Digital Image Processing.
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Lecture-24.
Fourier Transform.

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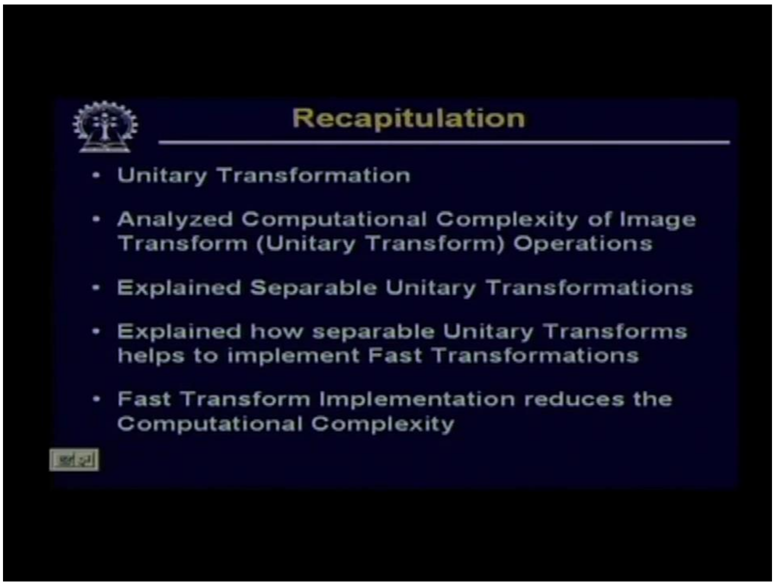

Hello, welcome to the video lectures series on digital image processing. In the last two classes, we have seen the basic theories of unitary transformations and we have seen uhh we have analysed the computational complexity of the unitary transformation operations particularly with respect to the image transformations. We have explained the separable unitary transformation.

We have explained how separable unitary transformation helps to implement the fast transformations and fast transformation implementation as we have seen during our last class. It reduces the computational complexity of the transformation operations. After giving the general unitary introduction to the general unitary transformations, in today's lecture, we are going to discuss about the Fourier transformation which is a specific case of the unitary transformation.

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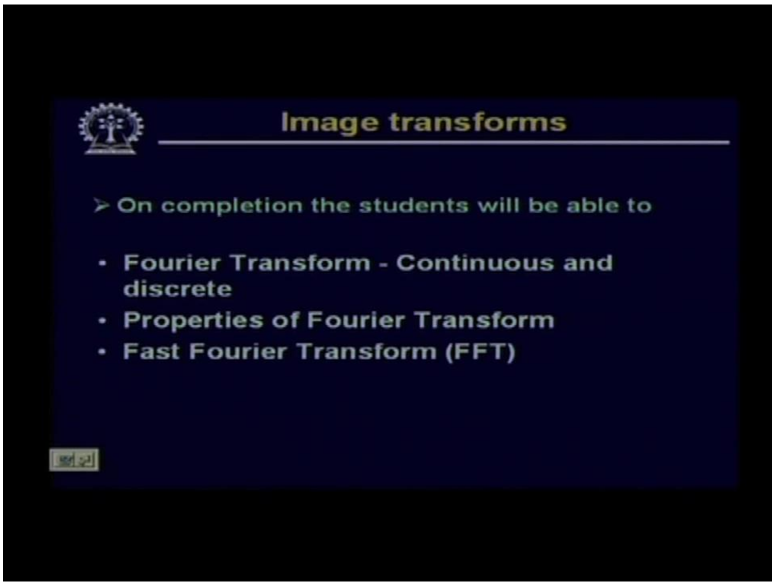

So during today’s lecture, we will talk about the fourier transformation and we will talk about fourier transformation both in the continuous domain as well as in discrete domain. We will see what are the properties of the fourier transformation operations and we will also see that what is meant by Fast Fourier Transform that is fast implementation of the fourier transformation operation.

Now this Fourier Transformation operation, we have discussed in brief when we have discussed about the sampling theorem that is giving given an analog image or continuous image while discretisation. The first stage of discretisation was sampling the analog image. So during uhh discussion on sampling, we have talked about the Fourier Transformation and there we have said that Fourier Transformation gives you the frequency components present in the image.

And for sampling, we must meet the condition that your sampling frequency must be greater than twice the maximum frequency present in the continuous image. In today’s lecture, we will discuss about the Fourier Transformation in uhh greater details. So first let us see what is meant by the Fourier Transformation. As we have seen earlier, that if we assume, a function say f(x).

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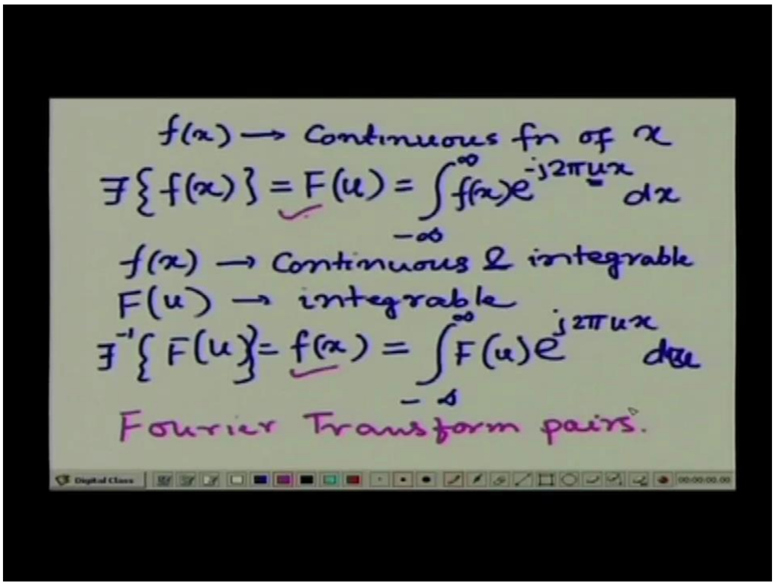

So we will first talk about the Fourier Transformation in the continuous domain, and if we assume that f(x) is a continuous function, so this f(x) is a continuous function of some variable say x, then a fourier transformation of this function f(x), we normally write it as, the fourier transformation of the function f(x). This is also written as capital F of u. This is given by the expression, integral expression f(x) e to the power minus j 2 pi u x dx.

Here the integration is carried over from minus infinity to infinity. Now this variable ux, this is the frequency variable. So given a function f(x), a continuous function f(x), by using this integration operation, we can find out the fourier transformation of the fourier transform of this continuous function f(x) and the fourier transform is given by f u.

Now for doing this continuous fourier transformation, this function f(x) has to meet some requirement. The requirement is, the function f(x) must be continuous, it must be continuous and it must be integrable. So if f(x) meets these two requirements, that is f(x) is continuous and integrable then using this integral operation, we can find out the fourier transformation of this continuous function f(x).

Similarly, we can also have the inverse fourier transformation, that is given the fourier transform F(u) of a function f(x) and if F(u) is integrable, f(u) must be integrable, then we can find out the inverse fourier transform of F(u) which is nothing but the continuous function f(x) and this is given by a similar integration operation and now it is F(u) integral e to the power j 2 pi u x dx and the sorry du and this integration again has to be carried out from minus infinity to infinity.

So from f(x) using this integral operation we can get the fourier transformation which is the F(u) and if F(u) is integrable then using the inverse fourier transformation, we can get back the original continuous function f(x) and these two expressions that is F(u) and f(x), the expressions for F(u) and expression for f(x), these two expressions are known as fourier transform pairs.

So these two are known as fourier transform pairs. Now from this expression, you find that because fr being the (express-) uhh fourier transformation, what we are doing is we are taking the function f(x) multiplying it with an exponential e to the power minus j 2 pi u x dx and integrating this over the interval minus infinity to infinity. So naturally this expression F(u) that you get in general complex because e to the power minus j 2 pi u x , this quantity is a complex quantity.

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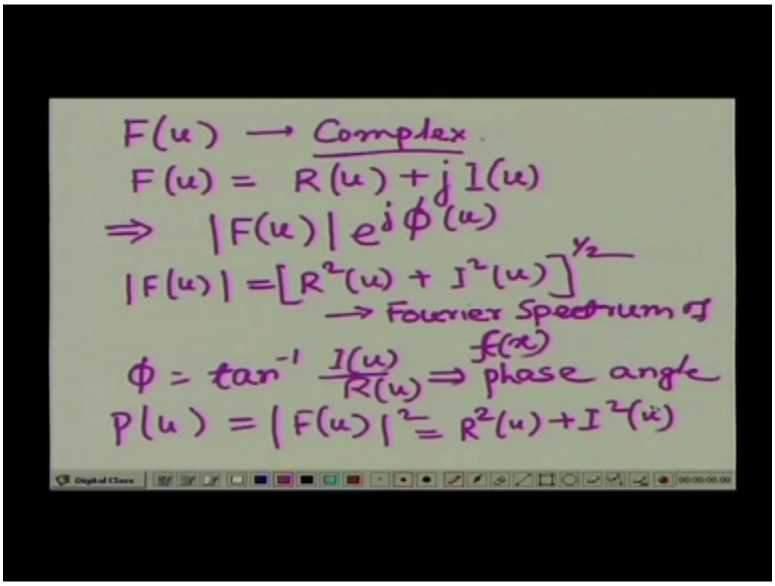

So in general, the function F(u), it is a complex function, in general it is a complex function and because this F(u) is a complex function so you can write this F(u) or you can break this F(u) in the real part; so the real part we write as R(u), and the imaginary part so it will be I(u). So the F(u) which in general is a complex quantity in now broken into the real part and the imaginary part.

Or the same F(u) can also be written in the form of modulus of F(u) into e to the power j of phi u where the modulus of F(u) which gives you the modulus of this complex quantity F(u), this is nothing but R R(u) square plus I(u) square and square root of this. Ok? And this is what is known as fourier spectrum of f(x). So this we call as fourier spectrum of the function f(x) and this quantity phi of u which is given by tan inverse I(u) upon R(u).

This is what is called the phase angle. This is the phase angle. So from this, we get what is known as the fourier spectrum, fourier spectrum of f(x) which is nothing but the modulus of magnitude of the fourier transformation F(u) and the tan inverse of the imaginary component I(u) by the real component R(u) this is that is what is the phase angle for this particular for a particular value of u.

Now there is other term which is called the power spectrum. So power spectrum of the function f(x) which is also represented as P(u). this is nothing but F(u) magnitude square and if you expand this, this will be simply R square u plus I square u. So you get the power spectrum, we get the fourier spectrum and we also get the phase angle from the fourier transformation coefficients.

And this is what we have in case of 1 dimensional image because we have because we have taken a function f(x) which is a function of single variable x. Now because in our case, we are discussing about the image processing operations and we have already said that the image is nothing but a 2 dimensional function which is a function of two variables x and y, so we have to discuss about the fourier transformation in 2 dimension rather than in single dimension.

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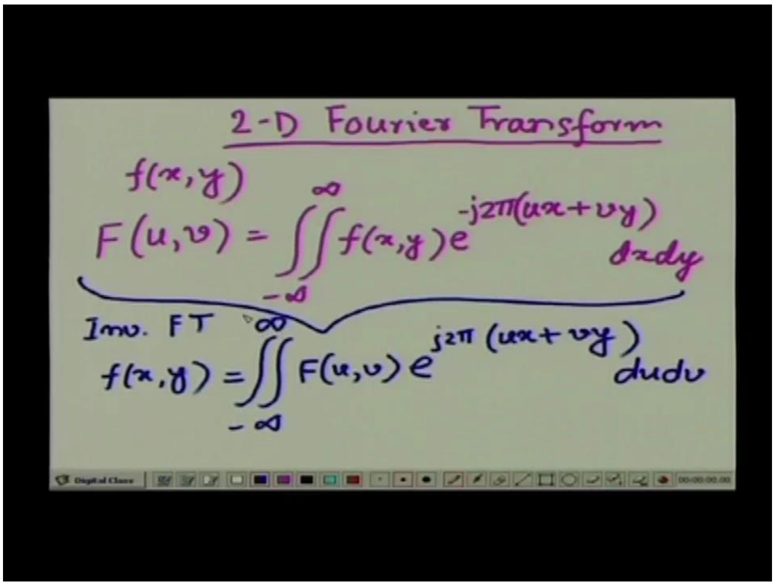

So when you go for 2 dimensional fourier transformation, so you talk about 2D Fourier Transform. The 1 dimensional fourier transform that we have discussed just before can be easily extended to 2 dimension in the form that now in this case, our function is a 2 dimensional function f(x,y) which is a function of two variables x and y and the fourier transform of this f(x,y) is now given by a F(u,v) which is equal to, now we have to have double integral f(x,y) e to the power minus j 2 pi u x plus v y dx dy and both these integrations have to be taken over the interval minus infinity to infinity.

So find that from a 1 dimensional we have easily extended that to 2 dimensional fourier transformation and now this integration has to be taken over x and y because our image is a 2 dimensional image which is a function of two variables x and y. So the forward transformation is given by this expression F(u,v) is equal to f(x,y) e to the power minus j 2 pi u x plus v y dx dy and integration has to be taken over from minus infinity to infinity.

In the same manner, the inverse fourier transformation, so you can take the inverse fourier transformation to get f(x,y), that is the image from its fourier transform coefficient F(u,v) by taking the similar integral operation and in this case it will be F(u,v) e to the power j 2 pi u x plus v y du dv and the integration has to be taken from minus infinity to infinity.

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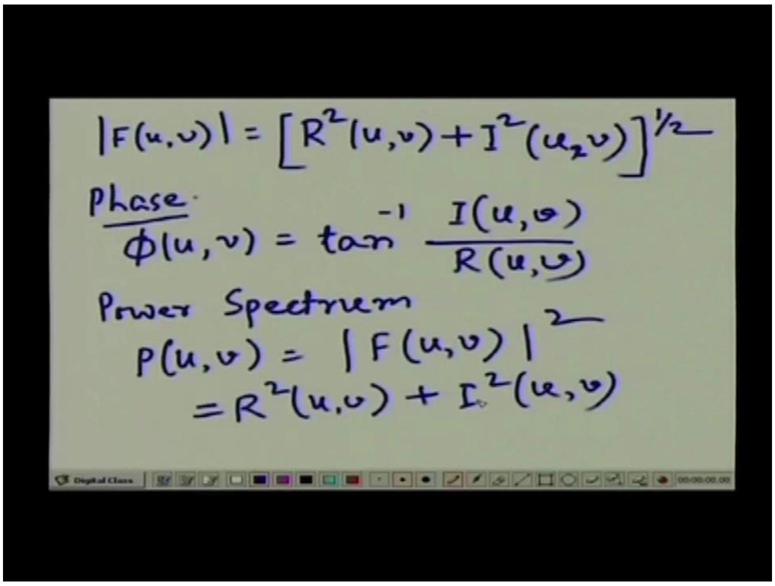

So in this 2 dimensional signal, the fourier spectrum F(u,v) is given by R square (u,v) so as before this R gives you the real component plus I square (u,v) where I gives you the imaginary component and square root of this.

So this is what is the fourier spectrum of the 2 dimensional signal f(x,y). We can get the phase angle in the same manner. The phase angle phi(u,v) is given by tan inverse uhh I (u,v) by R (u,v) and the power spectrum in the same manner we get as P(u,v) is equal to F(u,v) squared which is nothing but R squared (u,v) plus I square (u,v).

So you find that all these quantities which we had defined in case of the single dimensional uhh signal is also applicable in case of the 2 dimensional signal that is f(x,y). Now to illustrate this fourier transformation, let us take an example.

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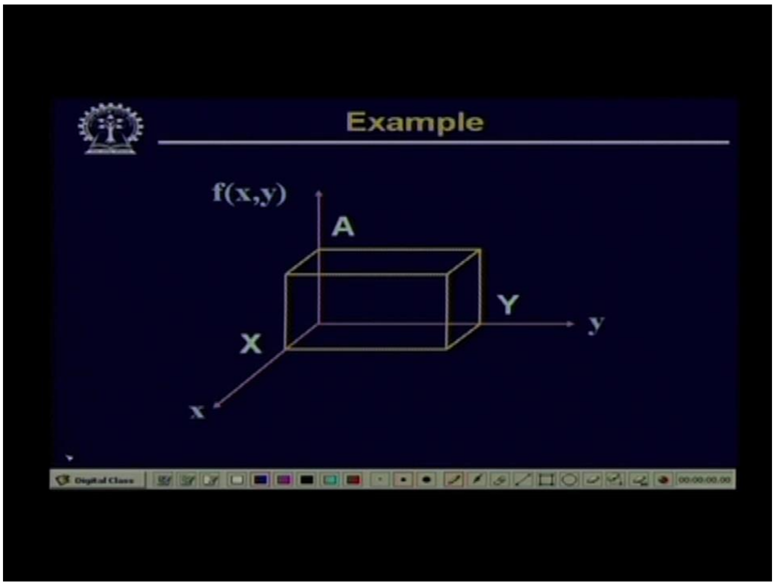

Suppose we have a continuous function like this, the function f(x,y) which is again a function of two variables x and y and the function in our case is like this that f(x,y) assumes a value, a constant value say capital A, for all values of x lying between 0 to capital X and all values of y lying between 0 to capital Y. So what we get is uhh a rectangular function like this where all values of x greater than capital X, the function value is zero and all values of y greater than capital Y, the function value is also zero.

And between 0 to capital X and 0 to capital Y, the value of the function is equal to capital A. Let us see, how we can find out the fourier transformation of this particular 2 dimensional signal.

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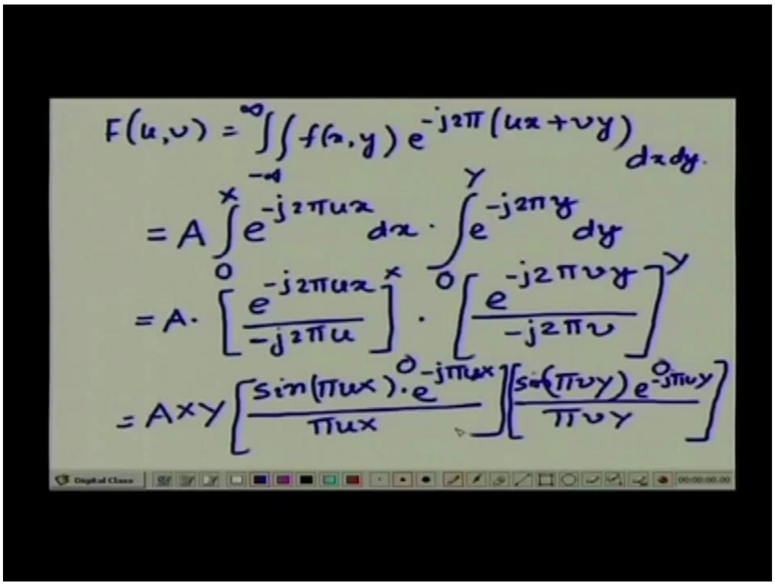

So to compute the fourier transformation, we follow the same expression. We have said that F(u,v) is nothing but double integration from minus infinity to infinity f(x,y) e to the power minus j 2 pi ux plus vy dx dy. Now in our case, this f(x,y) is equal to constant which is equal to A as long as x lies between 0 to capital X and y is in between 0 to capital Y.

And outside this region, the value of f(x,y) is equal to zero. So you can break this particular integral in this form. This will be same as capital A then take the integration over x which will be in this particular case e to the power minus j 2 pi u x dx. Now this integration over x has to be from zero to capital X multiplied by e to the power minus j 2 pi y dy where this integration will be in the range zero to capital Y.

So if I compute this, these two integrations, these two integrals, you will find that it will take the form something like this. And if you compute these two limits, you will find that, it will take the value A capital X into capital Y into sine pi u x into e to the power minus j pi u x upon pi u x into sine pi v y into e to the power minus j pi v y upon pi v y. So after doing all these integral operations I get an expression like this.

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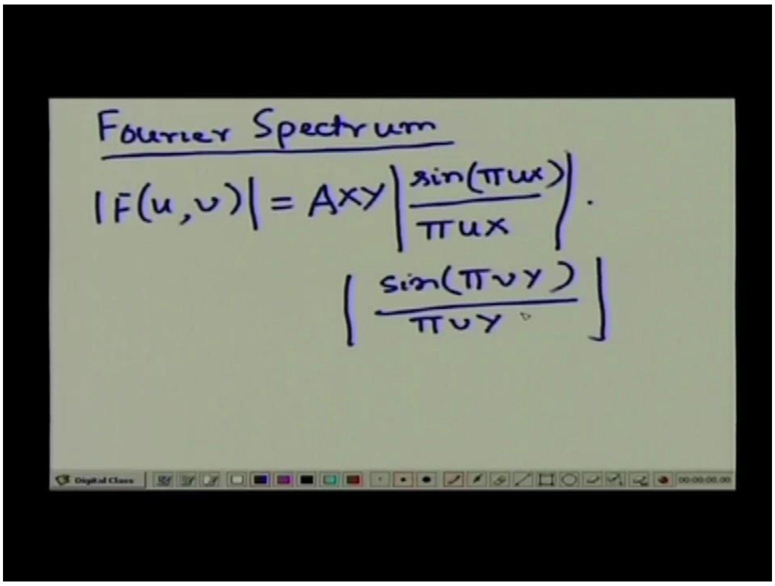

So from this expression, if you compute the fourier spectrum, the fourier spectrum will be something like this. So what we are interested in is the, fourier spectrum. So the fourier spectrum, that is modulus of F(u,v) will be given by A capital X capital Y into sine pi u x upon pi u x into sine pi v y upon pi v y. So this is what is the fourier spectrum of the fourier transformation that we have got.

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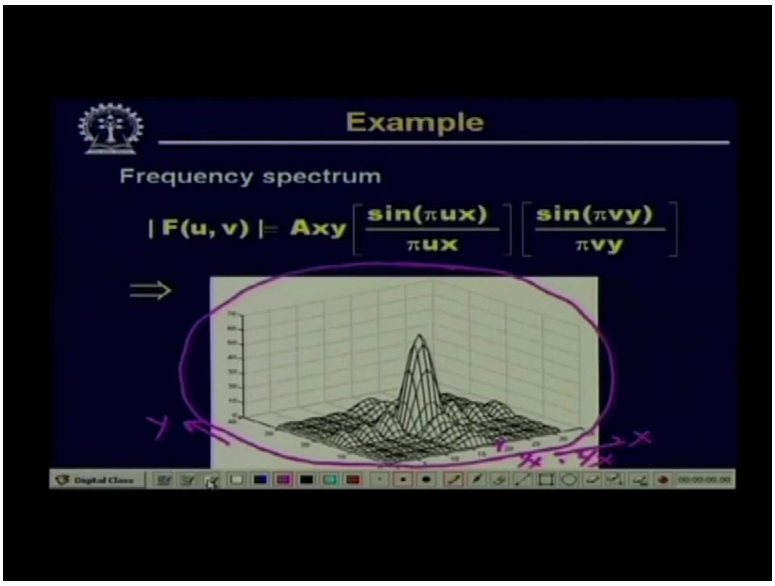

Now if we plot to the fourier spectrum, the plot will be something like this. So this is what is the plot of this fourier spectrum. So the fourier spectrum plot is this one. So you find that this is again a 2 dimensional function. Of course in this case the spectrum that has been shown is shifted, so that the spectrum comes within the range for its complete visibility.

So for a rectangular function, rectangular 2 dimensional function, you will find that the fourier spectrum will be something like this and you can find out that if I say that this is the x-axis and this is the y- axis and assuming the center to be at the origin, you will find that along the x-axis at point 1 upon capital X. similarly 2 upon capital X. The value of this fourier spectrum will be equal to 0. Similarly along the y-axis, at values 1 upon capital Y, 2 upon capital Y, the values of this spectrum will also be equal to 0.

So what we get is the fourier spectrum and the nature of the fourier spectrum of the particular 2 dimensional signal. Now so far what we have discussed, is the case of the continuous functions or analog functions. But in our case, we have to be interested in the case for discrete images or digital images where the functions are not continuous but the functions are discrete.

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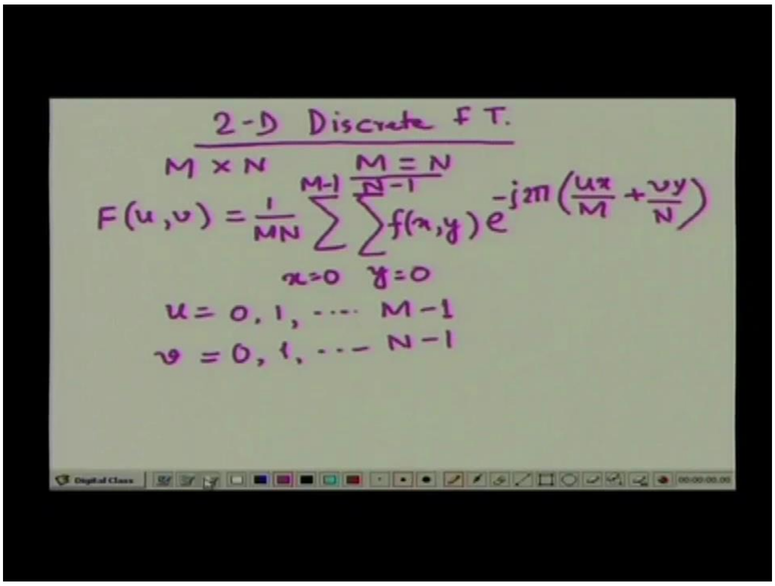

So all these integration operations that we are doing in case of the continuous functions, they will be replaced by the corresponding summation operations. So when you go for the 2 dimensional uhh signal, ok? So in case of this uhh discrete signals, the discrete fourier transformation will be of this form F(u,v), now this integrations will be replaced by summations.

So this will take the form of 1 upon m into n then double summation f(x,y), the expression remains almost the same minus j 2 pi u x by capital M plus v y by capital N and now the summation will be for y equal to 0 to N minus 1 capital N minus 1 and x equal to 0 to capital M minus 1. Because our images are of size M by N and the frequency of variables u, because our images are discrete, the frequency variables are also going to be discrete.

So the frequency variables u will vary from 0, 1 upto M minus 1 and the frequency variable v will similarly vary from 0, 1, upto capital N minus 1. So this is what is the forward discrete fourier transform. Forward 2 dimensional discrete fourier transformation. In the same manner we can also obtain, the inverse fourier transformation for this 2 dimensional signal.

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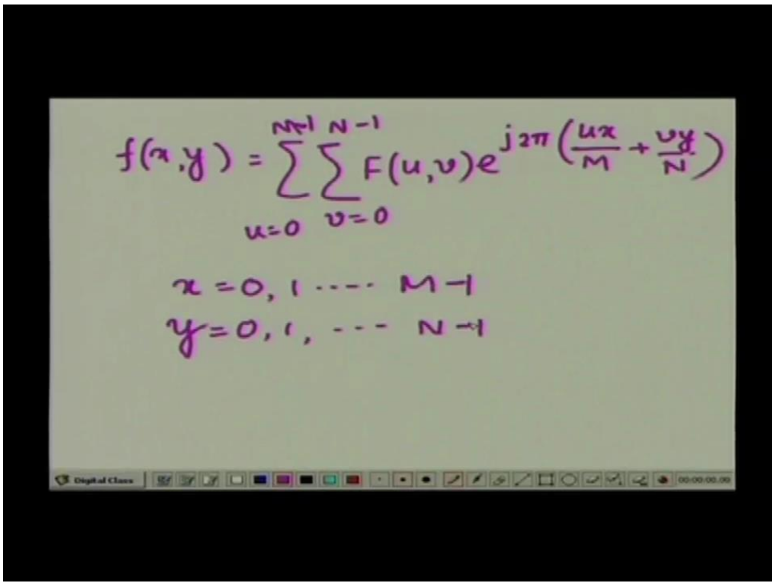

So the inverse fourier transformation will be given by f(x,y) will be double summation F(u,v) which is the fourier transformation of f(x,y), e to the power j 2 pi u x by M plus v y by N and now the integration will be formed v equal to 0 to capital N minus 1 and u equal to 0 to capital N minus 1. So the frequency variables v varies from 0 to capital N N minus 1 and u varies from 0 to capital M minus 1.

And obviously, this will give you give you back the digital image f(x,y), the discrete image where x will now vary from 0 to capital M minus 1 and y will now vary from 0 to capital N minus 1. So we have formulated these equations, in a general case where the discrete image is represented by a 2 dimensional array of size capital M by capital N. Now as is said, that in most of the cases, the image is mostly represented in the form of square array where M is equal to N.

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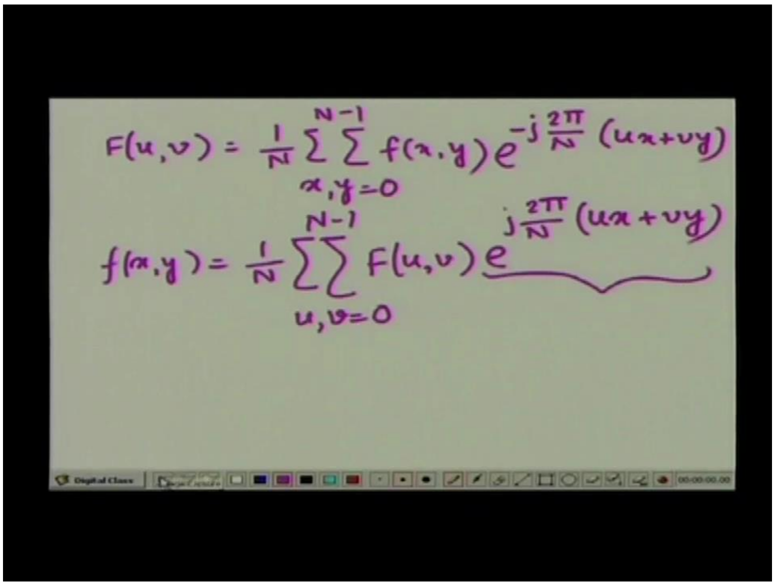

So if the image is represented in the form of square array, in that case these transformation equations will be represented as F(u,v) will be equal to 1 upon capital N, double summation f(x,y) and now because M is equal to N so the expression becomes e to the power minus j 2 pi by N u x plus v y where both x and y will now vary from 0 to capital N minus 1.

And similarly, the inverse fourier transform f(x,y) will be given by 1 upon capital N summation double summation F(u,v) e to the power j 2 pi by N u x plus v y but the variables u and v will now vary from 0 to capital N minus 1. So this is the fourier transformation pair that we get in discrete case for a square image where the number of rows and the number of columns is same.

And as we have discussed earlier, that e to power j 2 pi by N ux plus vy this is what we have called the basis images. This we have discussed with when we have uhh discussed about the unitary transformation. And we have said, we have shown that time that these basis images will be like this.

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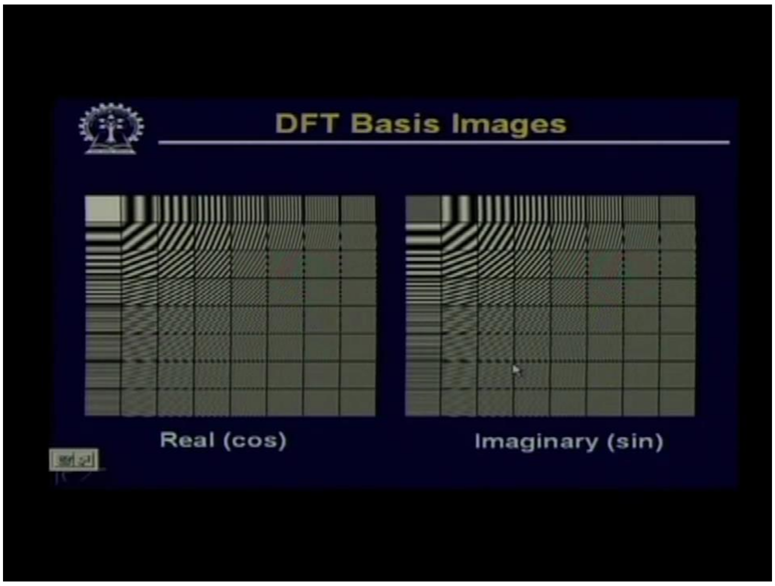

So as the fourier transformation as we have seen that it is a complex quantity, so uhh for the fourier transformation we will have two basis images. One basis image corresponds to the real part, the other basis image corresponds to the imaginary part and these are the two basis images, one for the real part and the other one for the imaginary part.

Now as we have defined the fourier transform uhh the fourier spectrum, the phase the power spectrum in case of analog image. All these quantities can also be defined are also defined in the case of discrete image in the same manner.

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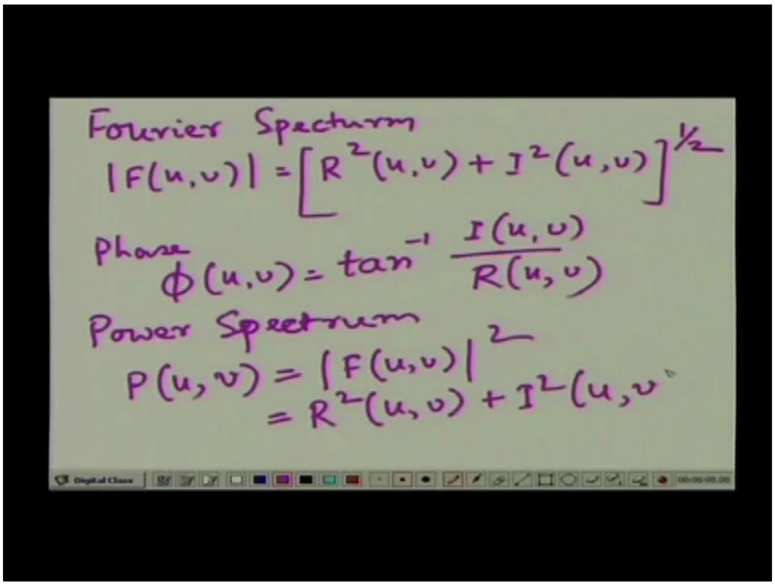

So in case of this discrete image, the fourier spectrum, is given by similar (**ex**-) expression that is F(u,v) is nothing but R square (u,v) plus I square (u,v), square root of this.

Phase is given by phi (u,v) is equal to tan inverse I (u,v) upon R (u,v) and the power spectrum P(u,v) is given by the similar expression, uhh which is nothing but F (u,v) modulus square which is nothing but R square (u,v) plus I square (u,v) where R is the real part of the fourier coefficient and I (u,v) is the imaginary part of the fourier coefficient. So after discussing about this fourier transformation both in the forward direction and also in the reverse direction. Let us look at how these fourier transform coefficients look like.

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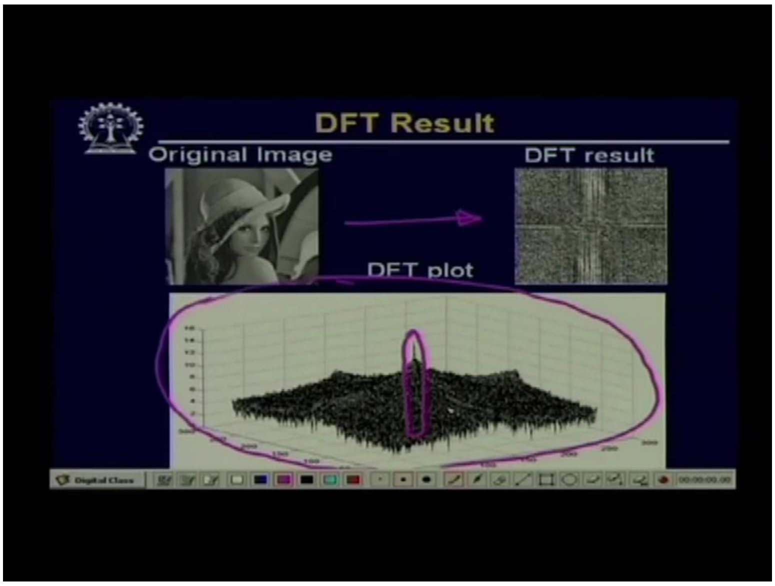

So here we have the result on one of the images and you find that this is a popular image, a very popular image which is uhh cited in most of the image processing textbooks that is the image of (())(30.40). So if you take the discrete fourier transformation of this particular image, the right hand side, this one shows that DFT which is given in the form of an intensity plot and the bottom one that is this particular plot is the 3 dimensional plot of the DFT coefficients. Here again when these coefficients are plotted, it is shifted so that the uhh the origin is shifted at the center of the plane so that you can have a better view of all these coefficients.

Here we find that at the origin the intensity of the coefficient or the value of the coefficient is quite high compare to the values of the coefficients as you move away from the origin. So this indicates that the fourier coefficient is maximum at least for this particular image at origin that is when u equal to 0 or v equal to 0 and later on we will see that u equal to 0 v equal to 0 gives you what is the DC component of this particular image.

And in most of the images the DC component is maximum and as you move towards the higher frequency components, the energy of the higher frequency signals are uhh less compared to the DC component. Thank you.