Networks and Systems
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Lecture- 22
Exponential Fourier Series

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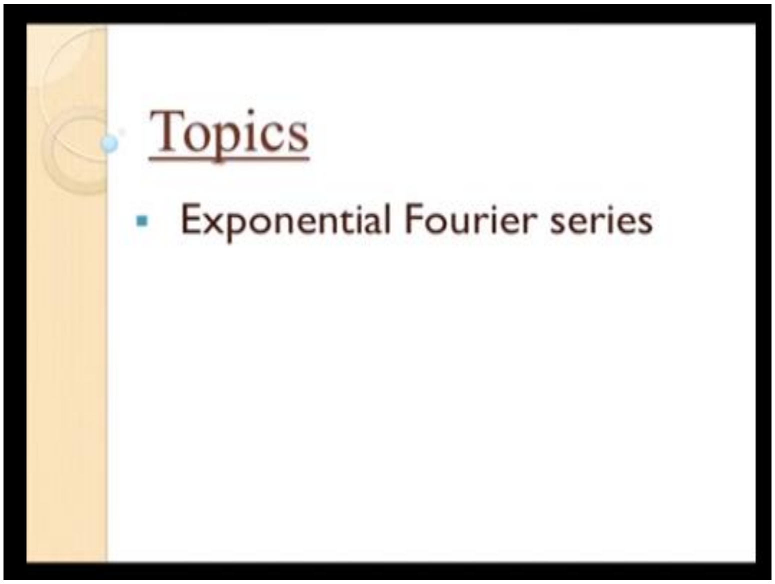

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We had set up the Fourier series earlier, in terms of trigonometric functions. Now, there is an alternative way of setting up the Fourier series. This will be in terms of exponential functions. Let us proceed to do that.

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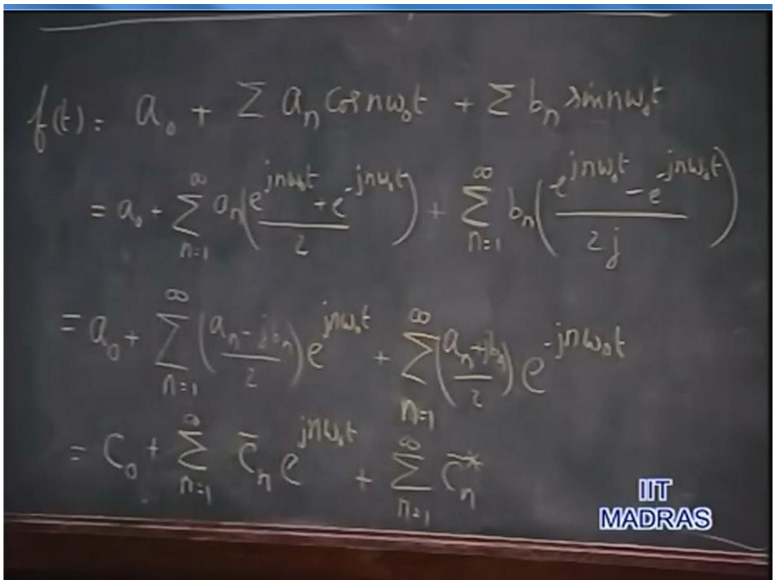

In terms of exponential functions, you recall that f(t). We had written in terms of trigonometric functions as a not plus the sum of cosine terms plus b n sine n omega not t sum of sine terms.

Now, we can express sine and cosine terms, in terms of exponential functions. So, I can write this as an e to the power of j n omega not t plus e to the power of minus j n omega not t divided by 2 plus b n e to the power of jn omega not t minus e to the power of minus j n omega not t divided by z j n from 1 to infinity.

Now, we have e to the power of j n omega not t terms here, as well as here. So, let us group them together, so that we write these series, in terms of exponential functions. So, you have a not plus what is the coefficient of e to the power of j n omega not t? An upon 2 plus bn upon 2j. So, I can write this as an minus j b n upon 2. This is the coefficient of e to the power of j n omega not t n ranging from 1 to infinity.

We also have terms like e to the power of minus j n omega not t and what is it is coefficient, a n upon 2 and then, because there is a negative sign here, it is plus j b n upon 2 n from 1 to infinity. So, what we have done is, express f(t) not in terms of trigonometric functions. But, in terms of exponential functions of the type a j, e to the power of j n omega not t.

Now, we would like to in the context of expansion, in terms of Fourier exponential functions. We would like to indicate the coefficients in a different way. So, we will call this C n, a complex number C n and this will be C n conjugate and since we are calling the exponents in the coefficients, giving the symbol C for the various coefficients, we may as well call this C not.

Therefore, I can write this as c not plus n from 1 to infinity just to indicate that, this is a complex number c v n i put a line on top. That is a complex coefficient. E to the power of j n omega not plus n from 1 to infinity. This is C n conjugate, because this is an minus j b n upon 2 is c n, this is it is conjugate the angle imaginary part has it is sign reversed, e to the power of minus j n omega not t.

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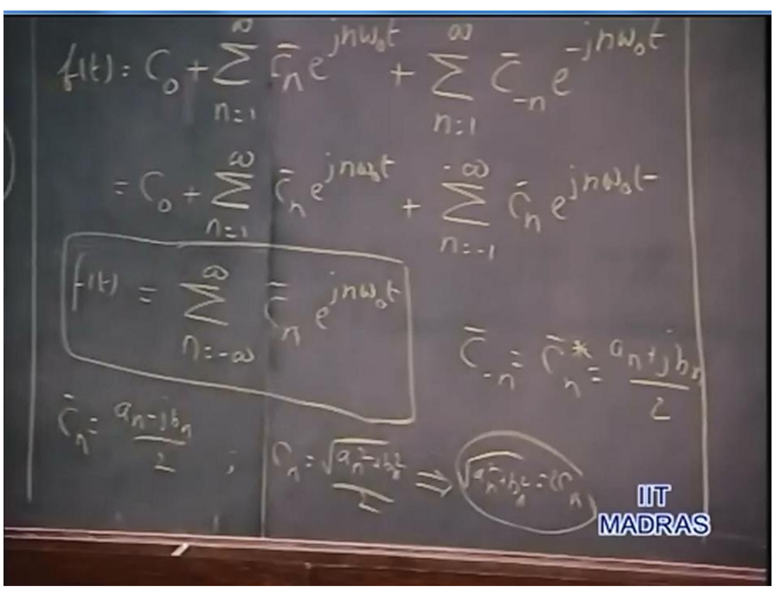

So, let us define C n star as c minus n. Then, we have f(t) as C not plus C n e to the power of j n omega not t n ranging from 1 to infinity. C minus n e to the power of minus j n omega not t. Now, we would like to combine these two, summations into 1. That can easily be done by substituting minus sign for n here, in which case, where you substitute minus n for n here.

Then, I can write this as C not n from 1 to infinity C n e to the power of j n omega not t. So, when you change the dummy sign of summation – n/n. Therefore, this n goes from minus 1 to minus infinity, this is C n e to the power of j n omega not t. Combining these two, I can write n from minus infinity to plus infinity, including this 0-1 to infinity minus infinity to minus 10 and 1 to infinity, C n e to the power of j n omega not t.

This is f(t), where we observe that c minus n is c n conjugate, which is a n plus j b n upon 2 because, C n conjugate is a n plus j b n by 2. So, this is a very compact way of representing the Fourier series you do not have groups of terms like a, coefficients and b. Coefficients you have only to deal with a single set of coefficients C n. The question is, how is the C n related to a n and b n, that we already have seen. That C n equals a n minus j b n upon 2.

And in anticipation of this, suppose I take the magnitude of C n, it is C n square root of a n square plus b n square upon 2. Or in other words, square root of an square plus b n square equals 2 C n. So, in anticipation of this notation only, when we put the Fourier series expansion of a function in trigonometric functions. We said the amplitude of the n th harmonic is 2 C n, rather than C n. It is in anticipation of this formula.

Now, the next question that we would like to ask is, how do we evaluate this C n coefficients? What is the formula for that, in the same way as we have done, for the trigonometric functions. How do we get this complex coefficients C n, directly from f(t).
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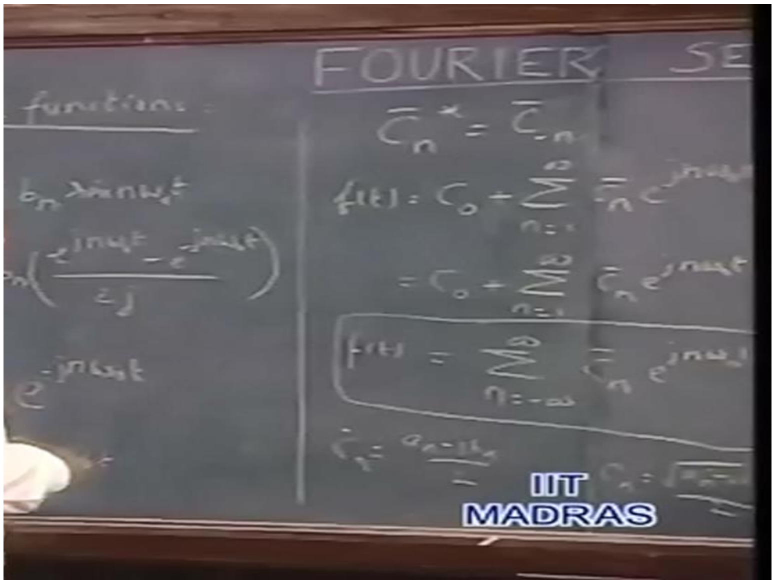

This can easily be derived as follows. C n we know is a n minus j b n upon 2 and we know the formulas for a n and b n, let us substitute that. This is half of a n is 2 upon t not f(t) cos n omega not t d t over a period and for d n, I can write 2 upon t not f(t) sine n omega not t d t. This j term comes from here. This half of course is conclud here. Now, these two integrals can be combined. First of all, I can take 2 by T not term outside.

So, 1 upon t not 0 to t not f(t) cos n omega not t minus j sine n omega not t and this we know is e to the power of j n omega not t. So, the integral that needs to be carried out to evaluate C n is like this. This is really the average of f(t) multiplied by e to the power of minus j n omega not t.

What do we observe here and what are the merits of this exponential function, the exponential form of the Fourier series. First of all we observe that, we have only one single formula for evaluating the various Fourier coefficients. We do not have separate formulas for a 0, a n and b n. And the notation is very compact.

More importantly, when we extend this concept of Fourier expansion of periodic functions to a periodic functions. What we will refer to as Fourier integral concept, which we will take up later. There these expressions for C n can be in a more straight forward fashion, extended to the Fourier integral concept, than you had persisted with a n and j b n. So, we have a single formula valid for all n.

The notation is compact and the notation can be extended in the Fourier integral quite conveniently. That is the important thing. So, in the Fourier expansion for this, we note that each term by itself, may not convey to us any physical signal. Because, when you substitute a value real value of time, this does not by each term by itself will not yield a real value of the function.

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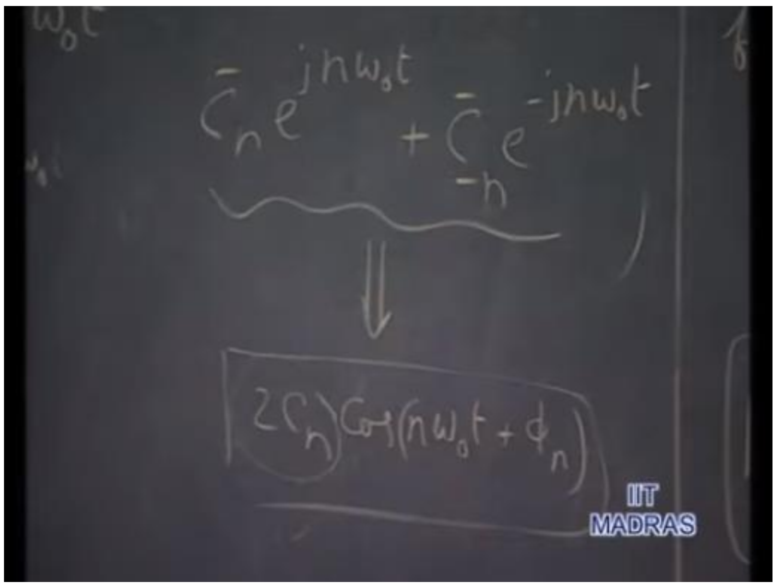

However, we have to bear in mind, that if you take C n e to the power of j n omega not t. And combine this with C minus n e to the power of minus j n omega not t. These two together will lead to a real function of time a sinusoid, which will be 2 C n cos n omega not t plus theta n. You can easily show that. So, the amplitude of the n th harmonic component is 2 C n and this real function of time comes by combining the two exponential terms, for plus n and minus n respectively.

So, individually it is not a physical signal, but when you combine these two, this is the conjugate of this you will get this. Now, let us work out an example, illustrating the exponential form of Fourier series.

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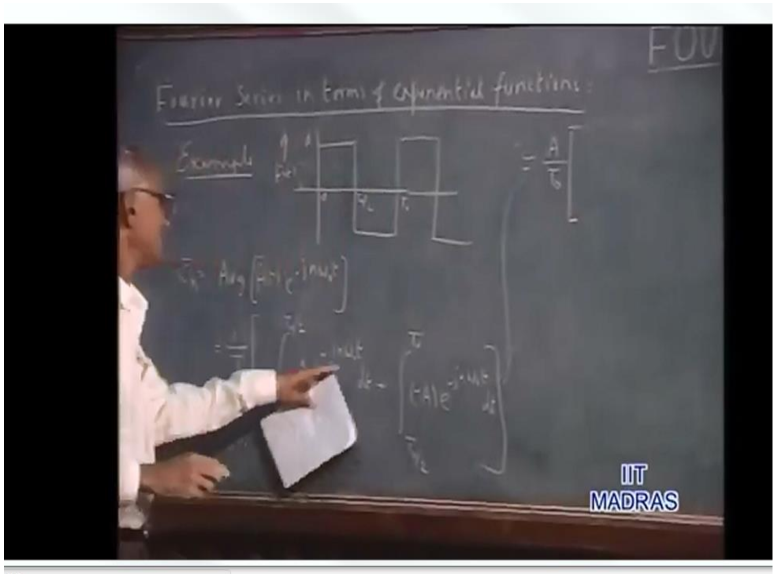

You take the same example that we have worked out earlier a square wave because, we can compare the results, so this is f(t). What we would like to find out is the Fourier series expansion in exponential form. So, C n would be easy way to remember would be average of f(t) multiplied by e to the power of minus j n omega not t. This is what we have to find out.

So, that will be 1 over t not and to find out the integral of product, we split this integral into two parts one from 0 to t not upon 2 and other t not upon 2 to t not, because the value of the function changes in these two intervals. Therefore, I can write this as 1 over t not 0 to t not upon 2. The value of the function is A e to the power of minus j n omega not t d t plus T not upon 2 to T not and in this interval the value of the function is minus A minus j n omega not t d t.

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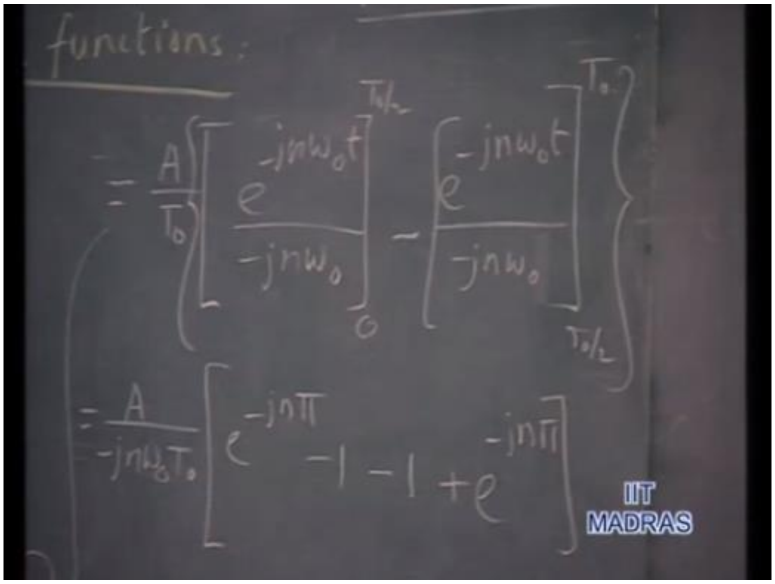

So, I take the A outside, A upon t not. The first integral will yield e to the power of minus j n omega not t by minus j n omega not minus, because of this minus sign here e to the power of minus j n omega not t divided by minus j n omega not and the first integral is evaluated between 0 and t not upon 2. The second integral is evaluated between t not upon 2 and t not.

So, you can take out minus j n omega not outside and if you evaluate this, it will be e to the power of minus j n at the upper limit, omega not t not upon 2 is pi minus and j n pi at the lower limit it is 1 minus 1 and minus at the upper limit, it is minus j n omega not t not, it is 2 pi. Therefore, e to the power of j n 2 pi, it is equivalent to 1, because it is an integral multiple of pi.

Therefore, 2 pi minus 1 and the lower limit, because of the minus sign, it becomes plus e to the power of minus j n pi. Because, T not upon 2 multiplied by omega not equals pi, leads to pi.

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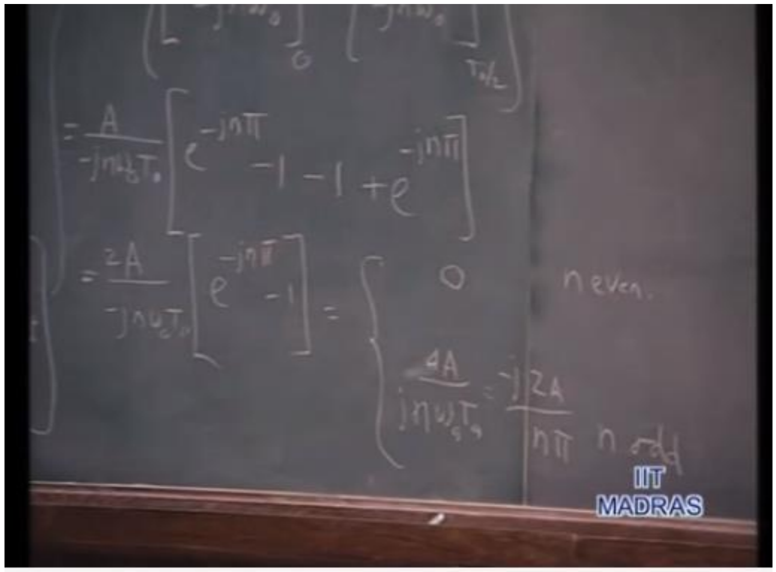

Therefore, this can be written as minus j n omega not t not this you can be combined to 2 times e to the power of minus j n pi and minus 2. Therefore, 2 a times e to the power of minus a n pi minus 1. And e to the power of minus j n pi is either plus 1 or minus 1, depending upon the value of n.

Therefore, if n is even this becomes 1. Therefore, this leads to 0 1 minus 1. If n is odd then e to the power of the angle is an odd multiple of pi. Therefore this minus 1, so you have minus 2 minus 1 minus 2. Therefore, it will become this j can be taken out. Therefore, it will become minus 4 A divided by n omega not t not and a j in front and omega not T not is 2 pi. Therefore, this will be minus j 2 A by n pi for n not.

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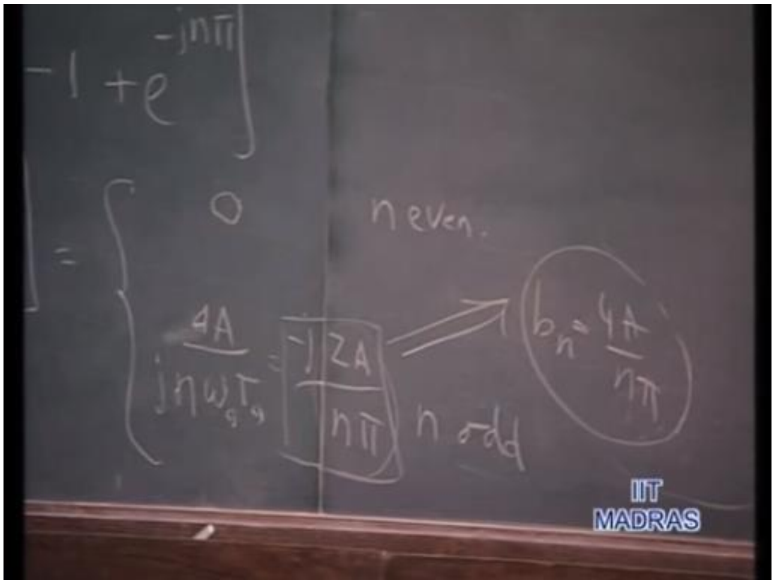

And since we know that C n equals a n minus j b n by 2. Therefore, what we now see is b n upon 2 is 2 A by n pi. So, from this we conclude that b n equals 4 A by n pi, a result which you have already obtained from the trigonometric form of Fourier. So, this ties up with that. C not of course, is 0 because the average value of this is 0.

And when we are carrying out this analysis, to find out the C n, it would always be advisable for us to arrive at the C not value independently, rather than the straight formula of substituting n because, sometimes when n equal to 0, it leads to some difficulties some degeneracy, because n may come in the denominator.

So, it is always advisable to calculate C not independently, rather than substituting n equals 0, in the general form. Sometimes it may work, sometimes it cannot. Now, we have discussed symmetry conditions in relation to the a and b coefficients. Now, what are the similar conditions, that are applicable to the C n coefficients. So, let us now discuss that.

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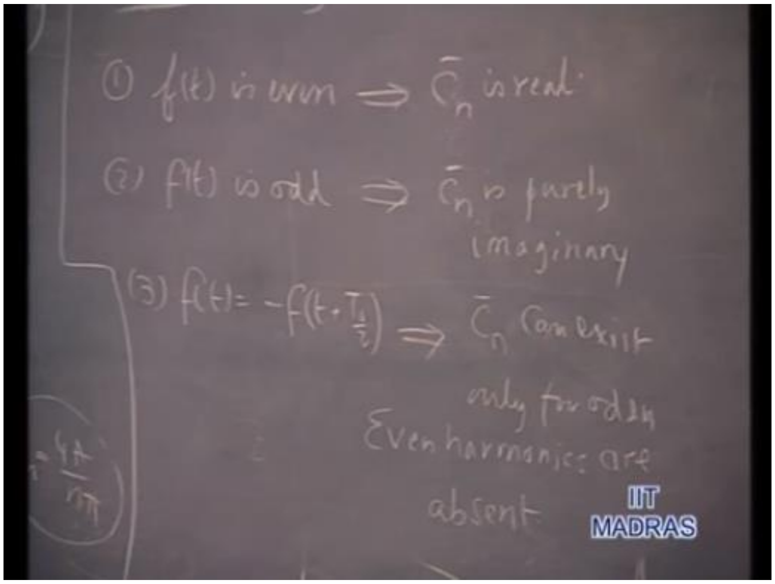

If f(t) is even, we said that only the cosine terms are present. That is an terms are present, not the b n terms. Since, we know that C n is a n minus j b n by 2, if f(t) is even it means that, C n is real. Because, C n is an minus j b n upon 2. So, if b n is absent, then C n is purely real. The angle associated with the complex number is 0. If f(t) is odd then of course, C n is purely imaginary.

We have the b n terms, but no a n terms and if f(t) exhibits half wave symmetry, then as before we have only the odd harmonics present. C n can exist only for odd n, even harmonics are absent. So, to summarize, what we have done in this lecture. We started with an example in which we have taken a half wave rectified sine wave, found out it is Fourier series, making effective use of the symmetric conditions.

We saw that a half wave sinusoid can be expressed, broken up into its even part and odd part. The odd part was a pure sine wave. The even part was a full wave rectified sine wave. Whatever waveform is there in one half cycle is reproduced and therefore, we have only the even harmonics present and we have found out the Fourier series expansion of the even part and odd part separately and use such a kind of waveform in a practical circuit consisting of a R and C, rectifier circuit and use that example to illustrate.

How we can make use of Fourier series in analyzing the steady state performance of a simple R C circuit. Then, we took up the question of Fourier series expansion, in terms of exponential functions. So, in terms of exponential functions we expressed f(t), as a summations of various terms each term being of the form C n e to the power of j n omega t, C n omega not t where C n is a complex coefficient in general, which is related to the a and b coefficients that we already talked about as C n equals a n minus j b n upon 2 and we found out that, the expression for calculating C of n is surprisingly simple.

It is simply the average of f(t) times e to the power of minus j n omega not t, valid for all values of n and we said that expansion of this form is useful for us, when we later go to the Fourier integral form, apart from its compactness and the fact that we have to calculate only one set of coefficients that is C n coefficients instead of having to calculate a n and j b n separately.

The price we have to pay for that is of course, we have to use complex algebra because, here is a complex number, whereas if you are calculating a n and b n, we have to deal with real functions only. So, we have a price for it, but nevertheless it leads to a very compact notation. So, we have two alternative ways of expressing Fourier series.

One in terms of the a n and b n coefficients, other in terms of C n and one can always convert, one set of coefficients into the other. We took up the square wave as an illustration and showed that the expansion in terms of C n will of course, naturally as we expect, leads to the same results but, of course, C n now is an imaginary quantity.

That means, this is related to b n which is 4 A by n pi which we already found out and lastly we talked about symmetric conditions, in terms of the Fourier coefficients of the trigonometric expansion, of the exponential expansion. And we said if f(t) is even, C n happens to be real. If f(t) is odd, C n is purely imaginary. And the half wave symmetry, which we already discussed with does not yield any new surprising results.

Of course, we know that once f(t) equals minus of f(t) plus t not upon 2, which means that the wave is reproduced with a negative sign, in the succeeding half cycle. Then, only eve harmonics are present and therefore, C n can exist only for odd n. Some of these, for some odd values of n, C n may not exist also. But, if at all it exists, it can exist only for odd values of n.