Digital Image Processing
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Module Number 01 Lecture Number 04
Signal Reconstruction Form Samples: Convolution Concept

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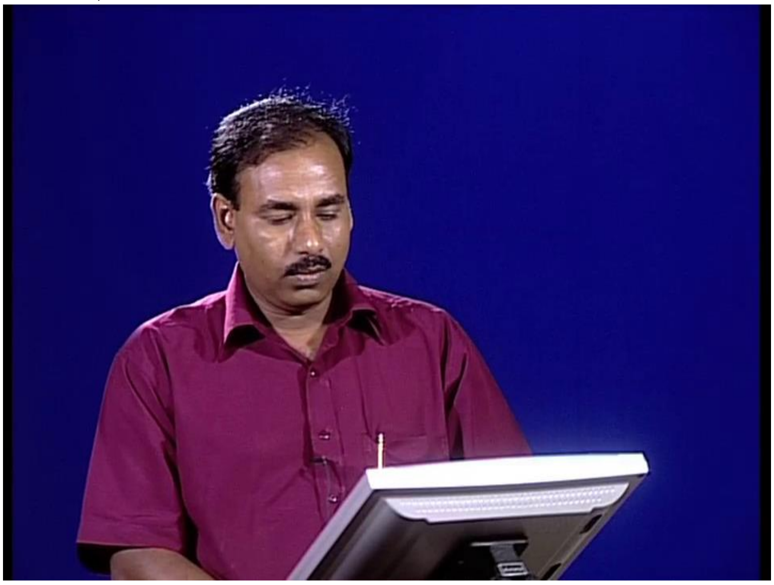

Welcome to the course on Digital Image Processing.

Convolution

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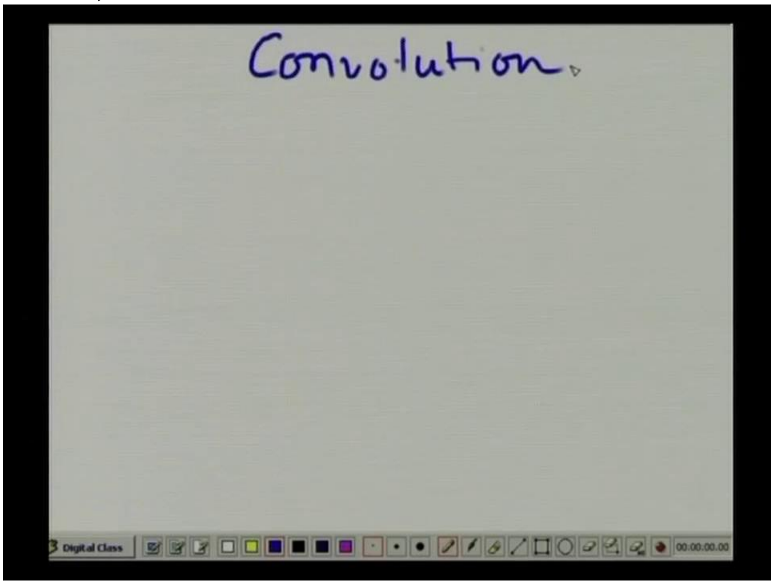

you will find that we have represented our sampled signal as "x s" t equal to x t multiplied by comb function t delta t, Ok.

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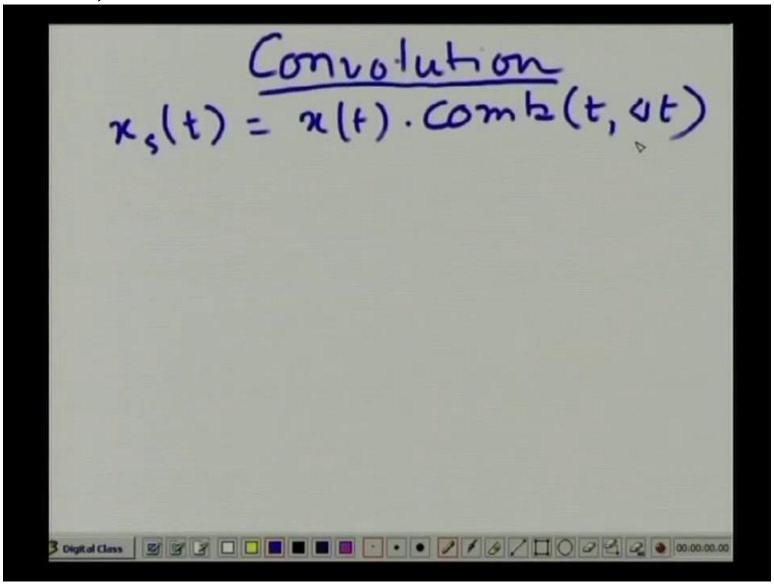

So what we are doing is we are taking 2 signals in time domain and we are multiplying these 2 signals. Now what will happen if we take Fourier Transform of these 2 signals? Or let us put it like this. I have 2 signals x t and I have another signal say h t. Both these signals are in the time domain. We define an operation called convolution which is defined as h t convolution with x t.

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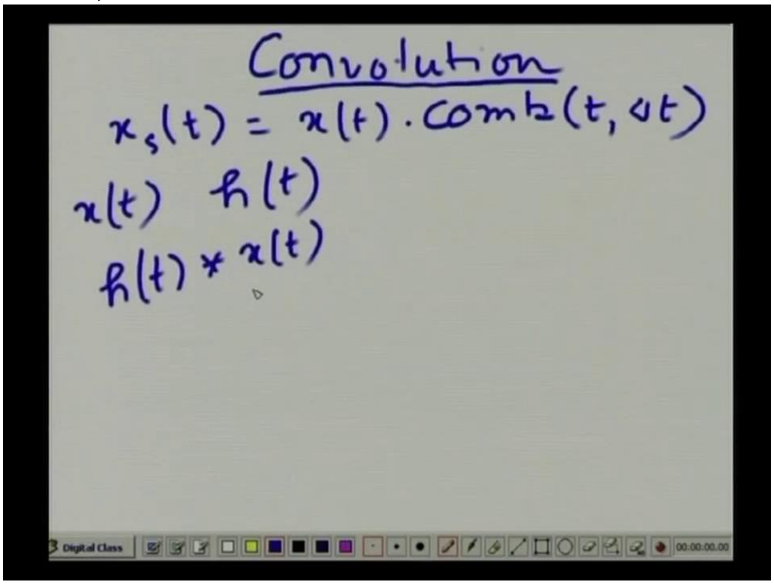

This convolution operation is represented as h of tau x of t minus tau d tau integration is taken over tau from minus infinity to infinity.

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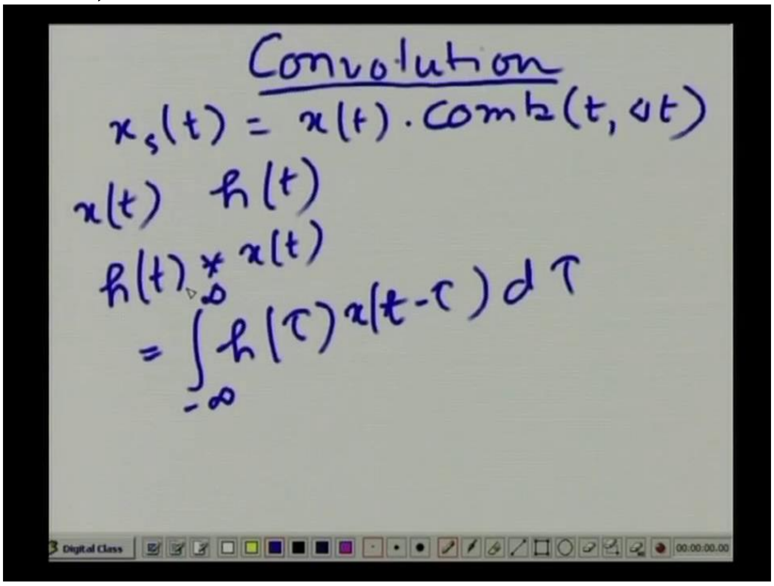

Now what does it mean? This means that whenever we want to take the convolution of two signals h t and x t then firstly what we are doing is, we are time-inverting the signal x t. So instead of taking x tau we are taking x of minus tau. So if I have 2 signals of this form, say h t is represented like this and we have a signal say x t which is represented like this

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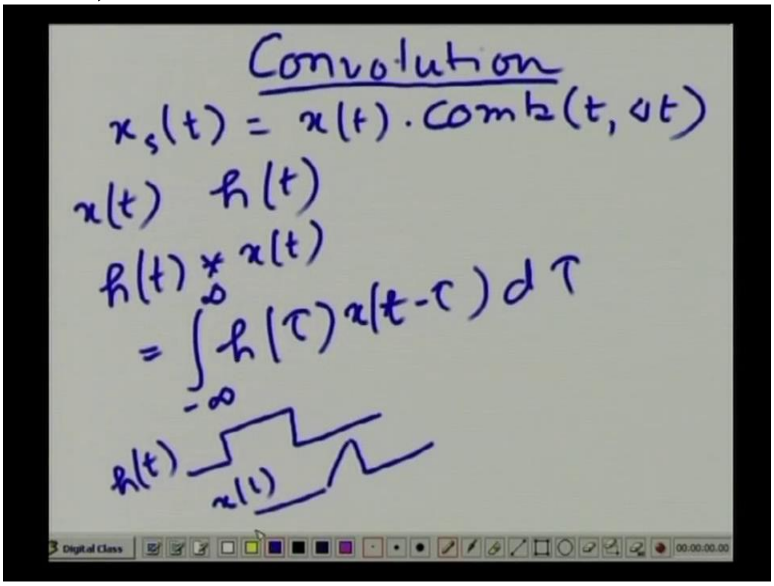

then what we have to do is, as our expression says that the convolution of h t x t is nothing but h tau x t minus tau d tau integration over minus infinity to infinity

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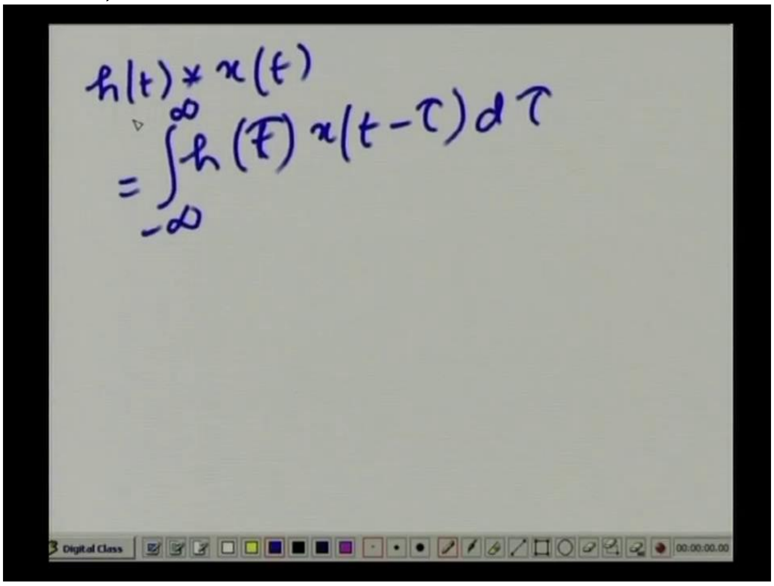

and h t is like this and x t is like this.

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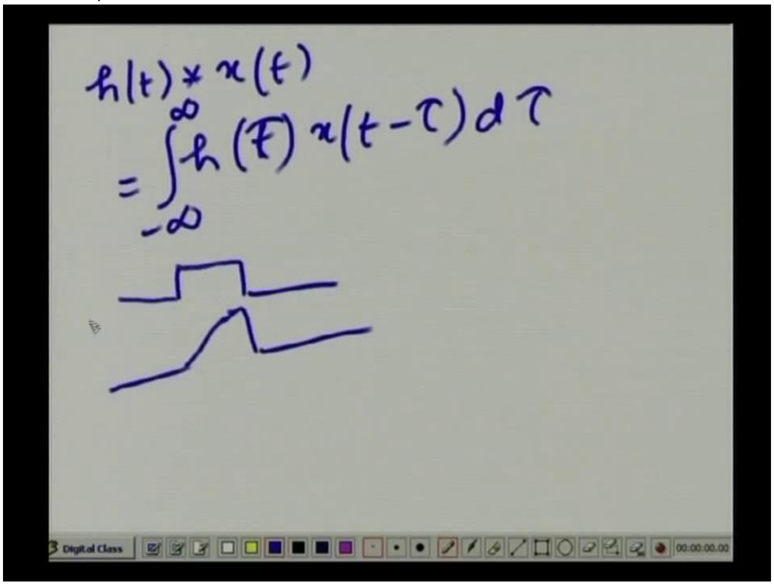

This is the h t and this is x t.

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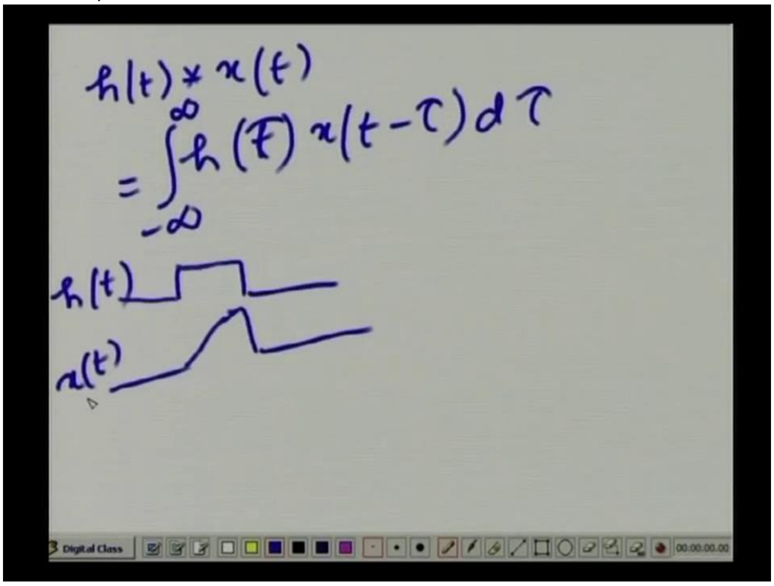

Then what we have to do is, for convolution purpose we are taking h of tau and x of minus tau. So if I take x of minus t, this function will be like this. So this is x of minus t.

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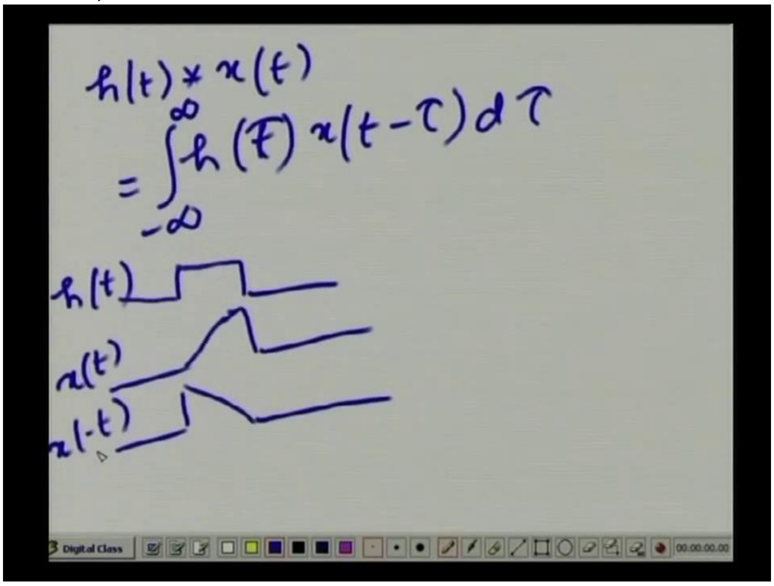

And for this integration, we have to take h of tau for a value of tau and x of minus tau, that has to be translated by this value t and then the corresponding values of h and x have to be multiplied and then you have to the integration from minus infinity to infinity.

So if I take an instance like this, Ok so at this point I want to find out what is the convolution value. Then I have to multiply the corresponding values of h with these values of x, each and every time instance I have to do the multiplication, then I have to integrate from minus infinity to infinity.

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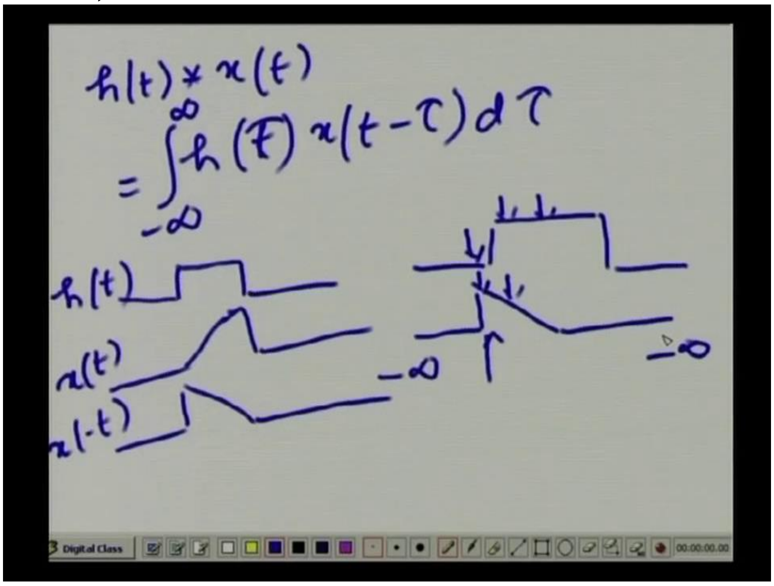

I will come to application of this a bit later. Now let us see that if we have a convoluted signal. Say we have h t which is convoluted with x t; and if I want to take Fourier Transform of this signal, then what we will get?

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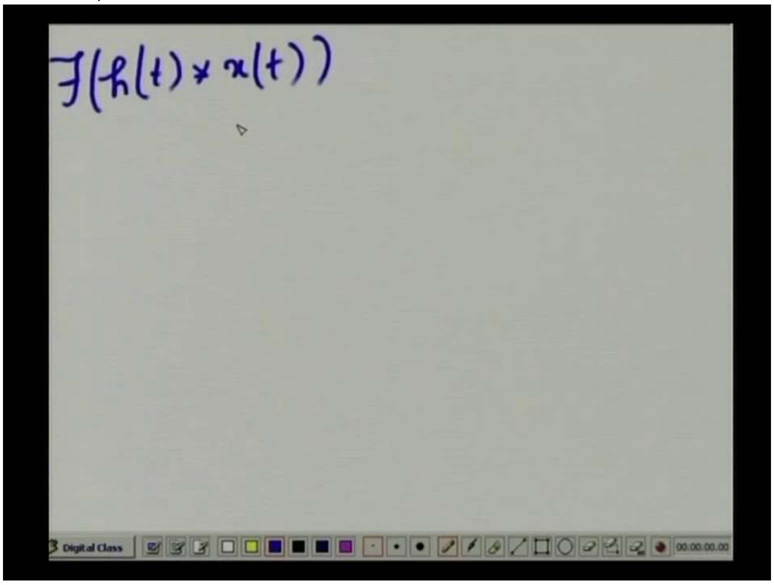

The Fourier Transform of this will be represented as h tau x of t minus tau d tau, so this is the convolution integration over tau from minus infinity to infinity and then for the Fourier Transform I have to do e “to the power minus j omega t” d t and then again I have to take the integral from minus infinity to infinity.

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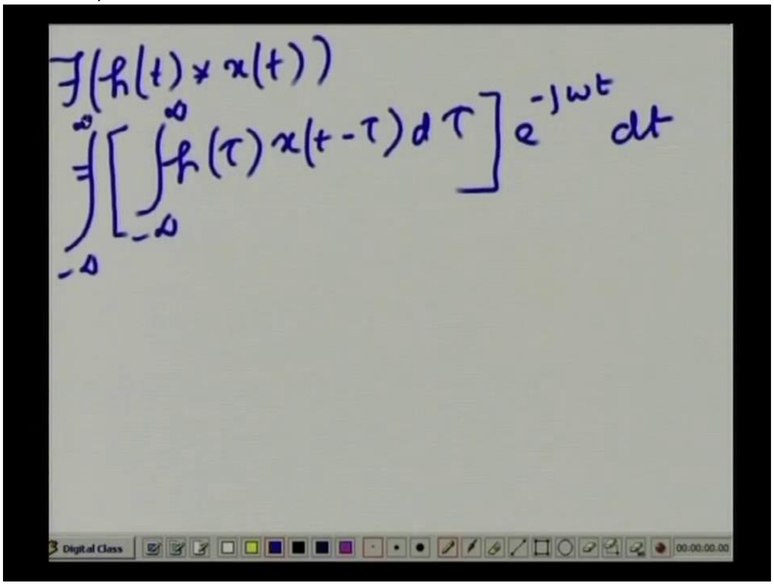

So this is the Fourier Transform of the convolution of those 2 signals h t and x t. Now if you do this integration, you will find that the same integration can be written in this form, I can take out h tau out of the inner integral. The inner integral I can represent as x of t minus tau e "to the power minus j omega t minus tau" d t. So I can put this as the inner integral. Then I have to multiply this whole term by e "to the power minus j omega tau" d tau and then this integration will be from tau equal to minus infinity to infinity.

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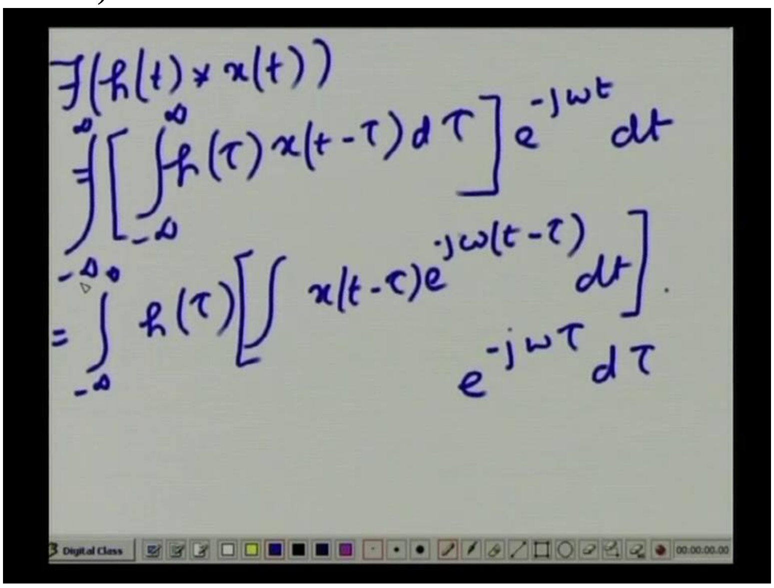

Now you will find that what does this inner integral mean? From the definition of Fourier Transform, this inner integral is nothing but the Fourier Transform of x t.

So,

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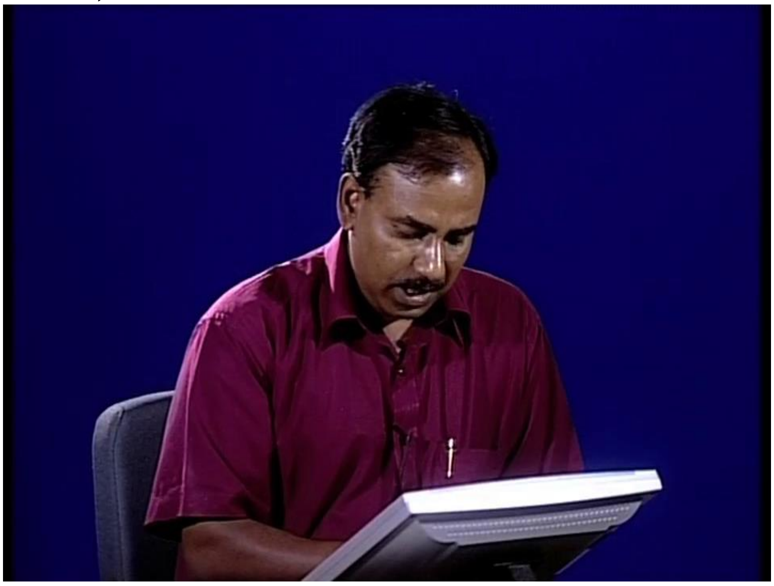

So this expression is equivalent to h of tau x of omega e “to the power minus j omega tau” d tau where this integration will be taken over tau from minus infinity to infinity.

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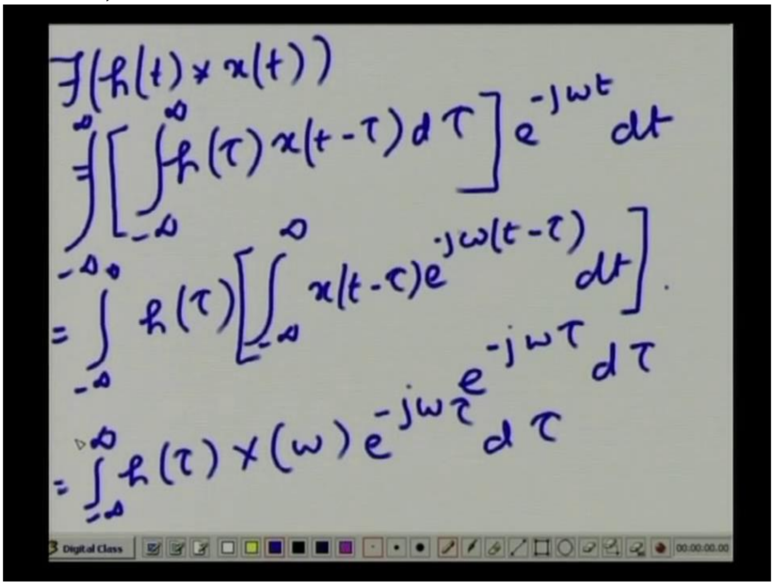

Now what I can do is, because this x omega is independent of tau, so I can take out this x omega from this integral. So my expression will now be x omega then within the integral I have h of tau e “to the power minus j omega tau” d tau where the integration is taken over tau from minus infinity to infinity.

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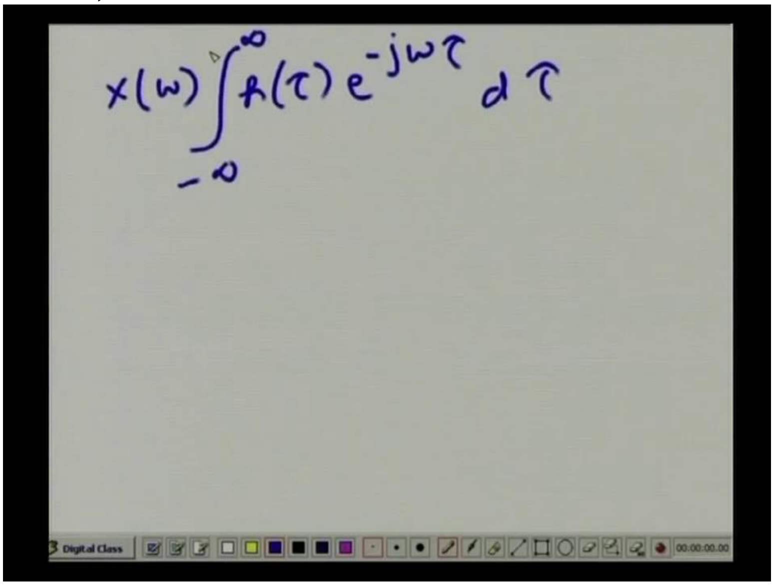

Again you will find that from the definition of Fourier Transformation, this is nothing but the Fourier Transformation of the time signal h t. So effectively this expression comes out to be X of omega into H of omega, where X of omega is the Fourier Transform of the signal x t and H of omega is the Fourier Transform of the signal h t.

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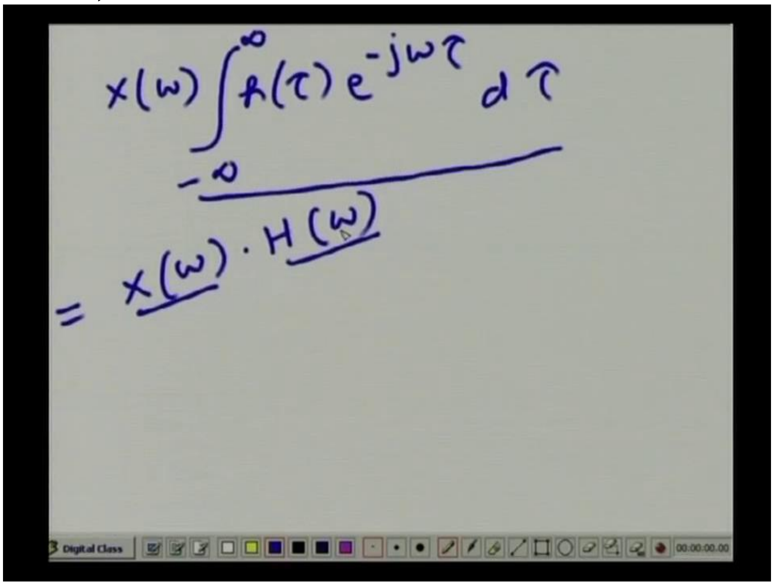

So effectively this means that if I take the convolution of 2 signals x t and h t in time domain, this is equivalent to multiplication of the two signals in the frequency domain.

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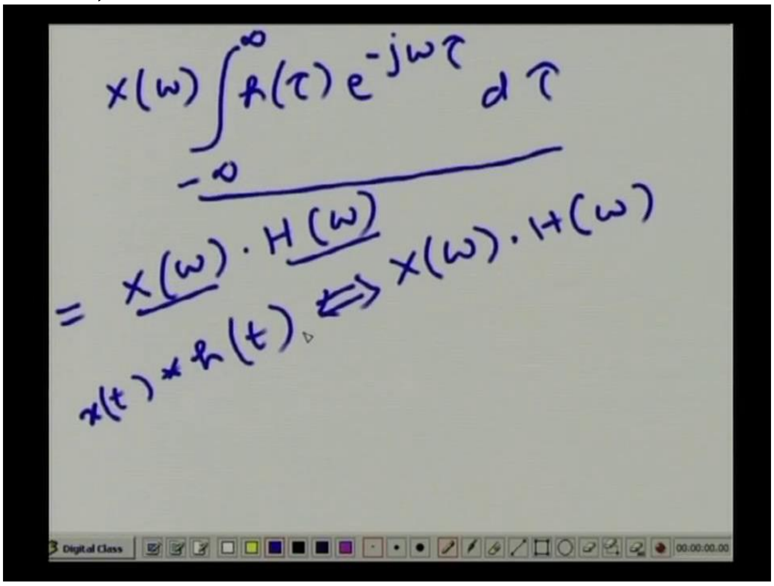

So convolution of two signals x t and h t in the time domain is equivalent to multiplication of the same signals in the frequency domain. The reverse is also true. That is, if we take the convolution of X omega and H omega in the frequency domain, this will be equivalent to multiplication of x t and h t in the time domain.

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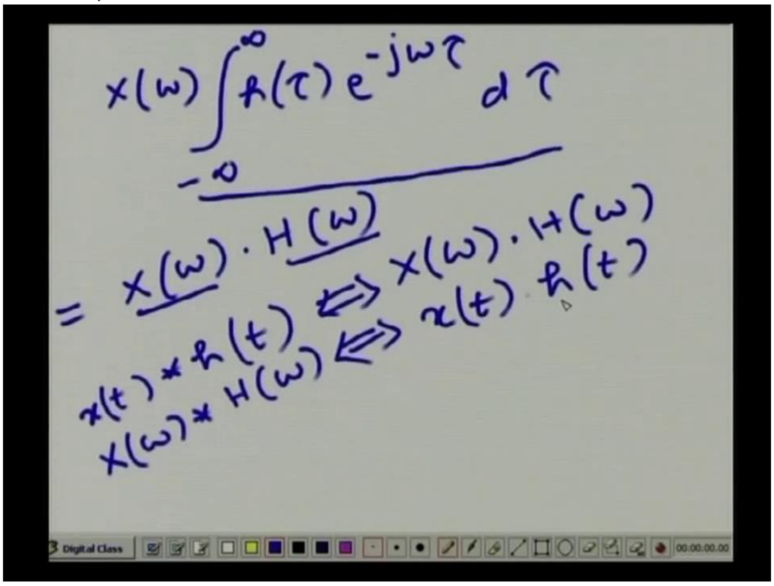

So both these relations are true and we will apply these relations to find out

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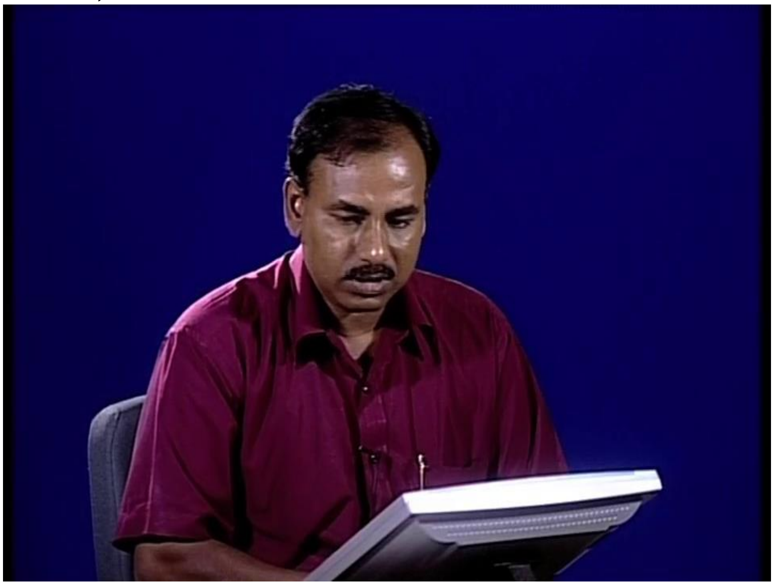

how the signal can be reconstructed from its sample values.

So now let us come back to our original signal. So here we have seen

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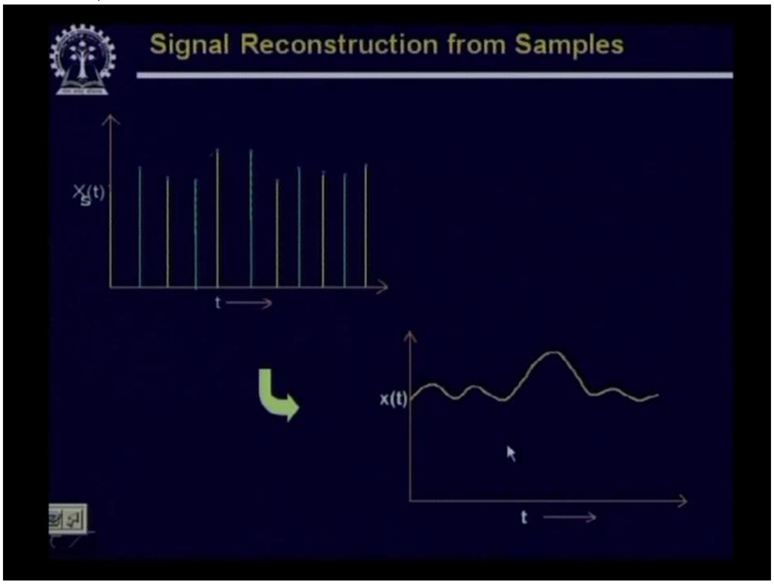

that we have been given these sample values and from the sample values, our aim is to reconstruct this continuous signal x t. And we have seen

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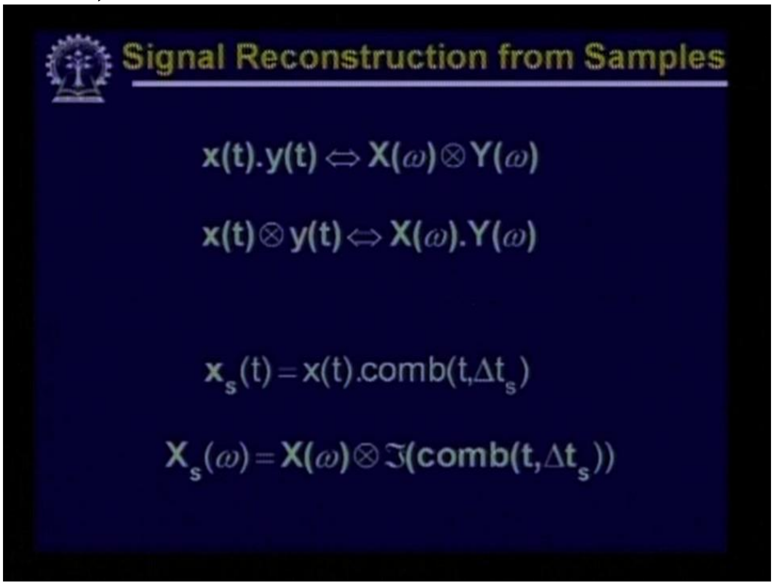

that this sampling is actually equivalent to multiplication of two signals in the time domain, one signal is x t and the other signal is comb function, comb of t delta t. So these relations as we have said that these are true that if I multiply 2 signals x t and y t in time domain that is equivalent to convolution of the two signals X omega and Y omega in the frequency domain. Similarly if I take the convolution of two signals in time domain, that is equivalent to multiplication of the same signals in frequency domain.

So for sampling when you have said that you have got "x s" of t

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that is the sampled values of the signal x t which is nothing but multiplication of x t with the series of Dirac delta functions represented comb of t delta t. So that will be equivalent to, in frequency domain I can find out "X s" of omega

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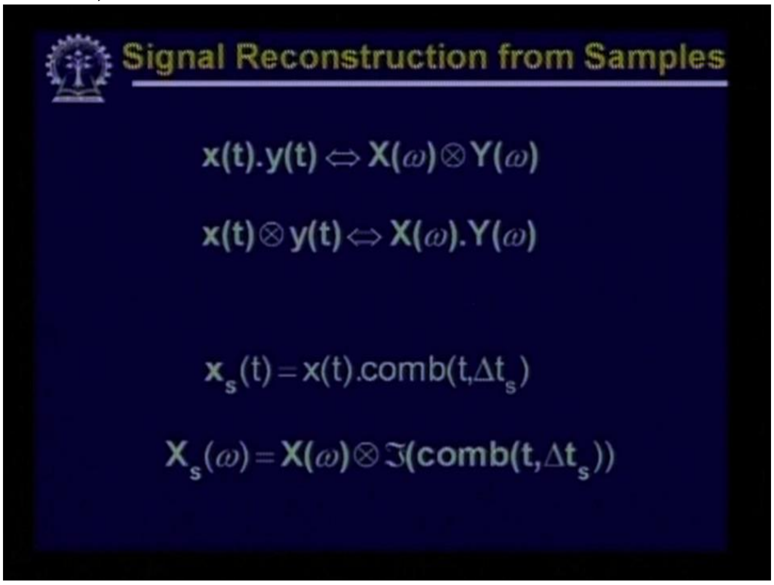

which is equivalent to the frequency domain representation X omega of the signal x t convoluted with the frequency domain representation of the comb function, comb t delta t and we have seen that this comb function, the Fourier Transform or the Fourier series expansion

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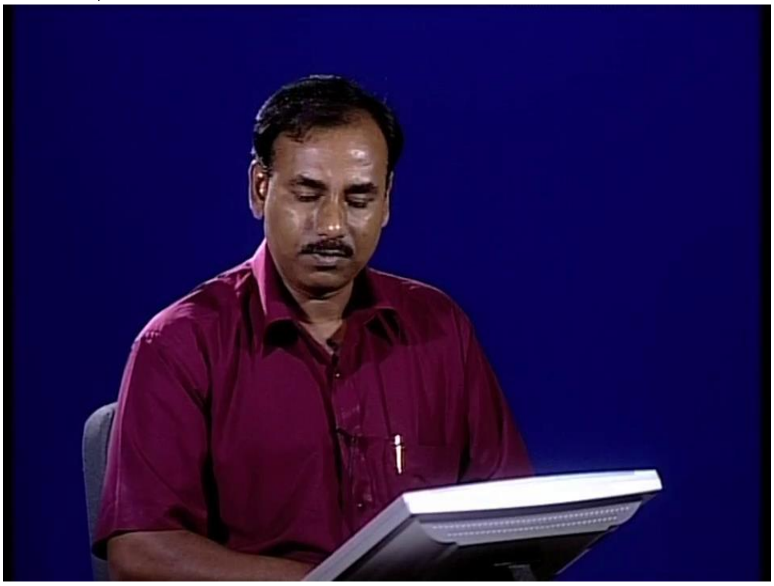

of this comb function is again a comb function.

So what we have is, we have is a signal x omega

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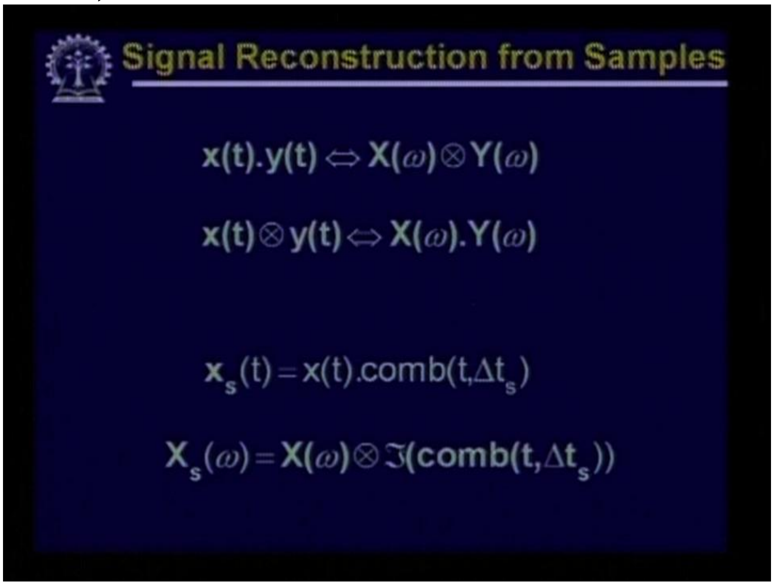

we have another comb function in the frequency domain and we have to take the convolution of these two.

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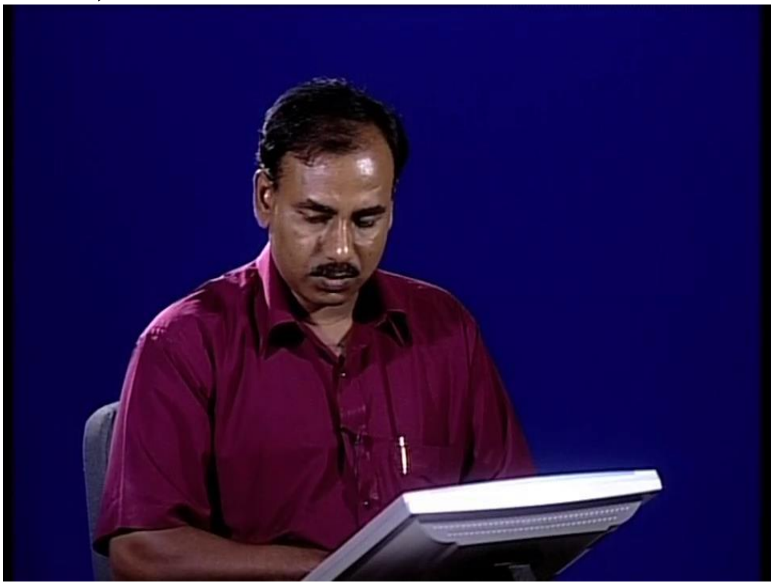

Now let us take this convolution in details

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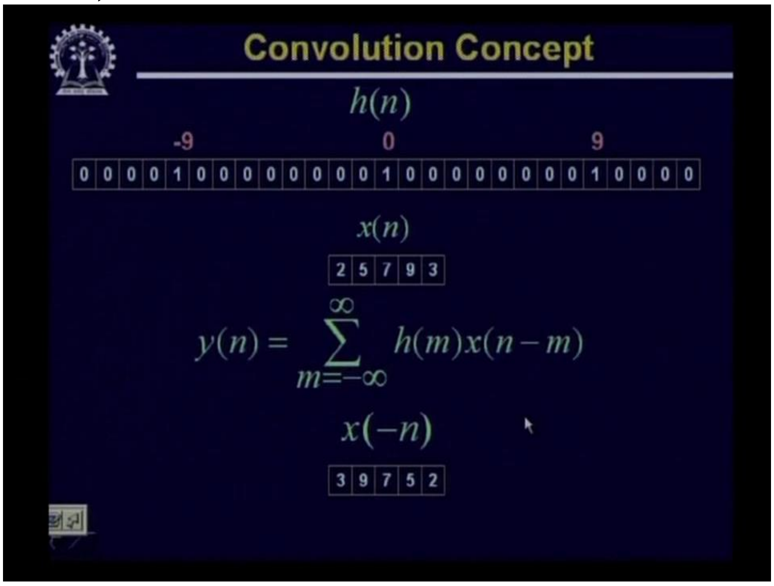

what does this convolution actually mean? Here we have taken 2 signals h n and x n, both of them for this purpose are in the sample domain. So h n is represented by this and x n is represented by this. You will find that this h n is actually nothing but a comb function where the delta t s in this case, we have value of h n is equal to 1 at n equal to 0, we have value of h n equal to 1 at n equal to minus 1, we have value of h n equal to 1 at n equal to minus 9, we have value of h n equal to 1 at n equal to plus 9. And this thing repeats. So this is nothing but representation of a comb function. And if I assume that my x n is of this form that is at n equal to 0, value of x n is equal to 7, x minus 1 that that n equal to minus 1 it is 5, n minus 1 minus 2 it is equal to 2. Similarly on this slide, for n equal to 1, x 1 equal to 9 and x 2 equal to 3; and the convolution expression that we have said in the continuous domain.

In discrete domain the convolution expression is translated to this form, that is y n equal to h m into x n minus m where m varies from minus infinity to infinity.

So let us see

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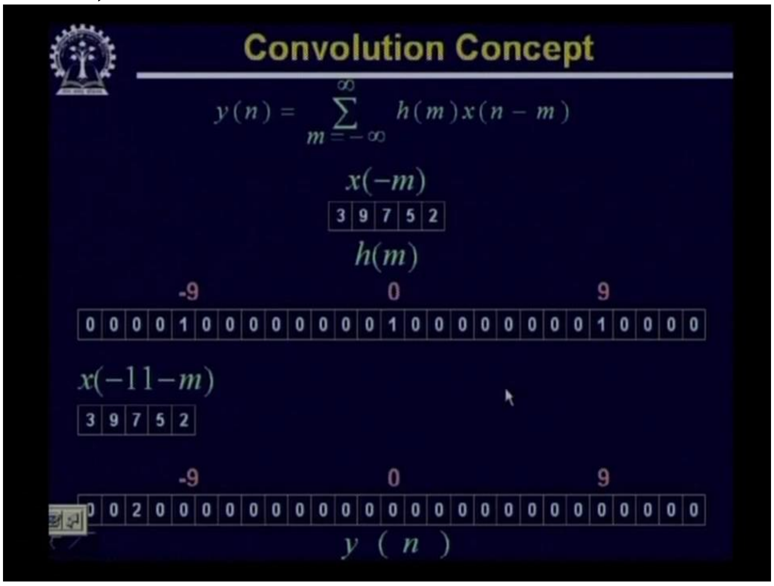

that how this convolution actually takes place. So if I really understand this particular expression that h m x of n minus m, sum of this from m equal to minus infinity to infinity we said that this actually means that we have to take the time inversion of the signal x n. So if I take the time inversion, the signal will be something like this, 3, 9, 7, 5 and 2 and when I take the convolution, that is, I want to find out the various values of y n that particular expression can be computed in this form. So if I want to take the value of y minus 11, so what I have to do is, I have to give a translation of minus 11 to this particular signal x of minus m, so it comes here. Then I have to take the summation of this product from m equal to minus infinity to infinity. So here what does it do? You will find that I do point by point multiplication of these signals. So here 0 multiplied with 3 plus it will be 0 multiplied with 9 plus 0 multiplied with 7 plus 0 multiplied with 5 plus 1 multiplied with 2, so the value I get is 2. And this 2 comes at this location y of minus 11.

Now for getting the value of y of minus 10, again I do the same computation and here you find that this 1 gets multiplied with 5 and all other values get multiplied with 0. And when you take the summation of all of them I get 5 here. Then I get value at minus 10, I get 7 here following the same operation, sorry this is at minus 9. I get at minus 8, I get at minus 7. I get at minus 6. At minus 6, you find that the value is 0. If I continue like this here, again at n equal to minus 2, I get value equal to 2. At n equal to minus 1, I get value equal to 5. At n equal to 0, I get value of 7. At n equal to plus 1, I get value of 9, at n equal to plus 2, I get value of 3, at n equal to plus 3, again I get the value of 0.

So if I continue like this, you will find that after completion of this convolution process, this h n convoluted with x n gives me this kind of pattern. And here you notice one thing, that when I have convoluted this x n with this h n, the convolution output y n, this is, you just noticed this that it is the repetition of the pattern of x n and it is repeated at those locations where the value of h n was equal to 1. So by this convolution, what I get is, I get repetition of the pattern x n at the locations of delta functions in the function h n.

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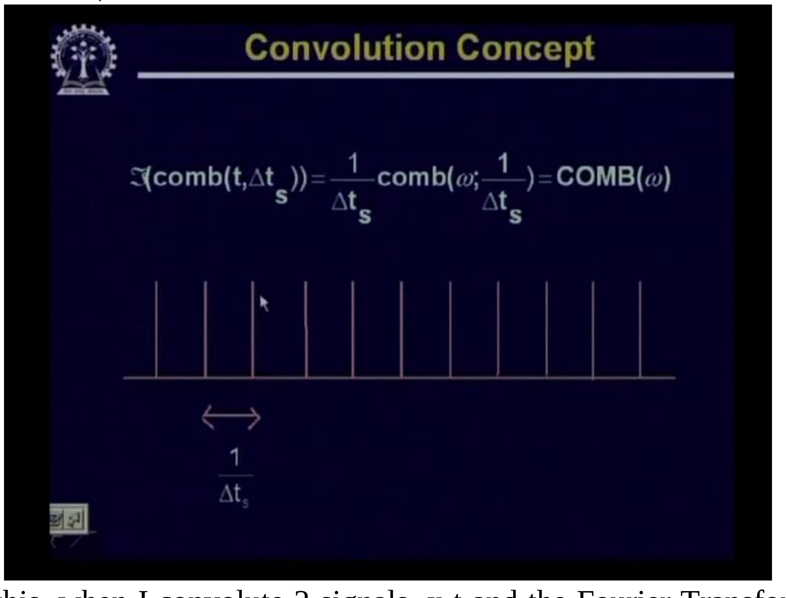

So by applying this, when I convolute 2 signals, x t and the Fourier Transform of this comb function that is comb omega in the frequency domain, what I get is

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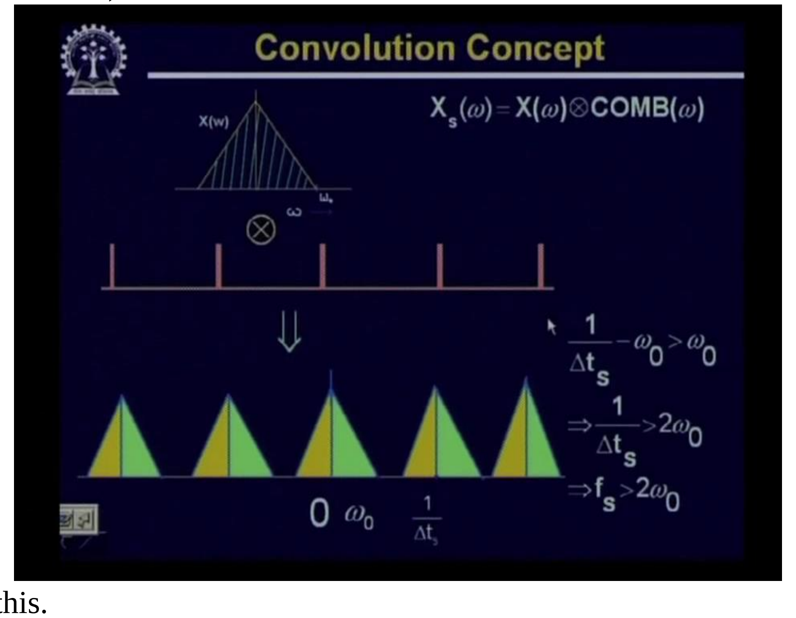

something like this.

When x t is band limited, that means the maximum frequency component in the x t is omega naught, then the frequency spectrum of the signal x t which is represented by X omega will be like this. Now when I convolute this with this comb function, COMB of omega then as we have done in the previous example what I get is at those locations where the comb function had a value 1 I will get just a replica of the frequency spectrum X omega. So this X omega gets replicated at all these locations.

So what we find here? You find that the same frequency spectrum X omega when it gets translated like this, when x t is actually sampled. That means the frequency spectrum of "X s" or "X s" omega is like this. Now this helps us in the construction of the original signal x t. So here what I do is, around omega equal to 0, I get a copy of the original frequency spectrum. So what I can do is, if I have a low pass filter whose cutoff frequency is just beyond "omega naught", and this frequency signal, this spectrum, the signal with this spectrum I pass through that low pass filter, in that case the low pass filter will just take out this particular frequency band and it will cut out all other frequency bands. So since I am getting the original frequency spectrum of x t so signal reconstruction is possible. Now here you notice one thing. As we said we will just try to find out that what is the condition that original signal can be reconstructed. Here you find that we have a frequency gap between this frequency band and this translated frequency band. Now the difference of, between center of this frequency band and the center of this frequency band is nothing but 1 upon "t s" which is equal to "omega s", that is the sampling frequency.

Now as long as this condition that is 1 upon "t s" minus "omega naught" is greater than "omega naught", that is the lowest frequency of this translated frequency band is greater than the highest frequency of the original frequency band, then only these 2 frequency bands are disjoint. And when these 2 frequency bands are disjoint, then only by use of a low-pass filter I can take out this frequency band. And from this relation, you get the condition that 1 upon delta "t s" or the sampling frequency "omega s", in this case it is represented as "f s" must be greater than twice of "omega naught" where "omega naught" is the highest frequency component in the original signal x t. And this is what is known as Nyquist rate. That is we can reconstruct, perfectly reconstruct

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the continuous signal only when the sampling frequency is greater than

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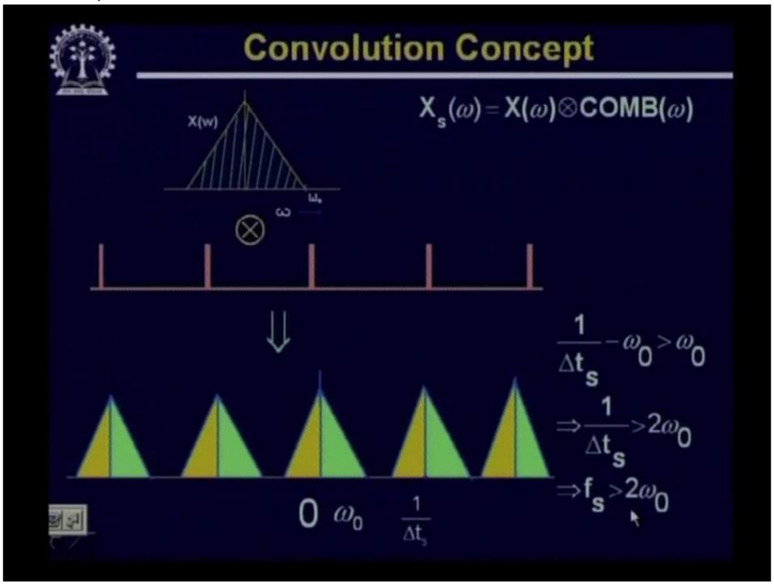

more than twice the maximum frequency component of the original continuous signal.

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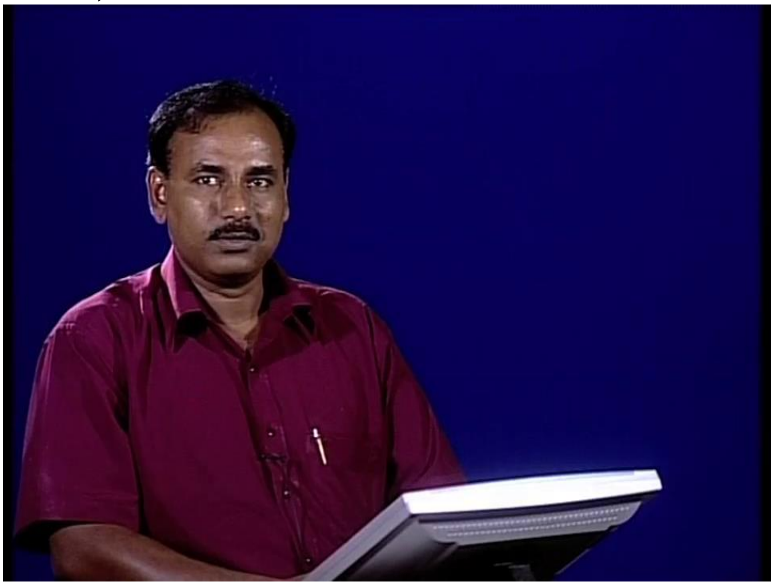

Thank you.