Discrete Time Signal Processing
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Lecture – 03
Linear, Shift invariant Systems

So, just let us try to recall what we did last time.

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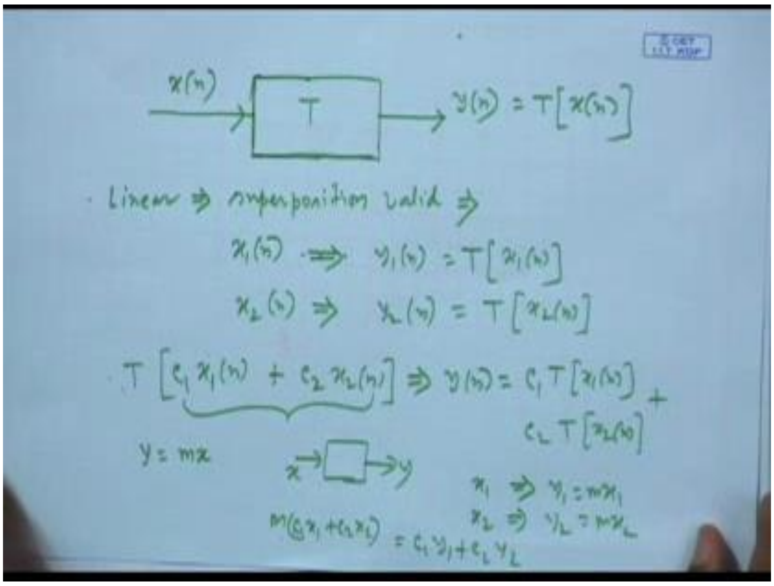

We discussed linear systems but linear systems we meant that principle of superposition should be valid that these are system who is x n as a sequence as an input produces y n system is repeated by an operator T just an operator which works on x n in somewhere rather and gives you y n. This system will be called linear if the principle of superposition holds good, what is superposition that if I give one input x 1 n, find the output. So, y 1 n which is T working on x 1 n.

Similarly, if I give another input x 2 n and suppose the output is y 2 which is T working on x 2 n then if I do not give x 1 or x 2 but if I give a linear combination of them that is x 1 n multiplied by C 1 there is all samples multiplied by constant scalar C 1 x 2 n multiply by C 2 all samples are multiply C 1 and C 2 could be 1 also. So, this is more general combination and then I add that 2. So, I can get a general new sequence.

If I input that at the input of the system the output should be if it is linear output should be this, that is this component and this component their addition what is this component there is a response of the system due to this part is here, but C 1 constant which comes out that is C 1 times a sequence if I give as input the output will be same as if I give x 1 as input and then at the output will multiply by C 1 there is at the output this much multiplied by C 1 this is a constant can come out.

So, it is one response due to this part and another response due to this part and, but the combined sequence the output response is a combination of the 2 additions of the 2 that is system output in of speaking in system output due to multiple inputs is a summation of the individual responses due to those respective inputs separated inputs this is a mathematical expression that if I give a linear combination up to 2 sequences x 1 and x 2 n system output will be what same C 1 times response due to x 1 and same C 2 times response due to x 2 their addition.

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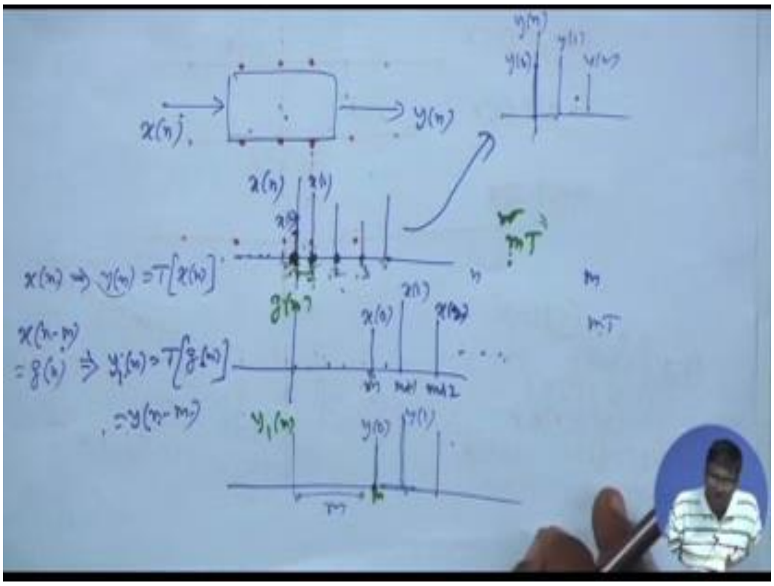

Another thing was very important that is shift invariant which comes is related to time invariants if the sequence is opted from time signal that is suppose system it takes x n as input y n as output and to for understanding the x n sequence which is shown here you imagine that it was obtained by sampling and analog function. So, this is after the corresponding to a sampling period 0, this is T, this is 2 T, this is 3 T, this is 4 T. So, though I do not write T, but in sequence told you we have only base are the index numbers we do not care about the time, but in your mind you know this is for 0, this for T, this is for 2 T, this is for 3 T like that on sequence such a sequence is given here and some output we are getting.

Now, question is if this sequence is delaying in time domain there is a shifted means if I shift it by say n number of points. So, 0 goes to m 1 goes to 1 plus m there is m plus m 2 goes to m plus 2. So, the same sequence you see at the shifted physically it means in time domain I am introduce you delay; 1 delay means T, 2 delay means 2 T, 3 delay means 3 T. So, m times since 0 goes to be m times T 1 T 2 T. So, m T similarly this point which is capital T that goes to m plus 1 input capital T to at another m into T delay.

So, m plus 1 T this is 2 T at another m into T delay m into T we go to here that means, in terms of time this sequence is delayed if as a sequence I say it just shifted because do not write the time here I just count the indexes. So, 0 goes to m-th point first goes to m plus 1 and second goes to m plus 3 like that. So, it is shifted, but actually in real time it is delayed by m into sampling period T that many sampling period that many sampling periods the entire thing is delayed if I give that to the input system and the system behavior suppose is not changing it is permanent it is constant it does not change.

So, if I give the current input I get a current output if it is instead of giving the input, now if I delay it by some amount of time and then give the same thing, then output also in nature will be same as the current output only thing is you have to replace from now what one that means, if input is shifted by some amount output current output also will be shifted by the same amount and that should not depend on the choice of m it should be true for any choice of m positive negative very high very low all if that be that is system property is not changing over time that is called shift invariant or time in time invariant directly show the time points 0 T, 2 T, 3 T, etcetera time invariant, but when I just write the index numbers and carry out by business like that then I call it is shift invariant mathematically it means.

Suppose, I give x n output is x n as input output is y n which is T x n and now I delay the shift this sequence by m x n minus m which is a shifted version. So, I call it is g n that is this sequence I call g n, but this is shifted version of original x n shifted by n everybody got shifted by m I call it g n if this g n input to the system the new output which I call y 1 n there is response of the system due to g n that should be nothing, but shifted version of original y and shift by the same amount m and this should be true for any m then it is called shift invariant.

So, now we will be considering systems which are both linear and shift invariant sometimes we call linear and time invariant because I mean we use this term time invariant for, of in any analog domain. So, by bad habit you can say we carry it over here also, but strictly this shift invariant.

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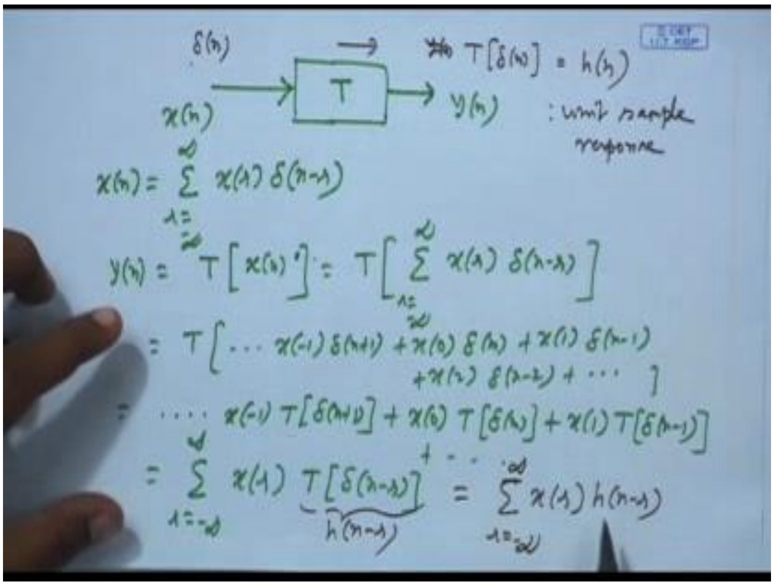

Now, consider a system this is x n, this is y n. Now, I remember yesterday in the last class I showed that any sequence x n can be repeated it as super in position of deltas delta sequences when we see if I can find out the slide is here.

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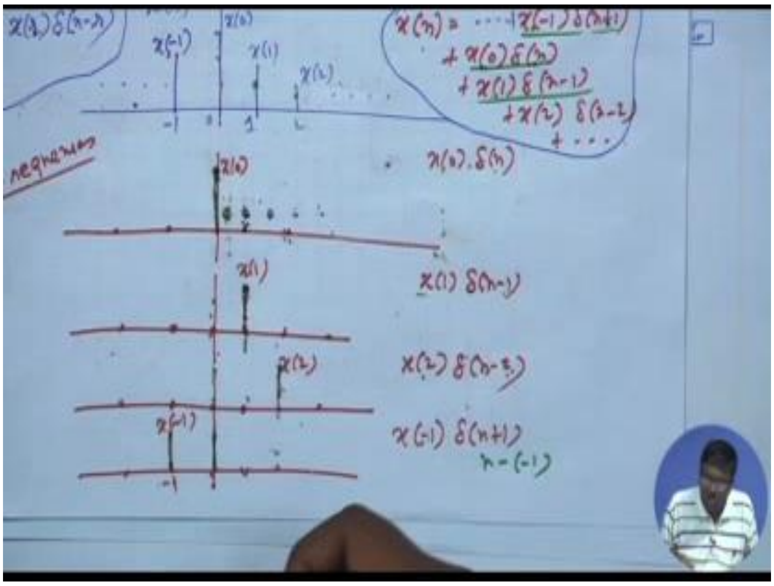

Suppose, this is a sequence x 0, x 1, x 2, x minus 1 like that, but this is same as start with 1 sub sequence with height with 1 sample here is x 0 and all that 0s then another sub sequence. So, this is nothing, but x 0 into delta n because delta n has 1 here and 0 everywhere. So, if you multiply it by x 0 1 becomes x 0, 0 remains 0.

Then another one I pick up the x 1 from here put it here all other points are 0 then this should be nothing, but x 1 into delta n minus 1 because delta n minus 1 means the 1 from here will be shifted here because it is delayed by 1. So, it will come here that 1 value multiplied by x 1. So, total height will be x 1 others are 0s then I pick up x 2 bring it here all other points are 0 this will be nothing, but x 2 delta n minus 2 that is delta n means 1 here that will be shifted by 2.

So, it will come here then multiply by x 2 0s everywhere I consider x minus 1 from here put it back this place all other points are 0 this will be nothing, but x minus 1 delta n plus 1 delta n plus 1 means n minus 1 that is n plus 1 that means, it will be shifted by minus 1 that is will go delta n had a 1 here that will be instead of going to the right it will go to the left or come here then multiply by x minus 1. So, for a light x minus 1 so on and so forth that you see if I add all of them you get x 0 plus 0, x 0 x 1 and 0 0 0 elsewhere, x 1 0 0 x 2 0 0 0 to x square.

So, get back your original sequence that means, original sequence x n is superposition of various sub sequences 1 is n 0 into delta n is 1 one is x minus 1 delta n plus 1 there is this one is x 1 delta n minus one there is this 1 x 2 delta n minus to the this one this. So, we can write in a compact manner is this way x r delta n minus r that is x 0 delta n minus 0 x 1 delta n minus 1 x 2 delta like that. So, there is all will be using now this here you see this same thing I am writing here. So, y n is T working on x n, right now x n I substitute by these.

Now, the system is linear and is a summation, is a summation of various sub sequences right. So, what all response will be summation of individual responses, because of linearity overall response there is for your signifying write the summation in the detail way in the in the you know in these manner x minus 1 delta n plus 1 x 0 delta n x 1 delta n minus 1 x 2 delta n minus 2, dot, dot, dot, dot these is summation. So, this T working on this total if is linear then if it same as T 1 working on the same individual ones and their summation and when T 1 works from the individual 1 this is likes the constant. So, it will go out this is the sequence part. So, it will be like, dot, dot, dot, dot x minus 1 will go out T will work on that then x 0 will go out T will work on that x 1 is constant again. So, it will go out T because this is a sequence it has n this does not n everything is a function of n is the sequence.

So, T will work on the sequence this is just a constant and, dot, dot, dot, dot in general; that means, in this a expression you can write T will work on this total and then this constant will go out we will work on these. So, x r T working on delta n minus r this is nothing, but this summation I am writing in this way all right, so far so good. Now, suppose if you give delta n here and the corresponding output y n corresponding output say T delta n you record and call it h n call it unit sample response and by use thing we impulse response, but actually it is a unit sample response because this is very much in by seeing this symbol looks like impulse function of a analog. So, we call it by mistake, but to the mistake is accepted as impulse response actually it is unit sample response.

So, if I say impulse response you try to understand that I am willing unit sample response suppose delta n has these that is unit sample response of this system there is you give him delta n as input what is the output we call it we call it h n which is unit sample response. So, T delta n h n, but in the system shift invariant that if delta n is shifted by r it was delta n now n minus r if it is shifted by r output view output T of this actually will be nothing original output shifted by the same amount. So, this will be nothing, but h n minus r T delta n was h n if delta n minus r that is if the sequence is shifted by r amount view out will be shifted or percent of the original 1 h n and amount of shift will be same r because it is shift invariant. So, then we get this result this is a famous thing it is called convolution between x and h there is we symbolically we represent these by like this.

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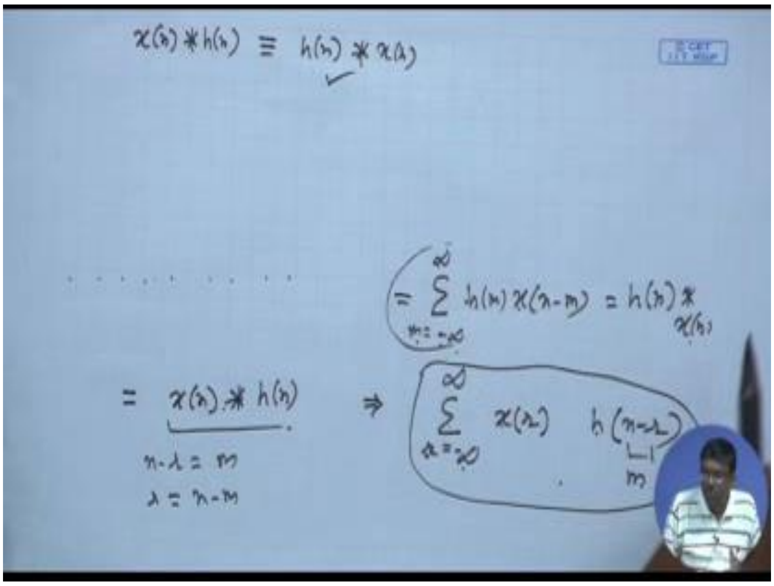

Let me explain the meaning of this symbol, whenever you come across this x n dot h n, first guy is x n second guy is h n then what we have to do you first bring a summation then bring in index r r l r a everything, but n not n because by this convolution you are trying to find out y n. So, n is your choice you wanted to find y at a particular n of your choice n is fixed by you than n in this summation is fixed r is only variable.

So, remember n is fixed by you from here from left hand size you want to find out the output at a particular n of your choice that n here r is variable. So, that is I am saying when I am writing in this form you do not bring n here bring some local variable r or l or m or k or I or anything and you should have the range from minus infinity to infinity I am telling if you if you get this expression what is the meaning first to bring a summation symbol bring a local index r a anything, but not n take it from minus infinity to infinity to be most general then first term will be first guy second term will be second guy, but the first term you make it a function of r this index, but the second term you bring it a function of minus of r and then plus n, n is your choice you want to carry out this at a particular n, so that n minus r.

So, first bring a summation then a local index r from minus infinity to infinity write first guy x second guy h within bracket for the first guy r will come within bracket for the second guy n minus r will come this is the convolution all right this convolution has some properties first is equal to commodity, commodity means x n convolve with h n will be same as h n convolve with x n commodity will like to; now, if you take 2 decimal number their addition is commodity a plus b is same as b plus a, but subtraction is not commodity a minus b is not same as b minus a.

Similarly, their product is commodity a into b, b into a they are commodity if you, but division is commodity a by b is not same as b by a by the way product of scalar is commodity decimal scalar a into b same as b into a, but if you give matrix x not matrix a 1 matrix b then a into b it not necessarily same as b into a. So, remember this product as long as just scalar decimal numbers scalar numbers their product is commodity, but instead of scalar you have matrices then there is a problem anyway there are plenty of commodity operations in real life convolution is 1 of them, how this have to show now in this summation you see n minus r I give it a name small m this n minus r I, call it m which means r n is fixed r is variable I am substituting I am bringing m to replace it.

So, what is r, r is n minus m. So, in this summation x of r instead of r I should have n minus m and h instead of n minus r I should have m and I can write h first, x second because this is x is 2 scalar numbers multiply. So, this into this same as this into this, I can write h m first x n minus m summation r now it will be over m now you see for an r goes to minus infinity it is n minus infinity, n plus infinity. So, your m goes to plus infinity m should be plus infinity and when r goes to minus infinity, sorry omega r plus infinity n minus plus infinity there is minus infinity m goes to minus infinity that means, it should be from plus infinity to minus infinity in the opposite direction from plus infinity where we will go back to minus infinity.

But it is a discrete summation. So, discrete summation if you sum from right to left like this or left to right like this, you cover all the points. So, it does not make any change therefore, summation from plus infinity to minus infinity or minus infinity to plus infinity, you cover all the discrete points on this time axis right. So, it does not change. So, I again write as m minus infinity to infinity now look at this thing there is a summation there is a local index m range from minus infinity to infinity like here summation local index r minus infinity, infinity first guy h, second guy is x in h within bracket this local index come in that is of x what comes n minus the local index.

So, it is like convolution between h and x evaluated at nth points. So, this is n like here also n this source that x n star h n is same as h n star x n you study the x n star h n I met this substitution and this immediately becomes h n star x n first guy is h transfer of the m second guy is x transfer of n minus m. Here, if read those function of r second guy was n minus r because r was replacing again all right and now I tell you always whenever you solve problem in all that instead of choosing the convolution this expression for convolution this expression you always choose these expression, this is a required expression, this is a rule of thumb then your problems become lots thing.

You will see I mean steps will become same guide and whole problems become same size if you strict to this original form things become little complicated that is why instead of this form, we always choose this form this is a rule of thumb it comes out of practice out of experience this what I observed. So, I am telling you, this as a tip I m giving you that whenever you have convolution between 2 sequences x and h like x n is the input h n is the impulse response of the system instead of taking here x n as h x n convolve with h n that is this formula you reverse it write it as x n star h n in this formula and then your sub sequence still become easy this is a rule of thumb.

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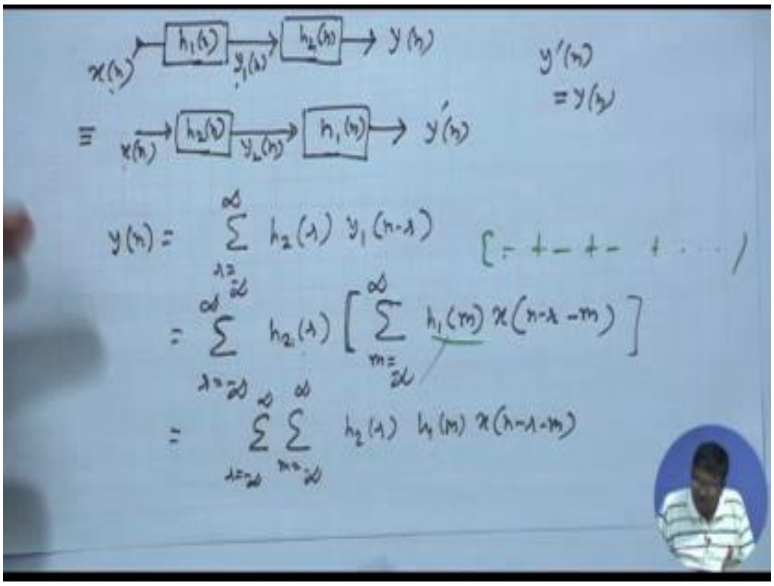

So, what we are proved here is that the convolution is commutative the next thing is convolution is distributive suppose I got x n I am got a system h 1 n. So, I it produce as an output y 1 n that goes into another system this is linear and shift invariant. So, for which is characterized by is unit sample response output is given by the convolution between the 2, this is a linear shift invariant system what you have seen a little while ago that for a little for a linear and shift invariant system the system is fully give him by characterized by it is unit sample response that is if we give delta n as a input the output whatever it is that you write in that case for any other input x n, we can carry out the output calculation that is nothing, but the convolution between them.

So, y 1 is a convolution between x n and h n and equivalent between h 1 and x n because convolution is commutative then this y 1 n goes to h 2 n again. So, this is the input now for h 2 n and new output is y n. So, y n is convolution between y 1 n and h 2 n or between h 2 n and y 1 n currently what is h 2 n it is another linear shift invariant system since linear and shift invariant it is fully characterized by it is unit sample response that is if I give delta n has input, here the output will be x 2 n that I record and put back here in that for any other input say y 1 n corresponding output will be a convolution between them between y 1 and h 2 and equivalent between h 2 and h 1 y 1, what I will show is that you can interchange the 2 and still you will get y n there is I can first bring h 2 n this input is x n.

So, this output will be defined now call it y 2 n and then you have first one h 1 n here and interchanging and still you will have y n this is by claim y n then if you call it y prime n we will show that y prime n is same as y n.

So, the 2 can be interchanged how to show that. So, let us work out y n, y n is a convolution between these two, but I told you always choose convolution instead of not between y 1 and h 2, but between h 2 and y 1. So, you write summation and index r h 2 first r y 1 n minus r what is y 1 there is a convolution between them convolution between x n h 2 x n h 1 or equivalently h 1 and x I will prefer h 1 and x. So, introduce an index m minus infinity to infinity bring m here and x not n because I want to find out y 1 and at n minus r. So, n minus r minus m instead of n I have his much index coming from here earlier we had y 1 n and now I have x that time I had x n now while n minus r. So, if it is x n minus r then minus here.

Now, what is this summation this is outer summation outer summation means every time r you take a value same is h 2 r 1 value 0 1 2 minus 1 minus 2 like that keep that r for that you carry out this full summation for various values of m multiply that by h 2 r then take another value of part hold that value and again carry out this summation put that you have to put that r always what about r you are choosing here you have to bring it here all right then distinguish same as.

So, this is a basically h 2 r for that you are carrying out the summation then h 2 another r you are carrying out a summation you are putting that those r's here you will write the same thing you can write like this you bring in h 2 r from outside there is first you have working out a summation then multiply by h 2 r it will be same thing if h 2 r is brought inside a summation it multiplies every term here for each m like if you expand this summation every term you multiply the h 2 r.

So, h 2 r every term h 1 m every term means for each m all right that is what is happening is these this summation you are expanding one term another term another term for various m every term you are multiplying it by h 2 1 same h 2 1 same h 2 1 for various that is a typical term is this is multiply by h 2 r and then you are bring for all m you will have the same thing that is instead of first carrying out a summation and then multiply by h 2 r you bring h 2 r inside multiply by everybody and h 2 r and then a you will get the same thing right, but now what will I do I will take h 1 m what I can do I can interchange the 2 summation why because how is the 2 thing going first you are choosing an r then you are carrying out this summation for all m then another r again you are carrying out a summation for all m you will get the same thing if I fix m and then carry out this summation for all r I get another n carryout the summation of for all r this entire thing is a function of r and m.

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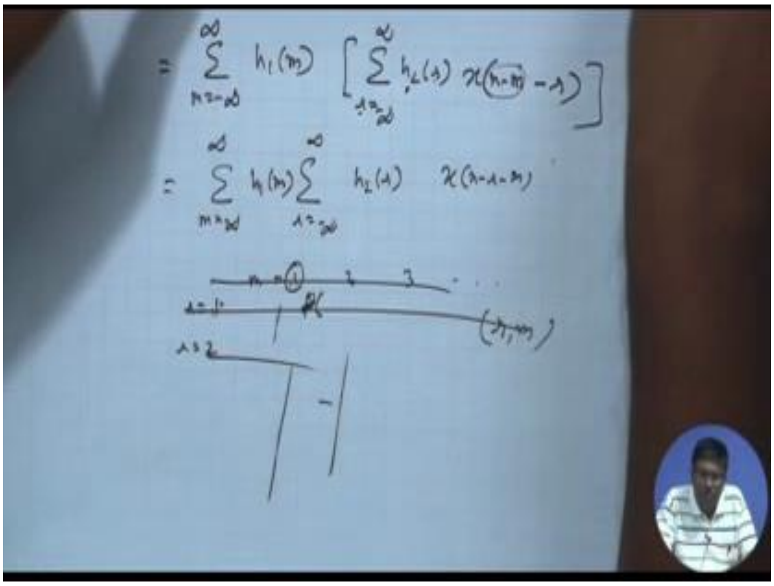

So, I am this summation means what r equal to suppose we are choosing 1 and then you are carrying out all summations f you know r equal to 1 and say m your say m equal to 1, 2, 3, dot, dot, dot, dot then again r equal to 2. So, r equal to 1 1, 1 2, 1 3 like that then again r 2, 2 1, 2 2, 2 3 like that instead you can hold m equal to 1 first take value, value r and I mean taken what the intern range then take what the intern range you will got the same points only.

So, either you first fix r for that r inside carry out this evaluation for all m's and add take another r again for that r inside you carry out this summation for all m and add alternatively you can fix m and vary r what the intern range then take another m again vary r what the intern range take another n vary r what the intern range you will cover all the points all r m p r's are from here in the m p r's will be covered r will be from minus infinity to infinity m minus infinity to infinity there is either r you fix and then variable another r again vary I over the same range another r again vary a over the all the range same range or alternatively every time is fix m then what the intern range.

Another m again what the intern range you will cover all possible of pairs of r m either way which means I can interchange the 2 summation, now I can bring out inner summation over m outside or you can bring summation over r inside and this entire thing from this these h 1 m can be brought out because that does not depend upon r it is like a common for every term every other terms is dependent on r this summation is over r these depends on r these depends on r they will be inside, but this guy was here, but it is common it does not depend on r it is therefore, everybody in this summation.

So, it can be push outside as a common you carry out the summation first then multiplied by h 1 n and x at x n minus r minus m. So, I write rewrite again x instead of n minus r minus m I can write as n minus m minus r same thing, but what is this is convolution between h 2 and x at n minus m r is the local index is a r here minus r here. So, n minus m m is fixed from outside every time that the. So, inside the summation m is fixed n is already fixed. So, this is a fixed number r is only varying. So, it is a convolution between h 2 and x what was convolution between h 2 h 2 n x y 2. So, this means this is y 2 this y 2.

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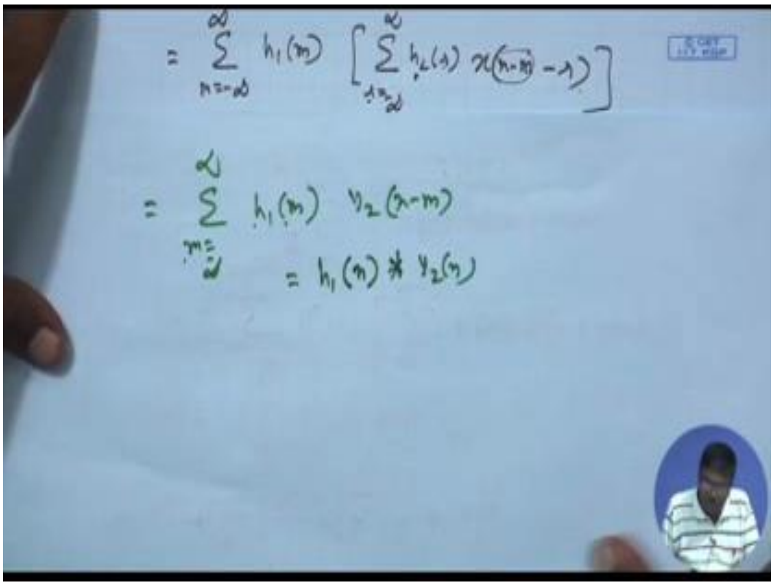

But at this point; that means, this is h 1 m y 2, but at this point again it is a convolution between h 1 and y 2 local index m come here with minus sign here n here. So, this is convolution between n star y 2 n. So, that is what we have here then we see what we are getting h 1 convolve with y 2 what same as y 2 convolve with h 1 and what is y 2, y 2 is a we have already seen convolution between the two. So, y prime n this is same as y n all right we started with y n and this is what this figure shows to be y prime n all right. So, this proves the equivalence which means you can interchange the two that is all for this session, now we will move to the next session.