Discrete Time Signal Processing

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Lecture – 7
Properties of DTFT

So, in the previous class, we give a definition of DTFT, we told you how it comes. There is if you, (Refer Time: 00:28) kind of sequence. At the input, and output becomes the same thing, but multiplied by a constant e to the power j Omega. And that is a function of Omega, but various Omegas, it will take different values, is complex in general, that is called the discrete time Fourier transform. And giving that, you can from that, you can get back the sequence h n, by inverse relation. And the question in point is I mean why only for h n?

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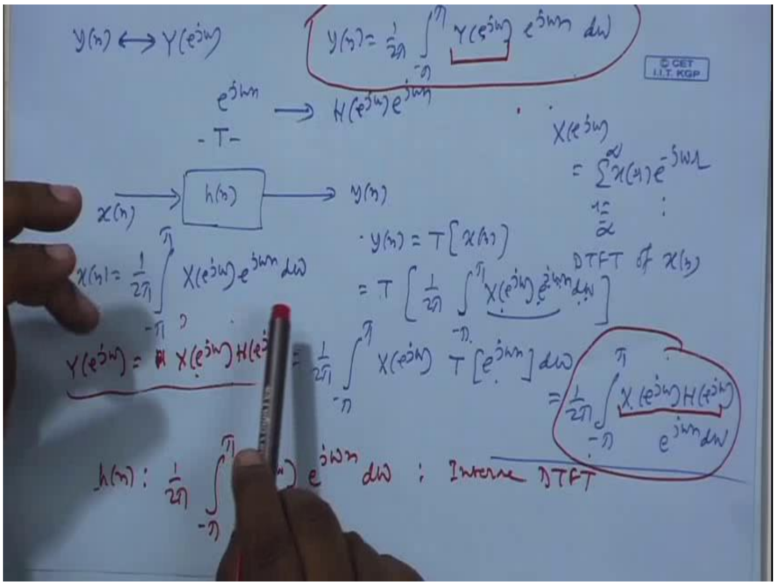

You can carry out a similar summation, for discrete time Fourier transform any sequence, not, not necessarily, unit sample is for a system. Because when I say, when I give a formula, summation, when I give a formula for this, this kind of formula for discrete time Fourier transform, where a is e to the power j Omega, why only h? It can be any sequence, for which I can run a sum like this, it will be called the DTFT, discretized time

Fourier transform, that sequence, or you can get back that sequence, by the inverse transform relation, like for h n, you can get it back from this DTFT, like this.

For yn, you can get it back like this, for xn you can get it back like this, capital Xe to the power j Omega DTFT, instead of capital H, capital X here, then the same Eigen sequence, e to the power j Omega, d Omega integral minus pi to pi 1, by 2 pi.

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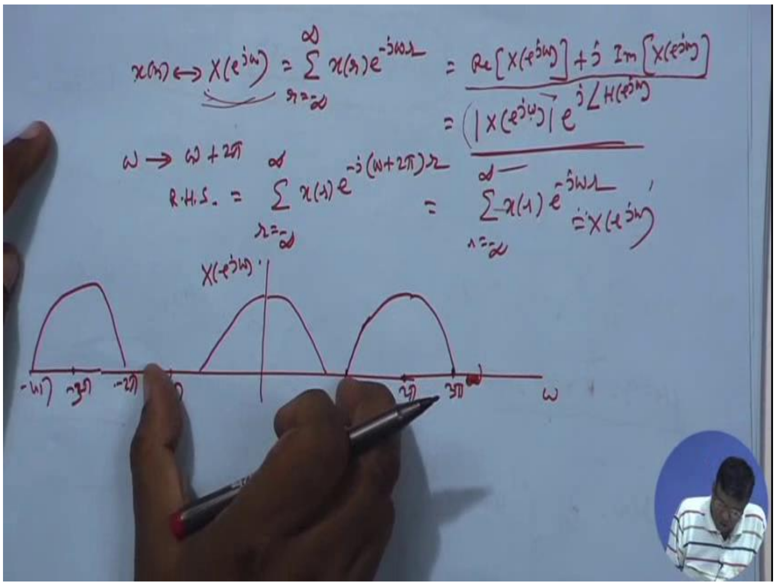

We have also seen, that, output DTFT, from this composition of the 2 integrals, this is yn directly and this is yn intersub, a similar integral by intersub of capital Xe to the power j Omega, capital He to the power j Omega, you can comparing, we could we wrote here, that output DTFT ye to the power j Omega is a product of the 2 DTFTs, 1 of the unit sample response, on the system, and the other of the input. With this convolution of the 2 sequences in time domain, that is in time domain. In digital frequency domain, it amounts to product of the DTFTs, and then you get the output DTFT, there is I stopped.

This function for any sequence, we write like this, if xn, that is, this given, you can see this entire function firstly complex. If you want to plot it versus Omega, this function has got real part, and imaginary part, that is real part of this DTFT, plus j times imaginary part of this total function right? Or alternatively mod is a polar form, this is a rectangular form a plus jb, that kind of form, for a complex number, and this is you can alternatively write this as polar form, that is magnitude times, e to the power j, angle of, this is a phase or polar form. Now in either way you see, if you want you plot this capital X, to the power j Omega, by vice versus -Omega, you need to plot 2 things, like here edit to plot the magnitude part, as a function of Omega, and phase angle is a function of Omega. 2 things are equal, then only you will get a complete justice for the x. Alternatively, in the this, for rectangular form, you again need to plot this real part of this DTFT, versus Omega, and again you have to plot imaginary plot of DTFT, versus Omega, then only you get a complete picture, ok?

Now, often, and specially in the case of this polar form, we might just you know, take of the magnitude plot, do not for the corresponding phase response plot, phase that is, phase plot versus Omega, but whatever logic we say, with respect to this magnitude response, same will be applicable, for this that can be easily seen. But technically, a complete plot of DTFT versus Omega, that is, frequency response as spectrum, spectrum equal to DTFT spectrum, of xm, which is this, this plot, plotting of this DTFT versus Omega, that actually always request 2 plots. One of magnitude versus Omega, another is phase versus Omega, or go (Refer Time: 05:08) one of the real part, from the imaginary part, versus Omega, although between the two, you normally prefer this polar form.

Now, this function DTFT, suppose Omega I replace by, Omega is radiant right is (Refer Time: 05:26) frequency here it is radiant, so this radiant, plus another 2 pi radiant. If I replace Omega, by that what happens to the right hand summation? RHS becomes, j Omega plus 2 pi, r, and then e to the power minus j Omega are as it is, but e to the power minus j 2 pi r to the power minus j 2 pi r, is always 1, for r, because r is an integer. e to the power j 2 pi e, to the power minus j 2 pi e, to the power j 4 pi e, to the power minus j 4 pi dot, dot, dot, they are all having value 1. These I believe you know. So, you will get back x of r, to the power minus j Omega r, which is nothing, but your, original DTFT, which means DTFT here, if I replace Omega, by Omega plus 2 pi, I get back the same thing. So, this is periodic, this is periodic over Omega. So, if you take the magnitude and phase, are normally plotting like this, but this is wrong, I should plot mod versus Omega I should plot angle versus Omega, but just to make my point I am just plotting it, e plot versus Omega. So, may be pi here, minus pi here. Therefore, extend from pi to 3 pi, with 2 pi here, and minus 2 pi, sorry minus 3 pi, minus 2 pi, minus 4 pi.

Here again, the same thing, minus pi 2 pi, next period is pi 2, 3 pi right. So, it will be repeated. Minus pi to pi 1 period, I think I will re-draw it. This, this I make some mistake here. The boundary is not correct not correctly done. Suppose, minus pi to pi that is 1 period. That pi 2 3 pi, minus pi 2, minus 3 pi. So, whatever you have here, that will be repeated to over this, over this. So, if you have things like this.

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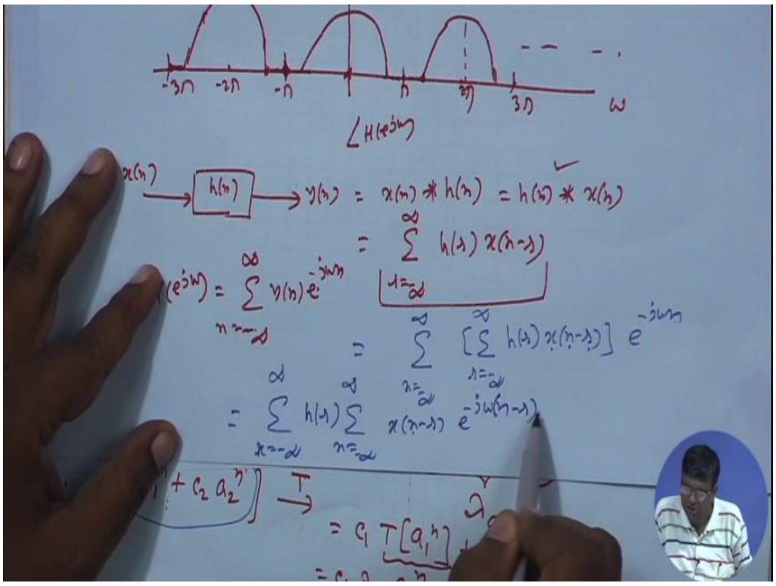

Suppose then again here, there will be a gap, it will be like this. Here it was 0, so next sent a frequency to 2 pi, here it was 0, next sent a frequency to minus 2 pi. So, there is a gap, there is a gap, because this gap, this gap, and this gap. So, on these said also this gap, this gap. The same thing, from minus pi to pi, that gets repeated here, here, like that. So, it will be like this, continue it is a periodic function of Omega, normally it should be magnitude versus Omega and parallely phase versus Omega, another plot, phase versus Omega, both will be periodic over Omega, over a period of 2 pi.

Normally we do not put a magnitude, sometimes just, just write these to indicate n power that I am, indicating both, but that is actually technically wrong, because I should have magnitude separately, phase separately, but never the less does not matter, because my point is here is to show, that periodical Omega. So, for that I just plot capital X e to the power j Omega right here, because argument is clear. The same logic if I write as magnitude here, and phase here, same logical you will it have there alright. So, there is one property, important property. Then another thing which you have already seen, that if there is a convolution suppose xn, convolute to h n, you get yn. Because yn is x n, convolute with h n, or equivalently h n convolute to with xn. We have seen, by comparing two integrals, we have seen by comparing two integrals that output DTFT is product of the 2 DTFTs, but this you can work out directly, what is, between the 2 convolution as I told you earlier, as I have been telling you, I will prefer these form.

So, if you take that form, it will be hr, x, n minus r. So, what is this is yn. So, what is ye to the power j Omega, if I compute directly yn e to the power minus j Omega n, by the DTFT formula, yn to the power minus j Omega n, there is directly you can compute the DTFT, but this yn now I replace, by this much. That is this summation comes, this comes, and then outside e to the power minus j Omega n. Now I have a double summation, as I told you in DSP, whenever you come across double summation, next step is to interchange the two. So, this summation suppose, comes out, and inside we have got summation over n, h of r, it does not depend on n. So, this quantity can come out as common, you have got x, n minus r, it remains same side first I interchange the two, that from inner summation h r comes out, because inner summation now is with this 2 n, h depends on r, not on n. So, with can come out as common and inner I have got xn minus r that depends on n. So, it remains e to the power minus j Omega n that depends on n. So, it remains, this is the thing.

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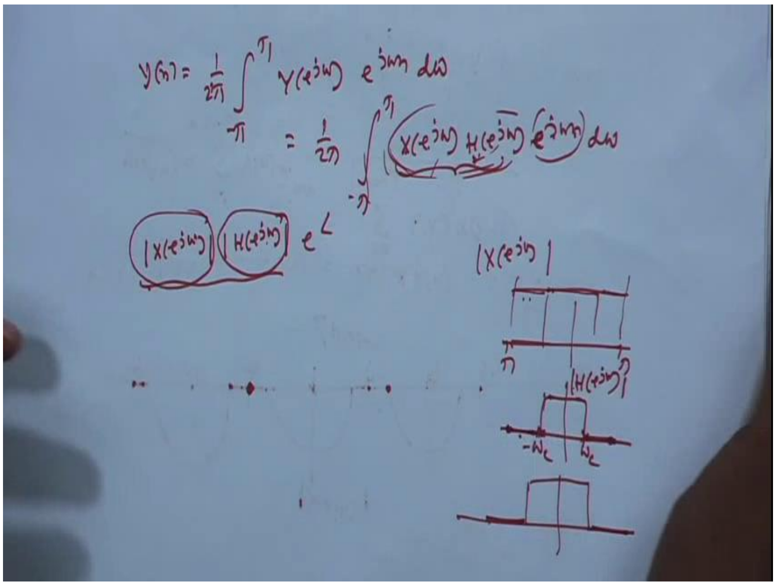

Now, I have got n minus r, and I have got e to the power minus j Omega n, suppose I want to make it n minus r. So, I have brought an extra term, e to the power minus j Omega, minus r, that is e to the power plus j Omega r, I have brought in here. So, I have to cancel it. So, I should have e to the power minus j Omega r. And this means, this total thing they depend on n, these depends on n, these depends on n, but this fellow does not depend on n. So, it can again go outside this first summation, inner summation, as common. So, it will be r, h of r, to the power minus j Omega r, and summation. Now n minus r, you can give a name m. So, if m is minus infinity, n is minus infinity, m also minus infinity, r is fixed from outside. So, it is a constant in the inner sum this r is coming from here, once you choose the r, when you go here r is fixed. So, r has fixed value.

N is varying, from minus infinity to infinity. So, as n goes to minus infinity, m also goes to infinity, minus infinity, and for n goes to plus infinity, these also goes to plus infinity. So, I write in terms of m, n minus r, I write has m, this is m now, now what is happening, in the outer sum first you choose the r, then carry, carry out the product h re to the power minus j Omega r, hold it, then for that r, you run this summation. Then again choose another r, carry out this product, or then hold it for that r again run this summation. But this summation is independent up r. Now it is I, m, for any, I mean this summation is what you can say, this is nothing, but DTFT of x m, but any m you choose, summation minus infinity to infinity, e to the power minus j Omega m. So, we get this, at this again does not depend on r, this summation is a constant, independent of r, independent any independent of any index. So, this will go outside.

So, this you know this it remains. This remains as it is. What is this? This is again DTFT of h n, h n, h n sequence. So, this is a product of, did I get a get the proof directly. This, this DTFT is nothing, but product of the two, this is a direct proof, earlier we just, arrived at that by comparison, of 2 integrals so; That means, output yn is a inverse. We have already seen this, I am re-writing. Is an inverse DTFT of; you can write as this. This has a good physical interpretation, you see, this is the sequence, because this is dependent on n. This is the sequence, the integral is with (Refer Time: 16:02) to Omega, there is, you are carrying out this, this is the amplitude of the sequence. This a, this like a carrier e to the power j Omega n, n is like the time axis, instead of time, this is the index, pointing to the time axis, just discretized, and it is multiplied by an amplitude what is independent of n. And you are super imposing, similar such carriers, by varying Omega continuously, not only 1 frequency, but various frequencies continuously, over this batch, and then adding all of them.

So, this is a super position of various carriers, of continuously varying Omega. Now this amplitude part, one is coming from input, another is by your design. So, magnitude of the amplitude is, this is the magnitude of the amplitude, then there is an angle part, angle part will get added with this, this is a magnitude part, here, you can make it large, you can make small, by choosing this large, or those small, for a particular Omega. For certain range of Omega, you can make the magnitude of this, see irresponsive, DTFT large.

And for certain densities of Omega, you can make this small. Which means, input DTFT will be is amplified, for certain densities of Omega, and they will disappear they will become 0, or very small, in certain other ranges of Omega where you design, your system to be such, that the corresponding DTFT, mod DTFT, has very low value almost 0 at those ranges of Omega. This gives raise to idea of filtering that, input we have, this mod xe to the power j Omega, high, at all values of Omega, from minus pi to pi is arranged, for any DTFT, minus pi to pi, suppose it has high value, same value, but I can design my h so that, may be over a range, it has high value and then remaining range it has 0 value.

So, when I multiply and then take the integral from minus pi to pi, only, maybe from minus Omega c 2 Omega c, some value only these matters. Because only during this part, input function, input DTFT at therefore, is magnitude will be multiplied by, the system response DTFT magnitude. So, this part is non 0, because these multiplied by these, but this range of Omega, whatever may be the input, DTFT value, we call this spectrum, that will be multiplied by 0. So, those frequencies will not go through in the integral, they will be I contributing nothing. So, output DTFT will be just a product of the two. So, this, this part is absent, this part is absent. This is idea of filtering, that passes certain frequencies, and do not pass certain frequencies, by designing this mod e to the power j Omega, accordingly. And thereby you, you know the idea of, band pass filter, low pass filter, high pass filter, these things come.

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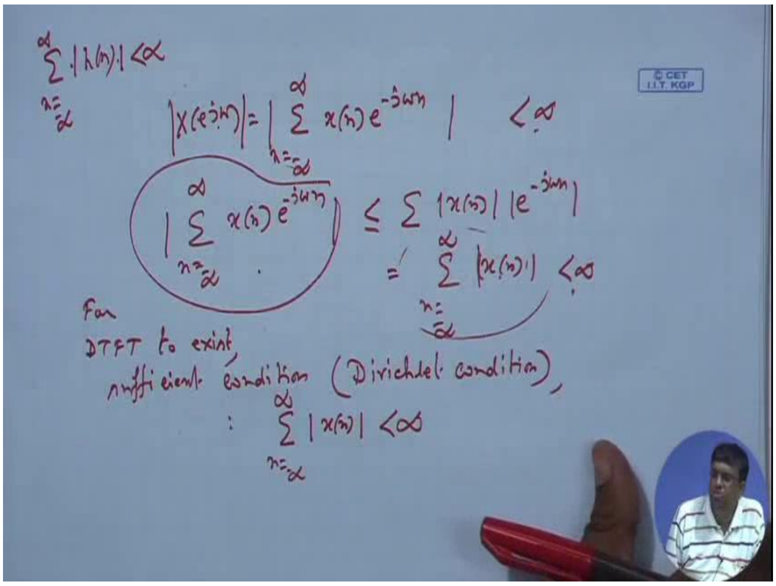

The next thing, I have assumed, or by this summation, DTFT exists alright? But you see this is a infinite summation. So, infinite summation does not mean that the magnitude of the summation may converge, or may suit up to infinity. If it does not converge, suit up to infinity, then DTFT does not exist. For that Omega here put in, it does not exist. For DTFT to exists, mod of this, for all Omegas, should be finite whereas, mod of this should be finite. Now I told you the other day, they have done mod of a summation using the triangle inequality, you can write, mod of a summation less than equal to summation of the mods. So, this, is less than equal to summation of mod over this, mod over this, and mod over product, is product of mods, and mod of e to the power j Omega n, but you see e to the power minus j Omega n, it is mod, it is magnitude is 1. So, this is nothing but. Now I want this left hand side to be finite, but left hand side is less than equal to this.

So, if this is fine made finite then; obviously, left hand side will also be finite, and therefore, DTFT will exists. So, sufficient condition, for DTFT to exist, is that, the sequence, it is mod value, and then if you sum it is to be finite, that is sequence should be absolutely summable. We have already come across this absolute summability in the context of stability. There is give a linear shift in various system of impulse response h, you will sample response hn, system will be stable as sufficient condition is that, it is unit sample response should be absolute summable, that is you take absolute value, and there some over all the n, it should be finite, if the sufficient condition if it is true. The system is always stable, though it if is not true, it does not mean it will always be unstable sometimes it can be stable also.

Similarly, if this finite, suppose this is finite, then this is finite. So, DTFT will always exists, but it does not mean, that if it is not absolute summable, there is absolute value and sum. If it is suit's up to infinity, it does not mean that, this (Refer Time: 22:16) is also suite up to infinity, because it is less than equal to. So, right hand side can still be infinity, this can still suit up to infinity but left hand side can be less than infinity there is finite. So, this is sufficient condition, that for DTFT to exist, for sufficient condition also called Dirichlet condition, is that you can take the absolute value of every sample of the sequence xn, and they are summed up the absolute values over all the indices, this sum should be finite.

This is summable, if it is satisfied, this mod is less than infinity, then DTFT exist, but it does not mean that if it is not satisfied, DTFT does not exist, because this is less than equal to this. So, it is it, can be the right hand side also, and if right hand side goes to infinity, still the less lesser side will imply that left hand side is finite, it is less, less than or equal to, equal to means, left hand side also remains infinity, then does not exist. But less than means, it may exist, but if the right hand side is less than infinity, it is finite I am very happy, then DTFT variable changes. This is sufficient condition.

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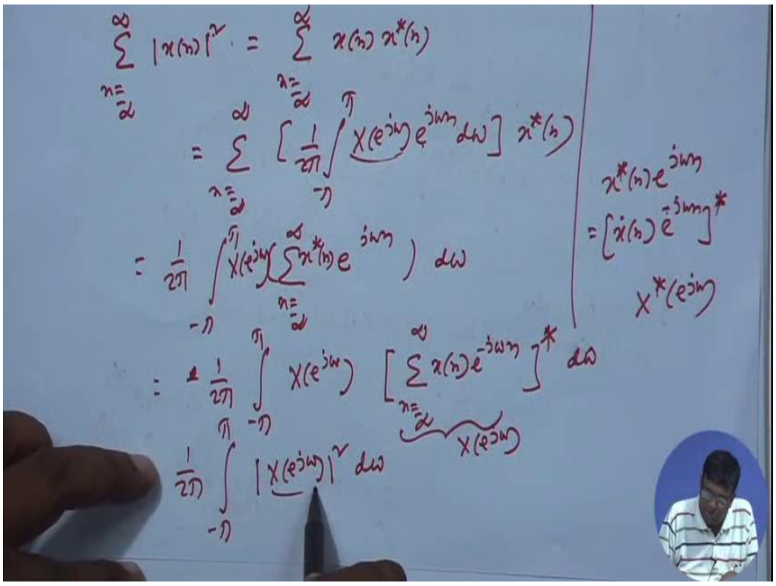

One important property is called Parsevals theorem. If I suppose take the mods value of every sample, square it up, it will give me the power, instantaneous power in that sample. And if I add over all them, I will get a total energy in the signal. If the energy is finite, it is called energy signal, if the power is finite, but summation is not finite I could the power signal. Now suppose this is given to you, and it is energy signal, and summation exists. Then I can write this xn into x star n, because mod xn square means, like mod j square means j into j star, xn into x star n. Then, xn I can replace by it is inverse DTFT relation, and outside x star n. Then there is a double summation. Next step is to interchange the 2 summations, 1 by 2 pi goes out, minus pi 2 pi outside, with respect to Omega, inner summation is with respect to n, anybody which has a value, which is dependent on n, will be inside. So, e to the power j Omega n, will inside, x star n, will be inside, this guy does not dependent on n, so it will be common outside, and d Omega.

Now, this summation, x star n, x star n, e to the power j Omega n, you can write it has xn; e to the power minus j Omega n star, conjugate of product is product is product of conjugates. We have seen the other (Refer Time: 26:20) I am reminding you, suppose z1 is r1, e to the power j theta 1, z2 another complex number r2, e to the power j theta 2. So, we have got z1, z2, as z, you have got r1, r2 e to the power j theta 1, plus theta 2. You have got z star, it is star of a product, it will be r1 r2, e to the power minus, this will become minus, theta 1, plus theta 2, which is same as r 1, e to the power minus j theta 1, r 2 e to the power minus j theta 2, which is z1 star, z2 star, right? z1 star means r1 e to the power minus j theta 1, r2, e to the power minus j theta 2. So, star of a, these are basic things, you should know, but I am not taking a chance, chances so, I am carrying it out. So, z1 z2 as z, star on that, that is star of the product, is product of stars.

So, x star n, e to the power j Omega n, I can write like this, because star of this product means star here, x star n, as star here. So, minus will become plus. So, this summation, star d Omega, but what is this summation? DTFT; it will be for any Omega, x, capital Xe to the power j Omega here, and it is conjugate here. So, this will be x star. With a star, it will be x star, e to the power j Omega. So, x e to the power j Omega, x stare to the power j Omega, it will be minus pi to pi.

This shows that, to calculate the total energy if it is an energy signal, you square up the mod value of every sample, take the mod and then square it up. So, that is an instantaneous power of that sample, and then sums it over all indexes, that are put on the energy. That you are carrying in, carry computing in time thing, you can do the same thing in frequency domain, take the mod value of the DTFT, square up, and integrated over, all the Omega from minus pi to pi, just multiplied by 1 by 2 pi. This called Parsevals theorem. It is very much useful in many contexts.

I stop here today, I will continue from here, in the next class.