, this example illustrates; how the Fourier coefficients can sometimes can be evaluated by taking the first or second derivative as the case may be and making use of a function which has only impulses as illustrated here.

So, in this lecture, what we have done is we have introduced ourselves to the concept of frequency spectrum, in particular the magnitude and phase spectra associated with this C n coefficients. We observed that these both are line spectra. The magnitude spectrum is an even function of n and the phase spectrum is an odd function of n and we have a series of examples.

We saw how this spectra can be constructed. In particular we have constructed this spectra of a train of impulses, the square wave and a train of pulses. And also we saw, how the facility provided by impulses in the integration procedures can be availed of, in situations where either the first derivative or the second derivative as this case contains entirely impulses.

So, in that case, it appears to take the derivative and find out the Fourier Series for that and use that information to find out, the Fourier Series of the original waveform as done in this case.