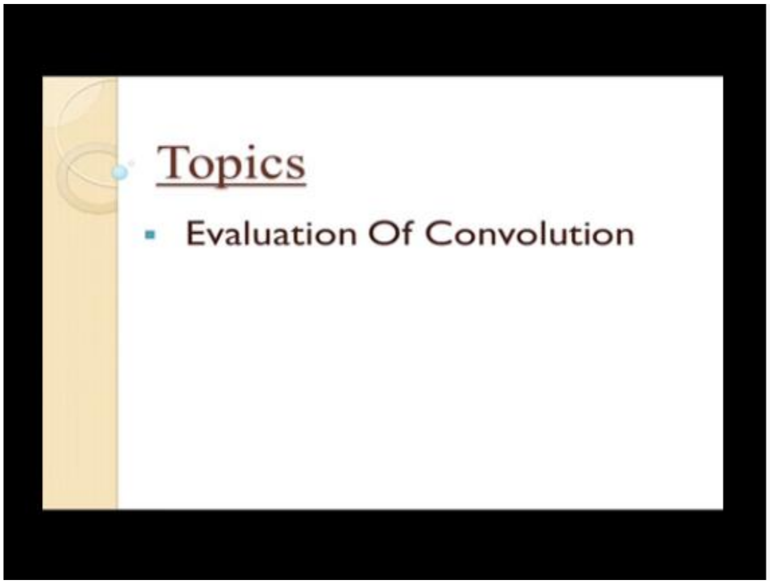
Networks and Systems

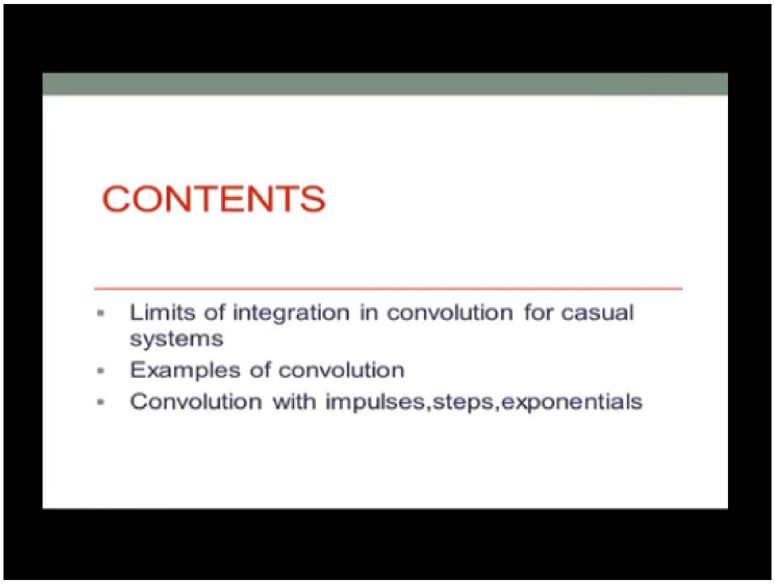
Prof. V.G.K. Murti
Department of Electronics Engineering
Indian Institute of Technology – Madras

Lecture-15
Evaluating the Convolution Integral

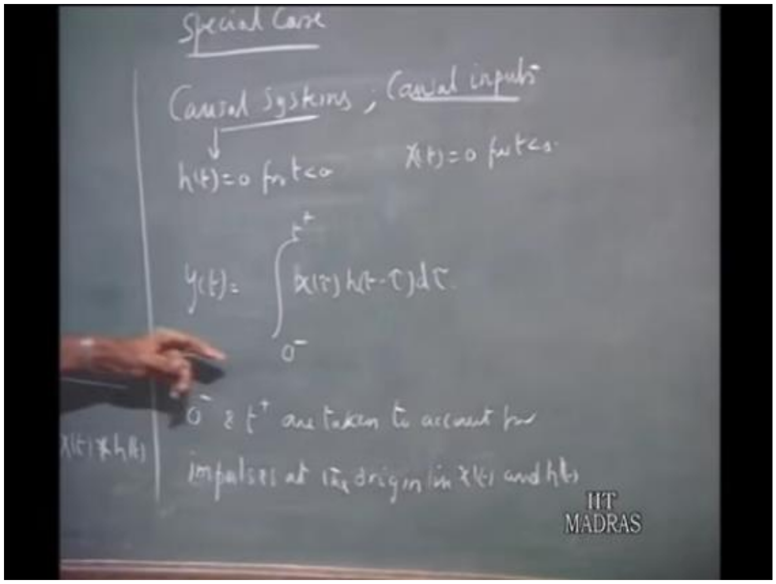
(Refer Slide Time: 00:15)



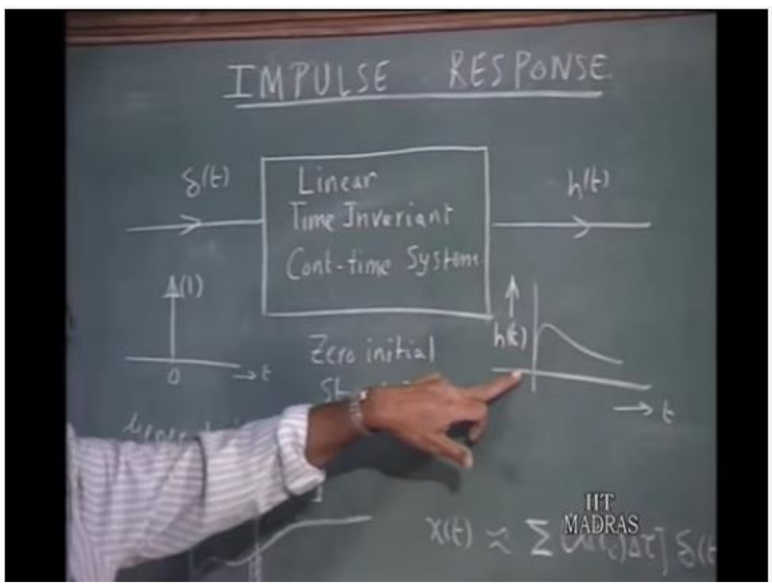
(Refer Slide Time: 00:20)



(Refer Slide Time: 00:45)



(Refer Slide Time: 01:15)



Now, let us look at these limits. This is the general expression for the response y t for any given x to u, any given x t. So, how are, when we talk about causal systems special case of causal systems and causal signals or causal inputs. If you take that then what are the consequences?

If the system is causal then an impulse x delta tou given here cannot have an impulse response for negative values of time. That means h t will be 0 for t less than 0. So, if the system is causal the impulse response must be 0 for negative values of time. So, the consequence of this causal system is: h t is 0 for t less than 0.

If the input is a causal causal input that means x t is 0 for t less than 0. Because of these 2 restrictions this impulse convolution integral that you are having here minus infinity to plus infinity these limits can be now simplified. Because x tou is going to be 0 for negative values of tou we need not start the integration from minus infinity. You can start from 0.

Similarly, tou is a running variable. When tou exceeds t then this h of t minus tou the argument becomes negative Therefore, that is going to be 0. Therefore, this integration needs to be carried out only for up to tou equals t instead of infinity. So, this lower limit and upper limit can be reduced now to 0 t. Then we can write x tou h of t minus tou.

So, for the special case of a causal system with a causal input, we need to carry out this integration only between the limits 0 and t not between minus infinity and plus infinity. In particular we, if x t for example may have an impulse at the origin. So, we must take that effect to the impulse to the account.

Therefore, it would be advisable for us to start the integration from 0 minus so that the impulse is fully taken into account in our calculation. Similarly, the impulse response may have an impulse at the origin. It may be possible that if you have an, you give an impulse then there may be impulse present in the response also.

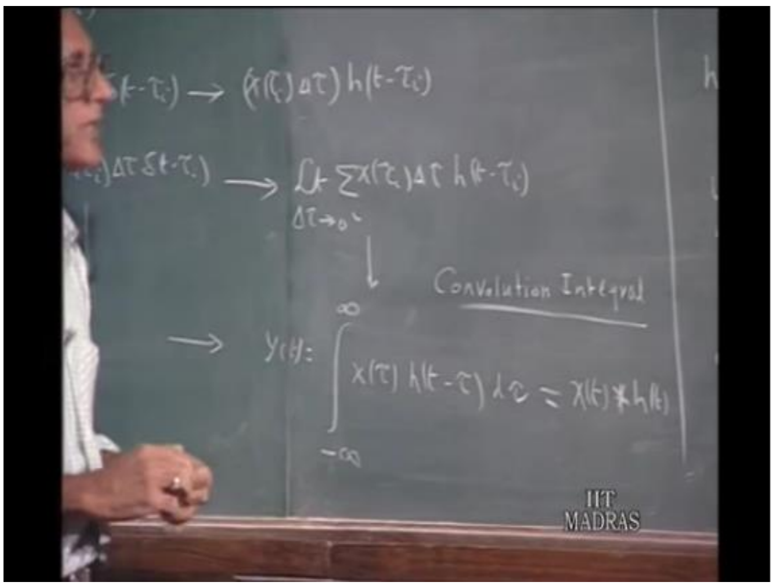
That means h of 0 may also have an impulse. Therefore, to take that into account you can take this t plus. So, 0 minus and t plus are taken to account for impulses at the origin in x t at 1 hand and h t at the other.

If there are no such impulses then, you can always take 0 and t. But, if you have impulses present it is advisable to take from 0 minus to t plus. That is how it goes. So, to summarize this discussion up to this point is given the response of a linear time invariant continuous system to an impulse, we describe this h of t h of t is called an impulse response.

This gives complete information about the system to enable us to, calculate the response to any input. So, any given input x t can be regarded as the summation of several impulses, as we have taken like this and each impulse gives rise to corresponding response, which can be calculated in terms of the impulse response.

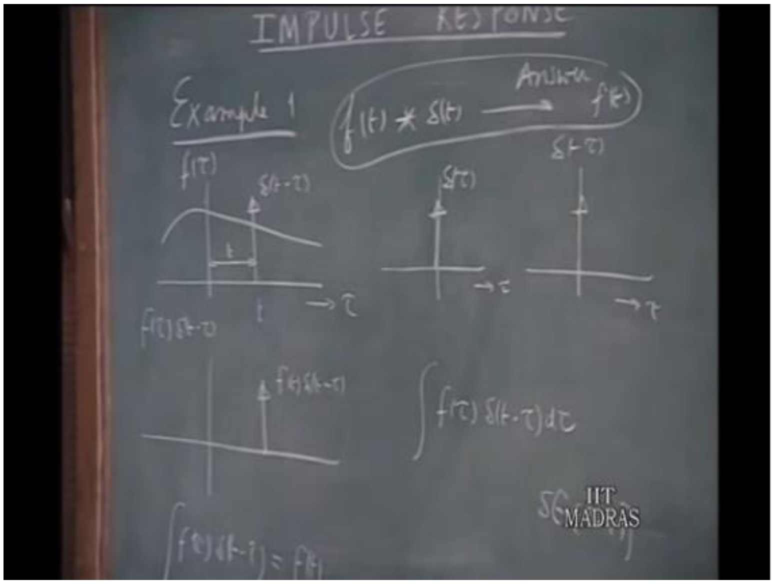
Putting all of them together, we come to the integration y of t is given by this integral which is called the convolution integral and the meaning of convolution we have seen.

(Refer Slide Time: 04:43)



And in particular when we are talking about causal systems and causal inputs, the limits need not be minus infinity to plus infinity as it would be in a general case. But, in the particular case of these pipe systems from 0 to t, in particular we may have impulses at the origin both in h of t and x of t. To take care of them, you must take the integration from 0 minus to t plus. Otherwise, only half the impulse will be taken into account.

(Refer Slide Time: 05:13)



Now, let us work out a few examples to illustrate these ideas. Let us take an example to illustrate the idea of convolution. Let us take a general function f of t. Try to convolve it into delta t and see what happens. So, you have 2 time functions and since, the final convolution is a function of time and you are doing some integration, it would be advisable for us to involve another variable which gets integrated in the process.

So, let me say that this is tou and this is f of tou. Now, delta tou is this. So, delta minus tou is also the same thing because, it is an even function. So, if you fold it becomes it becomes the same thing delta tou. So that means, delta tou is now here delta minus tou and in order to, what is it that we want to do? We want to take f tou delta t minus tou d tou and you want to integrate that.

So, instead of delta tou you must take delta t minus tou. That means you must take delta minus of t minus tou tou minus t. That means you must advance this delta by an amount equal to t. So, if you want to calculate the product or the convolution at a value t, you shift it by an amount.

So, this will be delta t minus tou, where this is the value of t. So, this shift depends upon the particular value t at which you want to evaluate. For 1 second you shift it by 1 second.

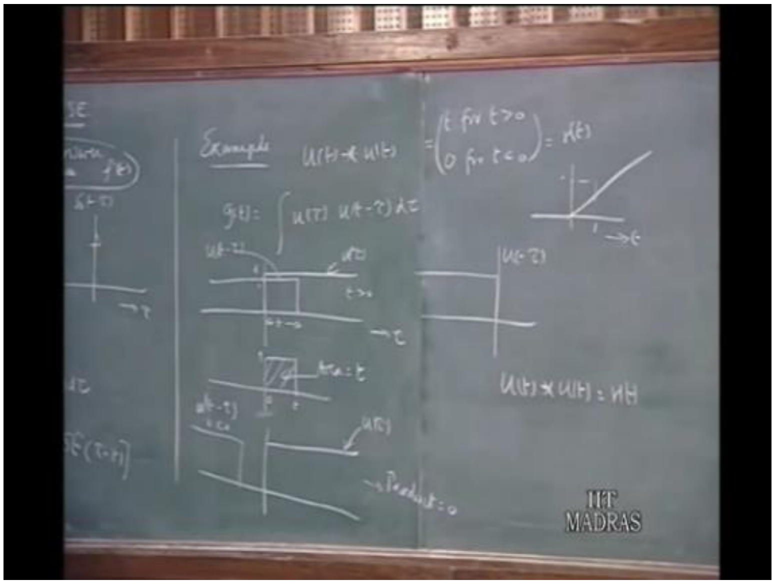
If, you want to evaluate it at 2 seconds it shift it by 2 seconds and so on and so forth. Now, you have folded this signal, then shifted this signal, then multiply this 2 out.

If you multiply this 2 out, you have f tou multiplied by delta t minus tou what you get is because delta is 0 everywhere else except at this point, you have a small delta here whose value is f at the value t delta t minus tou. And when you integrate that, f tou delta t minus tou that will be equal to, the area under the curve is f of t. So, the result is that, f t convolved with delta t is f t itself. Answer is f of t itself.

This is a very interesting result that the delta function when it convolves with any time function yields the same time function identically without any change. It is something like what of course, you may not have delta in priority, but if you have a narrow slit for example, instead of a delta function, suppose you are having a small slit like this and you do the same convolution then, you get an output which is very closely resembling this f of t.

Something like you are for example, the sound track on a film passing through a sensing head or a sound signal on a tape passing through a head. Then if you have a narrow slit then, you pass it through this and if our response is something like similar to the product and then integration over the small area then, you get more or less essentially the same signal as the output, something similar to that.

(Refer Slide Time: 08:51)



Now, let us take a second example. Let us see what happens when, a step function is convolved with another step function. u of t convolved with u of t. So, the result is suppose result if, this is called g of t. According to the formula this will be u tou u of t minus tou that is the second signal d tou and you integrate it over the limits which are appropriate to this. Let us see what they are.

Now, you take first of all u tou. This is u tou. This is 0 here and this is equal to 1. Now, u of minus to take care to picturize this u of tou is like this, u of minus tou will be something like this. This is u of minus tou. But then, you do not want u of minus tou. You must shift it by an amount equal to t. u of t minus tou would be something like this. That will be u of t minus tou where, this amount is equal to t.

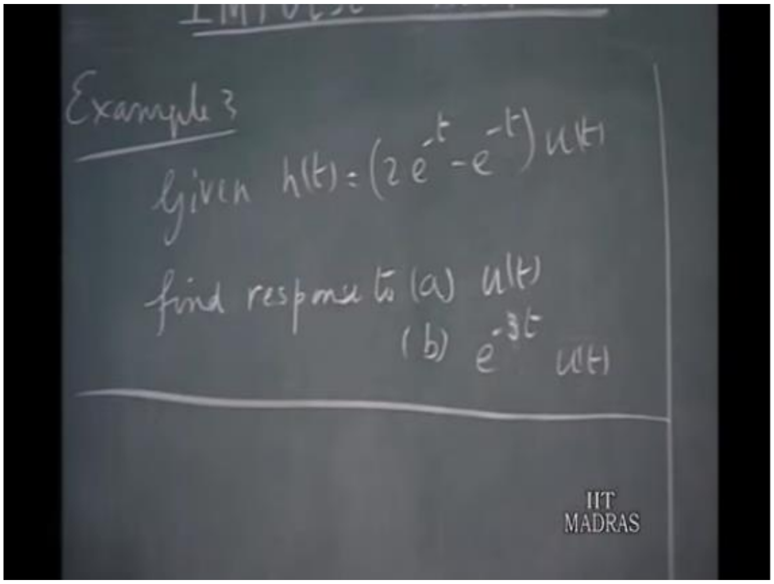
So, you multiply these 2. This is also equal to 1. This is multiplied by this. You will get this curve 0 to t. So, the area under this curve, after all when you want to integrate this it means you are finding out the area under this curve. The area under this curve is t units. Therefore, the result of u t star u star with u t or the convolution of that is going to be t for positive t.

So, we can say this is t for t greater than 0 because we have shifted in the forward direction. Now, let us imagine what happens for negative t. This is t what you have pictured here, is for t greater than 0. What happens when t is less than 0? So, this is u tou. This is u of minus tou. For negative t you shift it in the other direction. Therefore, you have a curve like this that will be u t minus tou for t less than 0.

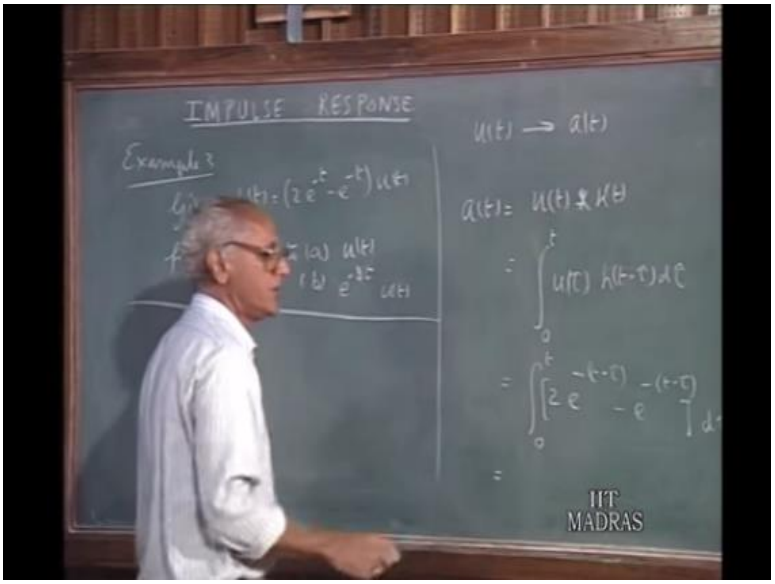
So, when you make the product of these 2 it will be 0 because, when this is non 0 the other is 0. When this is non 0 this function is 0. Therefore, the product is 0. Product of u tou times u t minus tou is going to be 0 identically. Therefore, area is going is going to be 0. Therefore, the convolution product is 0. So, the product is described by r of t. That is a ramp function.

That means u t it will be like this. If it is 1 this is 1. So, the conclusion is u t star u t convolved with u t equals ramp function. So, the idea of convolution is relatively easily seen in simple cases, by plotting these graphs in this manner. Again key words are: fold, shift, multiply and integrate. Integrate means finding out the area under the curve. For simple geometries the area to the curve can be easily found out without doing integration and you can picture what is the final result it is going to be using these concepts.

(Refer Slide Time: 12:54)



(Refer Slide Time: 13:43)



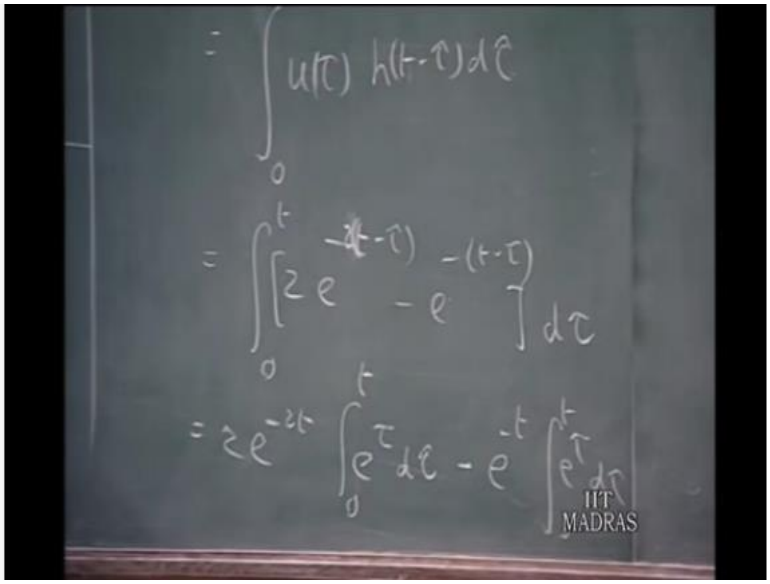
Now, 1 more example. Given: h of t for a type of system that we are talking about is 2 e to the power of minus t minus e to the power of minus t u t, for an initially relaxed linear time invariant continuous time system. Find response to a u t b e to the power of minus 3 t u t. So that means, we are asked to find out using this impulse response the response to 2 arbitrary inputs.

Now, let us see the first 1. If u of t gives rise to a response a t then, a t can be found out by convolving u of t with the h of t. So, I will write this as the input now is u t and that is convolved with the impulse response h t. So, the meaning of this would be both now this is a causal system.

The impulse response is 0 for negative values of time and the input is also causal. Therefore, we can straight away say this is 0 to t u t. So, I will call it u tou h t minus tou d tou. Then, this can be regarded as 0 to t. In the interval 0 to t when tou takes from positive values, this is going to be 1 anyway.

Therefore, we can disregard this. So, h we can write this 2 e minus of t minus tou minus e of minus of t minus tou d tou. Again u tou is there, u of t minus tou is there u of t minus tou is going to be 1 in the interval from 0 to t because, as long as tou is less than t then, t minus tou is going to be positive. Therefore, both these u tou here and u of t minus tou that is coming here are going to be 1 in the range of integration.

(Refer Slide Time: 15:11)

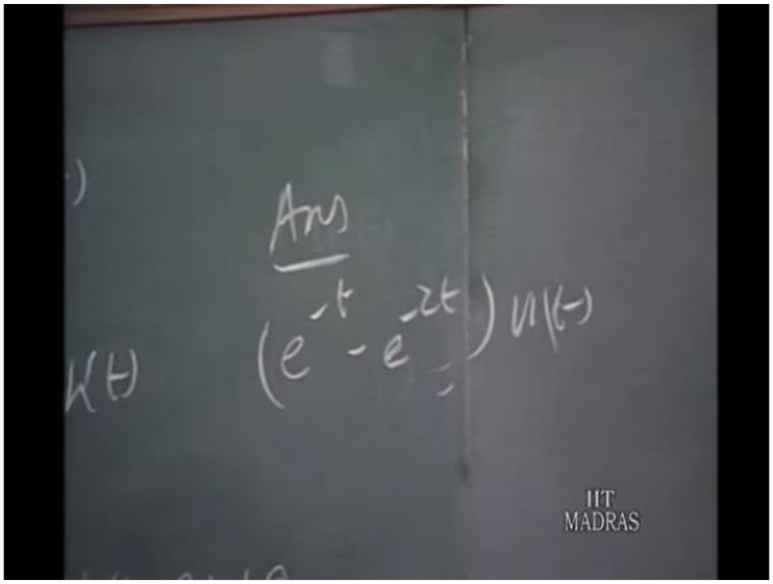


Therefore they can be omitted. So, this will be what you are having here and this can be written as u in the integration e to the power of minus t comes out with the first term. Therefore, you will have 2 e to the power of minus t and then when you are integrating e to the power of, I should have put this minus 2 t.

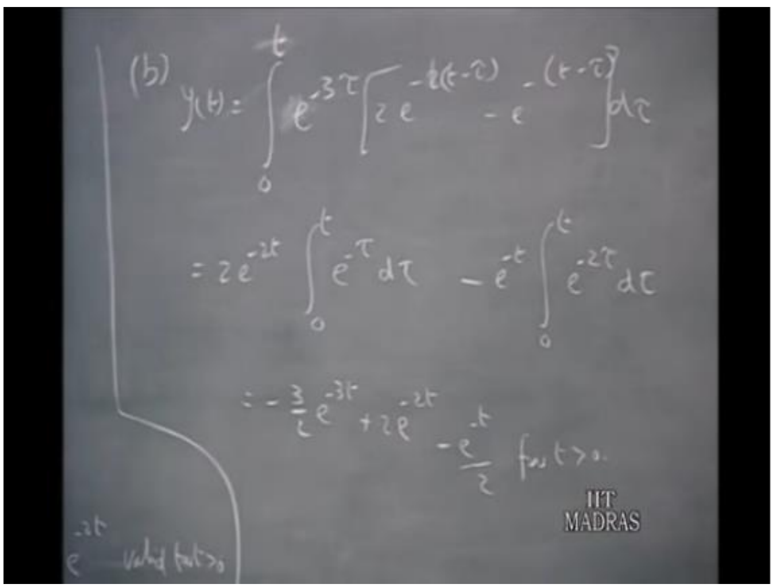
Kindly note this. That is what this is minus of 2 of t minus tou. Otherwise, there is no meaning here writing 2 t. 2 minus t minus tou. Therefore, you have 2 e to the power of minus 2 t and 0 to t e to the power of tou d tou. That is 1 tou and then minus e to the power of minus t that is, the common factor that comes out.e to the power of tou d tou 0 to t.

You can carry out these 2 integrations and final result will be e to the power of minus t and minus e to the power of minus 2t. This is valid for t greater than 0. So, a of t is now obtained as answer is e to the power of minus t minus e to the power of minus 2t. That is the answer as far as the first part is concerned.

(Refer Slide Time: 16:45)



(Refer Slide Time: 16:57)



The second part the y t is now 0 to t again. u tou not u tou. This time it is e to the power of minus 3 tou that is, the input. This is going to be 1 for the range of integration. h t minus tou that is 2 e to the power of minus 2t minus tou minus e to the power of minus t minus tou.

That is the integration that is to be done and working out as before, this is integration with respect to tou. Therefore, e to the power of minus 2t can be brought outside 2 e to the power of minus 2t. Then, 0 to t e to the power of minus 3 tou and e to the power of minus plus 2 tou.

You have got e to the power of minus tou d tou and then, for the second part you have e to the power of minus t that can be pulled outside and the integral from 0 to t e to the power of minus 3 tou and e to the power of plus tou. Therefore, e to the power of minus 2 tou d tou is what you are getting.

And if you carry out this work the answer will be finally, minus 3 upon 2 e to the power of minus 3t plus 2 e to the power of minus 2t minus e to the power of minus t upon 2 for t greater than 0. Of course, for t less than 0 it is going to be 0.

Notice that in this particular case, we have found out the complete solution. After all the input now is e to the power of minus 3t. That means the force part of the response will have only this particular complex frequency which is equal to minus 3.

So, this is the first part of the response. This is the response which you would obtain by taking the particular integral solution of the differential equation. And this part is the natural response which contains the response when there is no particular input, no sustained inputs.

Actually as a matter of fact you can see, the impulse response what do you mean by impulse response? An impulse appears at t equals 0 and afterwards there is no input. That means whatever follows is kind of natural response of the system. Therefore, the impulse response here, whatever frequencies are present here is the natural frequencies of the system.

And the force response for e to the power of minus 3t will have also e to the power of minus 3t. So, you can recognize this to be the force response for the particular integral solution and this is the natural response or what comes as the complementary solution of the differential equation.

So, at this stage let us we have, let us summarize what we have learnt so far. In the question of introductory system concepts, we have seen how systems can be analyzed particularly transient behavior of the systems.

We have looked at various methods the classical differential equation approach which, entails a lot of involved calculations to find out both the complementary function as well as the particular integral solution and even more so finding out the initial conditions in a differential equation of a large order.

Then we said equivalent information of the differential equation can be obtained, through the system function concept h of s or through the frequency response function h of j omega. We will exploit these 2 particular functions, when we are discussing the Laplace Transformation and Fourier Transformation techniques at a later point of time.

We also said that equivalent information can be provided by the impulse response method impulse response h of t or the step response a of t. We have looked in some detail only the impulse response as far as our previous discussion goes. A similar analysis can be given in terms of the step response as well.

Step response is called a of t as we have mentioned here, almost identically we can carry out the analysis. We have also some kind of convolution integral involved there also, but we will do this later after we study Laplace Transformation methods. So, impulse response and step response also can be used to characterize the linear system and here again we are calculating all this in time domain only.

So, these are all different ways of describing the input output relations of a linear system. The differential equation approach, system function approach both as a function of j omega or in time domain using either the impulse response or the step response sometimes called indicial response.

The step response is also called indicial response. Some of these techniques we will return to later, when we are talking about Laplace Transformation techniques. And so we will close our discussion of the introductory system concepts at this stage. Let me now give a set of examples for you to work out as a in the form of an exercise.