Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Optical Communications

Week – XI
Module-II
BER determination

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Hello and welcome to the mook on optical communications. In this module which is actually a continuation of the previous module we will complete the discussion and derivation of the bit error rate or the probability of the bit error for the on-off keying systems. And then compare different receiver techniques as to which one will be better in terms of the sensitivity of these receivers okay. So if you remember what we were looking at in the previous module we had arrived at an expression for probability.

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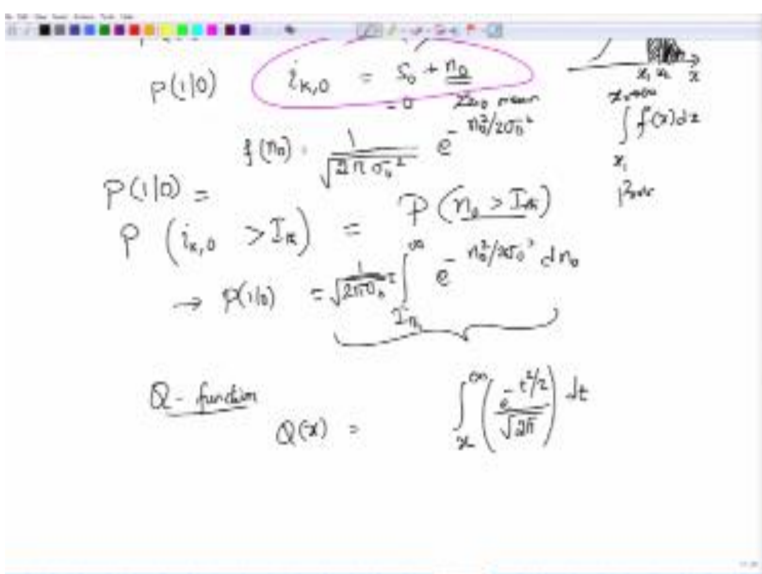

That the receiver would declare a bit as 1, when the bit that has been transmitted is actually 0. So this of course, is an error right, so when the bit is declared as 1 when the input that has been or when the bit that has been transmitted it 0 would lead to an error, there will also be an error of the same form when the received bit has bit 0 while the bit that was transmitted is 1.

While arriving at an expression for the probability of mistaking a 0 with a 1 which is given by this probability of 1 given that 0 was transmitted, we found that this actually is given by this integral of this particular function okay. Because this integral, you know is very popular it comes up in many, many places there is a specific short hand notation that we actually use and this notation is what we call as the Q function.

The Q function is defined by this expression which is given by 1/ square root of 2Л and area of the curve from X to infinity, which curve is that, that is this particular curve of course, you can actually push this 1/ square root 2Л into it, so you will actually be looking at the area of this particular curve which I just wrote down in the bracket okay. And this area from X to infinity what kind of a curve is this.

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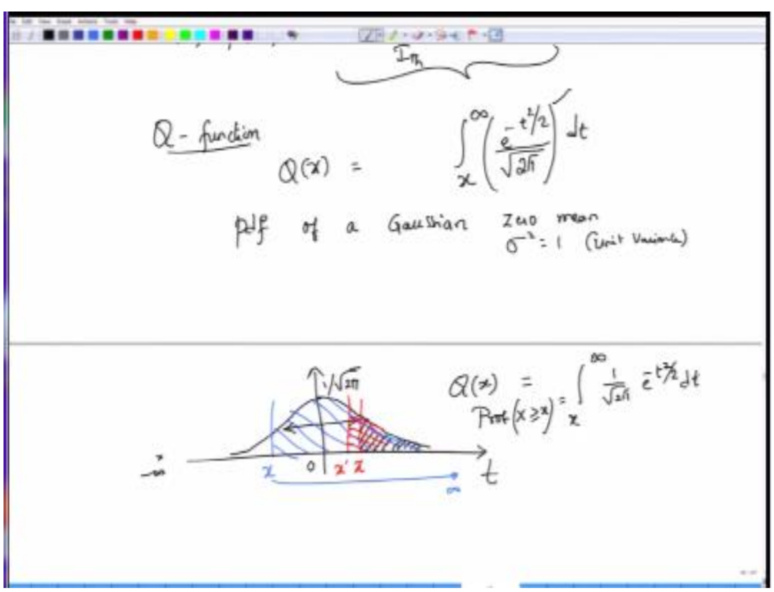

If you look at this curve you will realize if you compare this one with the Gaussian random variable, you will realize that this curve is actually, you know an expression for a probability density function of a Gaussian random variable okay, that is a random variable which has a Gaussian distribution or Gaussian density function with 0 mean and variance equal to 1 this is called as unit variance.

So it has 1 unit of variance and it is 0 mean, which means that if you were to sketch this function as a function or T, so if you sketch this function that is in this bracket as a function of time, you would see that it is actually centered at 0 at which point it will actually be 1 and thereafter, it would go decaying as you go, you know for both in terms of higher values of T in positive side as well as in the negative side.

The variance is actually the difference between the points where the function value drops to 1/e of this one or $e^{-1/2}$ or something like that okay. So it does not really matter at T=0 this actually has an amplitude of 1/ square root 2Л which we forgot okay. So if it drops to 1/e or this value or ½ e of this value that is essentially the variance of this one.

But it is important to note that this curve right never touches this P axis, the horizontal axis because the curve goes to 0 only at T=+infinity as well as T= -infinity. This curve is also symmetric, now on this curve let us pick a certain point okay. Let us call this point as X okay, for example X could be 2.3, X could be -1.5, it could be anything.

If you look for the area under this portion of the curve right to the right side of the X that you have picked that would actually tell you the Q function. So in fact the Q is defined by lighting this lines okay and then evaluating the area for example you might try a different x´ value and then find the area under this curve you are feeling adventures you can try this value of x okay this time x would be negative but this would the area under the curve that you are looking at so by writing different lines.

And then looking at the area under the curve beyond to that particular line you know this vertical and that we are drawing from that point all the way up to ∞ that would be the definition of Q(x) function okay if you make x=-∞ which is good all the way up to ∞ here and then look at the area under this particular curve although way up to t=∞ that value will be equal to 1 right because the area under the probability density curve must integrate itself to 1 okay so this Q(x) is basically the area under the curve from x to ∞.

So it is partial area and in terms of the probability what it simply means is that the probability that the random variable that you have considered so call this random variable as x it will take on value that is greater than equal to x is given this Q function okay of a 0 mean unit variance random variable okay so this is $1/\sqrt{2\pi}\ e^{-t2/2}$ dt now what is the relationship of this Q function to our probability of the error.

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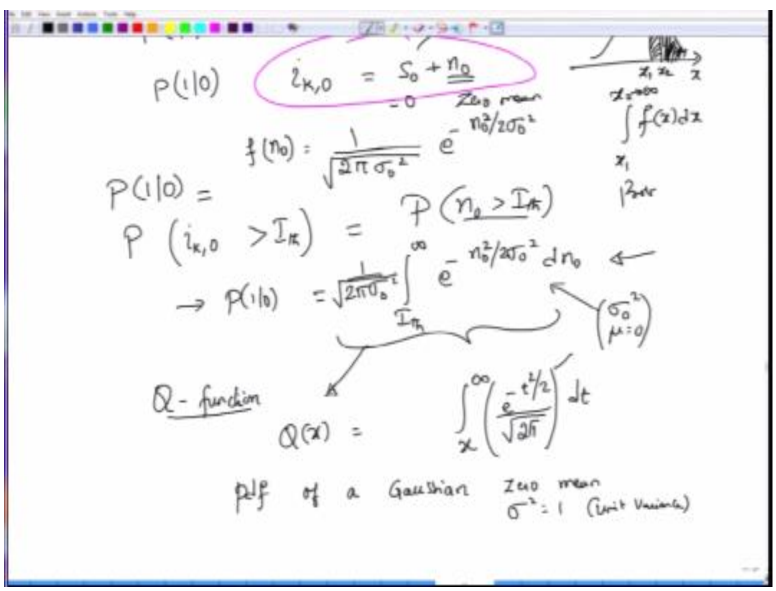

Well look at this expression here you have an integral which goes from Ith which is a threshold current to ∞ and there is equation or there is function which is very similar to a Gaussian function except that the variance of this one is given by σ_0 then mean is still 0 so the variance is

σ_0 or the variance is σ_0^2 while the mean is still equal to 0 so if I can rearrange this equation such that it looks in the form a Q function and Q function is well tabulated in the literature.

So if I can rearrange this function to look like Q of something okay then I can use those values which people have already computed many years ago to do that I need to just change the variables of the integral okay so if I change the variables out here and normalize this one then we will be able to put this expression p(1) given 0 in terms of Q function so let us do that will do this an exercise for n_0 and I leave this as an exercise for n1 when we get to the discussion of n_1 okay.

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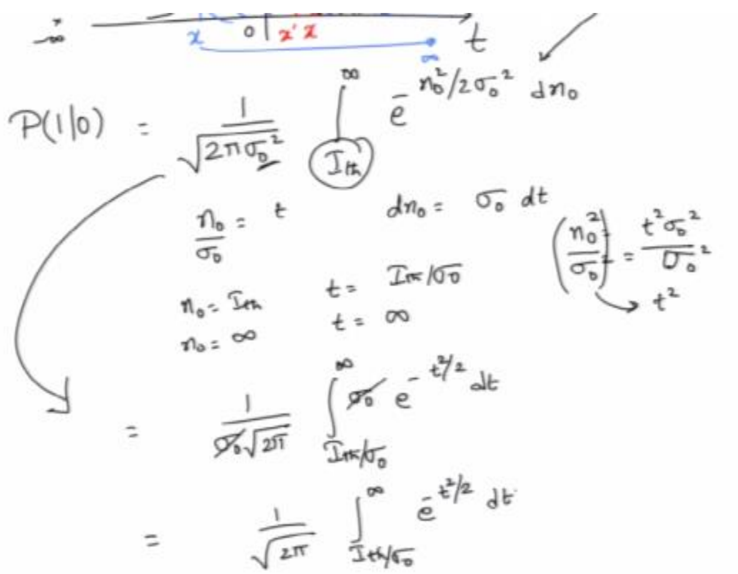

So the probability of the receiver declaring 1 given that 0 was transmitted is given by the area under this probability density function where you are going from the threshold current to ∞ e- we have consider n_0 here so let us consider that n_0 so $n_0^2/ 2 \sigma_0^2$ dn_0 so clearly if you look at these two expressions' in place of t I have put n_0 right so let me define n_0 / σ_0 as anew variable t then I know that d n_0 will be given by σ_0 (dt)and when n_0 = I threshold t = the lower limit t will be = to Ith / σ_0 and when $n_0 = \infty$ t will be = ∞ therefore the integral limits will change okay.

so the integral will change and then we can write down over here the expression so this expression will now become $1/\sigma_0 \sqrt{2\pi}$ I simply pull this σ_0^2 from this under root to outside okay and the international limits will change Ith will now become Ith/∞∞ remains the same the scale up ∞ by a factor of ∞ nothing really happens and in place of n_0 I have to write $n0^2$ is t^2 $\sigma0^2$ and $no^2/\sigma0^2$ is given by this fellow so define both sides by σ0 so this $n0^2/t0^2$ is nothing but t^2 so this becomes $e^{-t2}/2$ and then dn0 is nothing but dtσ0 but this σ0 being a constant cancels out after coming out of the integral cancels out with a σ0 that is during the denominator.

And what you get now is 1/√2Π the lower limit of the integral is $I_{th}/\sigma0$ going all the way to infinity and then you have $e^{-t2}/2$ dt, now do you see the equivalence with respect to the Q function.

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Yes you do, this is nothing but Q function of $I_{th} / \sigma0$ okay, so if I know what is the threshold current right then this threshold current divided by σ0 will be the expression for probability that you get 1 when you are actually transmitted a 0 so the receiver mistakenly identifying 0 as a 1 is given by this expression.

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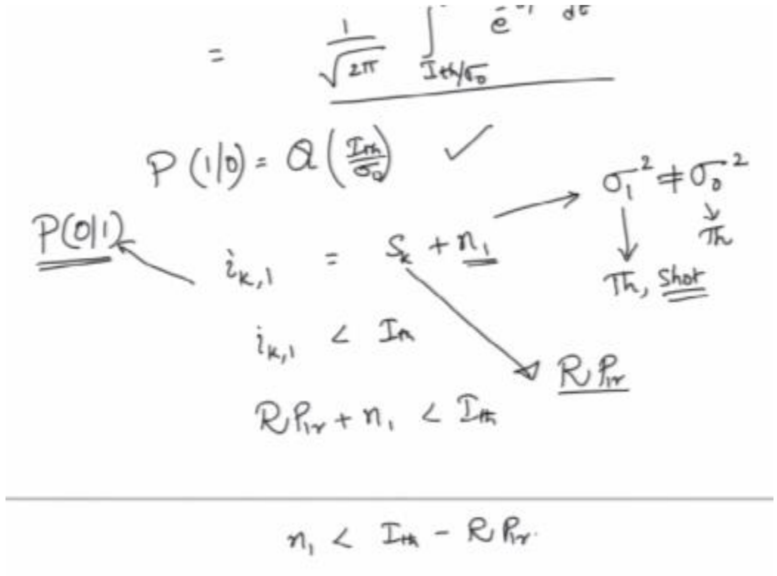

Now let us proceed to find out an expression for the similar probability of error which is probability that the receiver will think have received a 0 while the bit that was transmitted was 1 okay, so this error if I want to find out have to look at the sample current at you know the sampling time i_k but this time I know that the bit that I have transmitted is actually 1, right and what would be the photo current when you have transmitted bit 1?

Well the photo current will involve whatever the transmitted component the photo current corresponding to that plus the noise that corresponds to bit 1 let us call this as N1 now you must remember that the variance of this noise source will be different the variance of $\sigma 1^2$ is different and in fact larger compare to $\sigma 0^2$, why? Because $\sigma 0^2$ is only thermal noise whereas $\sigma 1_2$ will involve thermal noise as well as short noise right.

And short noise is actually dependent on how much input power you have for how much optical power you have transmitted, so this would be the condition but when will I make this error, when will the receiver think that you have transmitted a 0 when the photo current value actually falls below the threshold current value, right. So when the photo current value falls below the threshold current value then we declare this as a bit 0, okay. Although in practice actually what has happened is you have received input was supposed to be higher than the threshold if there was no noise.

But the noise has gone so negative that it actually dips the value there right, so it actually dips the value and that is the probability that we are trying to find out, so when this i_{k1} becomes less than i_{th} and what would be S_k oval Sk should be RP_1 r right, so this is the photo current that you have obtained is R x P_1 r where P_1 r is the optical power that you have received r stands for the responsivity.

In the absence of the noise this is what you would have received but in the presence of noise you are going to get N 1 as well okay, so let us write downs that one so you have RP_1 r + n1 being less than the threshold current, right. When will this happen in terms of the noise variable, when $n1 < i_{th}$ - RP_1 r okay.

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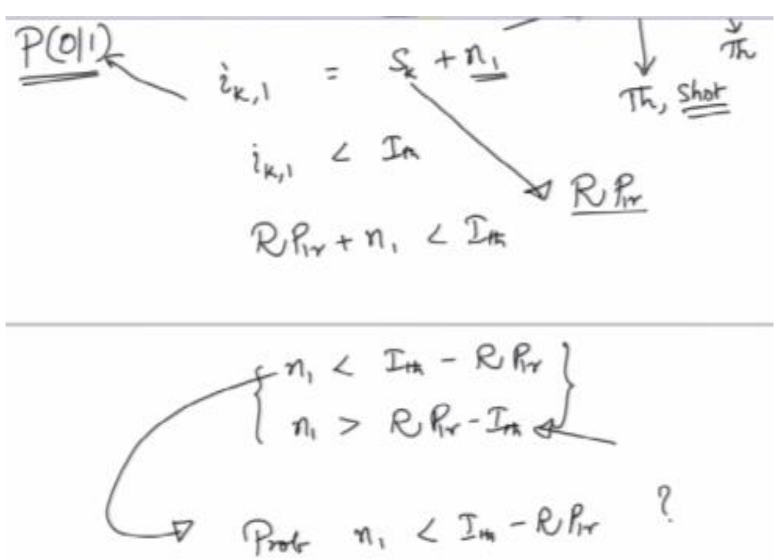

When this happens you are going to get an error right, you can turn around this equation but multiplying by -1 on both sides and when you do that you are going to have to also change the integral limit over here so this fellow equivalent of n1 being greater than RP_1 r – I threshold, okay so when this noise process actually happens to be greater than this one then you are actually going to be in the situation where you are looking at the error there, okay. You can work in both domains you know both expression does not really matter for simplicity it will might take this one as an example going from this expression to this expression is fairly simple for you so I will leave that as an exercise, okay. So what is the probability that the noise n_1 would be less than Ith-Rp_1r what is the probability that this noise would be less than this.

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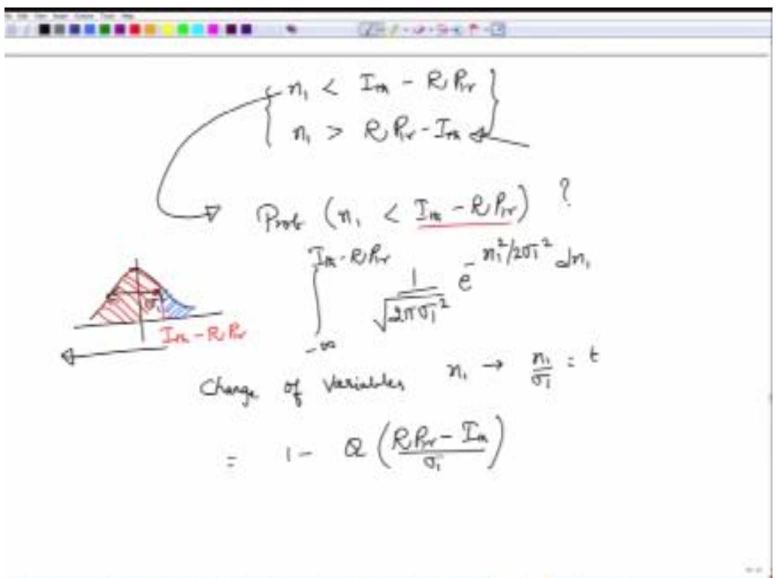

Well, this is would the situation when I have to integrate so if I look at the probability density function, the probability density function for bit 1 will have a variance of σ_1^2 okay, and it would be the area that you are looking for is when the noise actually is less than this particular value, so value let us call this as Ith-RP_1r and you are looking for the area under this one. But area under this is actually equivalent of finding the area here, right and then subtracting from 1 so the area in the blue lines is actually 1 minus the area of this.

If you want to find the probability of this event itself that is when the noise is less than this one well you can do that but this time you have to integrate from this fellow all the way up to $-\infty$, right or you have to integrate from $-\infty$ all the way to this Ith-RP_1r, so integrate this one $-\infty$ to Ith-

RP_1r the function will be $1/2\pi\sigma_1^2$ it is not σ_0^2 and then you have $e^{-n12/2\sigma12}dn_1$, okay. you can do a change of variables over here I will just introduce you do that one and I will leave this as an exercise to you so change of variables from n_1 to n_1/σ_1 define it as t and then change the limits appropriately change the integration variable appropriately you do all this simplification and you will see that this is simply $1\text{-}Q(RP_1r\text{-}Ith/\sigma_1)$ so in terms of the Q function this is what you are going to get, okay.

This is of course make sense right, because this one is $RP_1r\text{-}Ith/\sigma_1$ okay, and from that area 1 minus of that is what you are going to get as the area under the other position. So you have to do this one little carefully to see that this expression is alright, okay.

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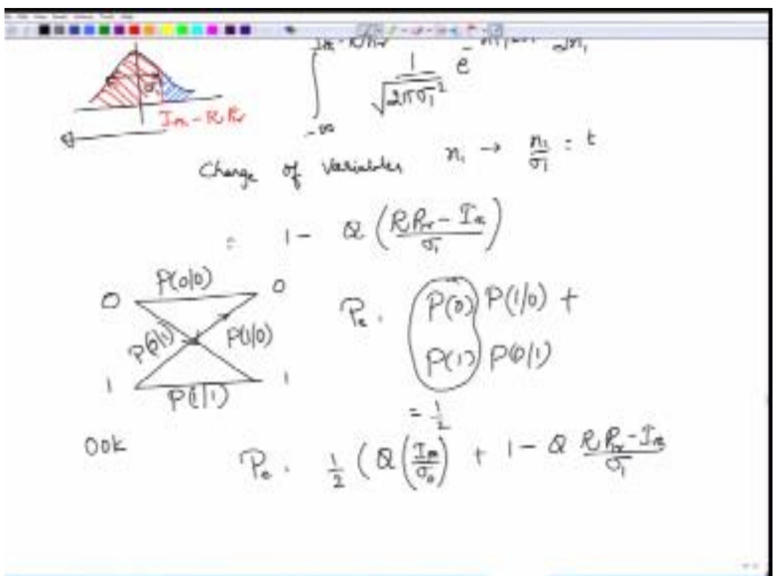

Once you have done that now you can put together the total error, how do you put together the total error suppose you have bit 0 and 1, right in the on off keying that is what we have. When you transmit 0 there is a certain probability that the receiver will identify the 0 as 0, okay that is given by probability of 0 given 0. However, this is what you are interested in, the probability that so this is 0 and 1 the probability that the receiver has identified it as 1 although you transmitted a 0, okay.

Similarly this link or this line will tell you the probability that the receiver thinks you have transmitted a 0 but you have transmitted a 1, so these are the error components and this is the probability of the correct decision that is probability that you have send a 1 and the receiver also thinks that you have send a 1, okay decides in favor of 1. The probability of error is of course what is the probability that you have send a 0 but you end up being 1 and that would be probability of 0 times probability of 1 given 0, okay so this is where you end up making an error plus probability of 1 and probability of 0 given 1 okay.

Of course we have also assume that these two are equvi probable if they are not they you have to wait the threshold current appropriately but if you assume that these two are equal so each of them will be equal to half and the probability of error will be equal to $1/2(QIth/\sigma_0)$, okay + 1-Qrp1R-ITH /σ1 okay.

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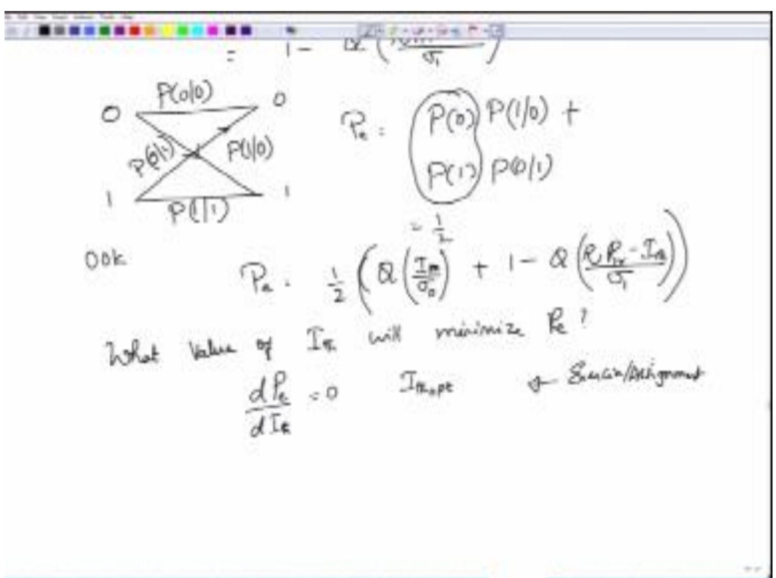

So this is what the probability of error is okay. So with this as the probability of error our next question is what value okay of ITH will minimize Pe what value of ITH should we chose such that the probability of error will be minimized well the straight forward answer to this one is that you have to differentiate this pe with respect to ITH set this to 0 and then find out the value of the threshold current which would minimize the probability of error okay.

Doing this is unfortunately not so simple you have to do some you know some rules that you have to know for the differentiation that you normally are not aware so I will leave this as an exercise or as an assignment for you okay and I will tell you how to solve this one in the assignment right so if you excuse this step.

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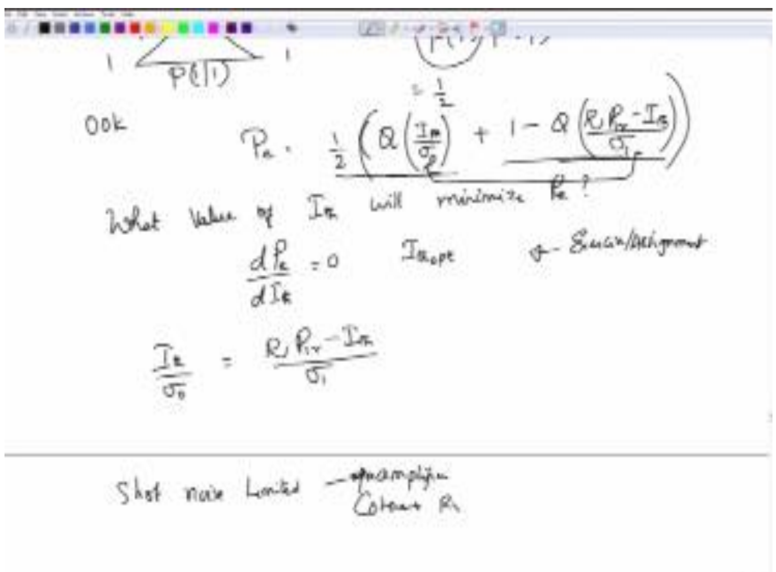

Then kind of a hand waving argument can be made okay we have this Q functions right so when you differentiate this one they are going to go to 0 right but if you look at the probability of these error this would be the probability of making an error when you have transmitted a 0 but you think it is a 1 and this portion will be the probability that you make an error when you transmit a 0 and a 1 right.

If you take the arguments of each of these Q function and equate them then you have kind of distributed the errors equally so the both the errors are equally probable and that happens when the threshold current by σ0 is actually equal to Rp1R- ITh / σ1 okay. This is what when happens when your short noise limited okay you do not have thermal noise but when you have short noise limited when do you get short noise limited. Well when you get amplifier in the loop right so you get a pre amplifier in your circuit or you are looking at the coherent receiver okay when you have coherent receiver okay.

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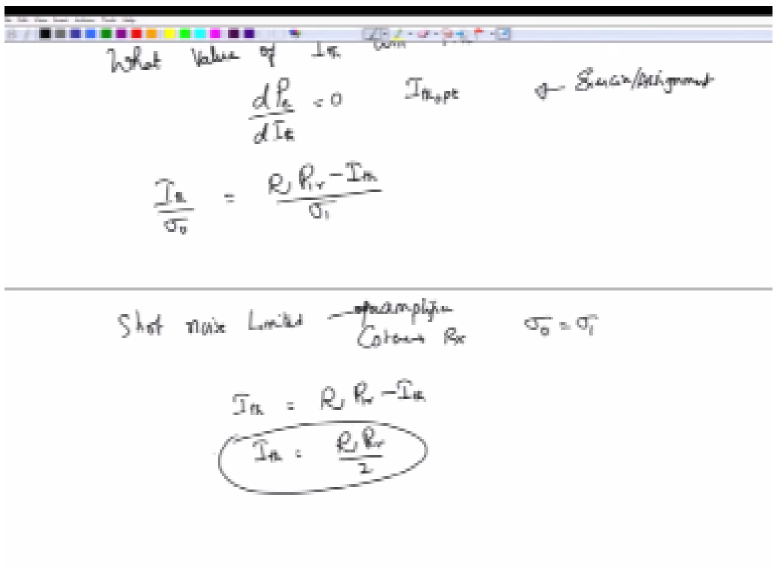

When you have a coherent receiver this are short noise limited, for which case σ0 =σ1 and what will happen to ITH, ITH = Rp1R-Ith or the threshold value that minimizes the probability of error is given by Rp1R/2 so this is the half way mark between what you received as bit 1 and what you receive at bit 0 but this threshold works only when you have a short noise limited okay.

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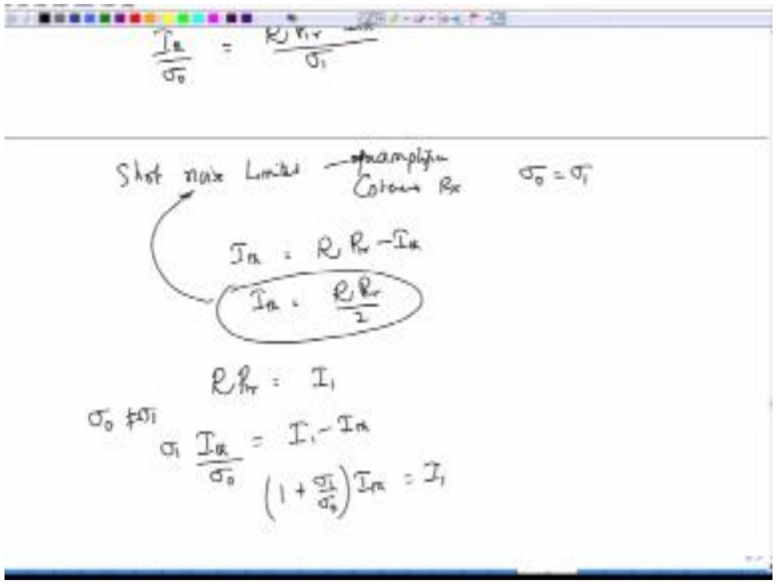

And since Rp1R is nothing but the current i1 that you have received okay and then you can go back to this expression in the case when σ 0 is not equal to σ1 and find out what would be the value of I threshold okay. I threshold is given by or rather I threshold divided by σ0= i1-ithreshold and there is a σ1 here I pulled out the σ1 that was there in the denominator on to the numerator you can you know simplify this equation you get 1+σ 1/σ0 over here when you pull this ith pack on to this one times ITh = i1 okay.

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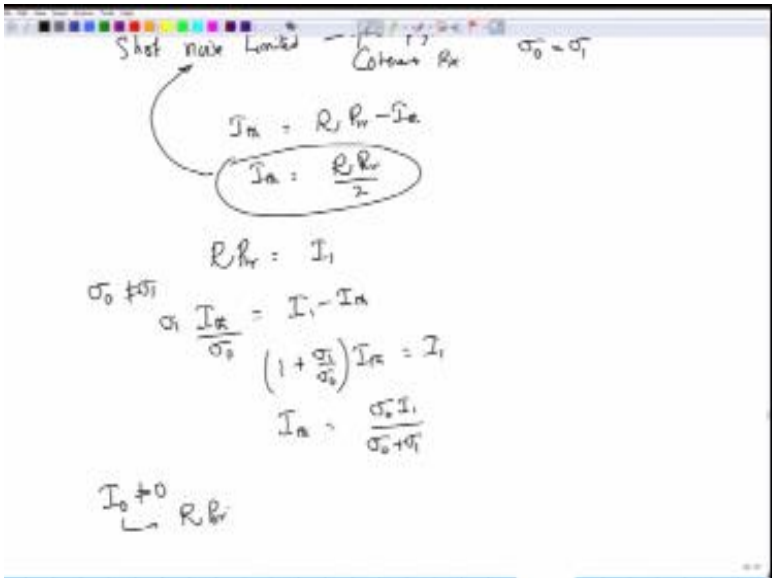

Simplifying this relationship I threshold which would be the optimum value will be given by $\sigma_0 i_1/\sigma_0+\sigma_1$. You further can consider the case when i_0 is not 0 i_0 being the current that you would receive when you have transmitted a bit 0.

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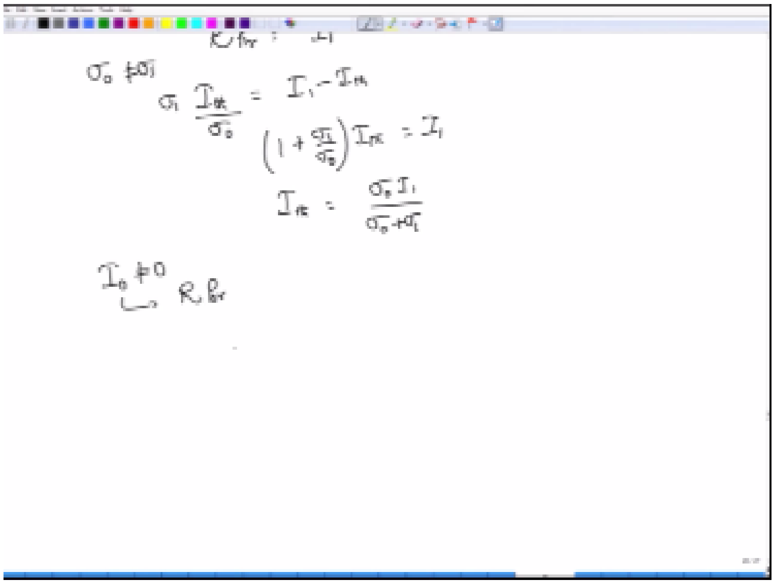

That is when the laser is suppose to produce 0 power but in fact it produce a non 0 power because of the problems with the laser itself then the current that you receive the signal photo current that you received when the bit is 0 is not exactly 0 and that can be also in corporate if you go back to this expression.

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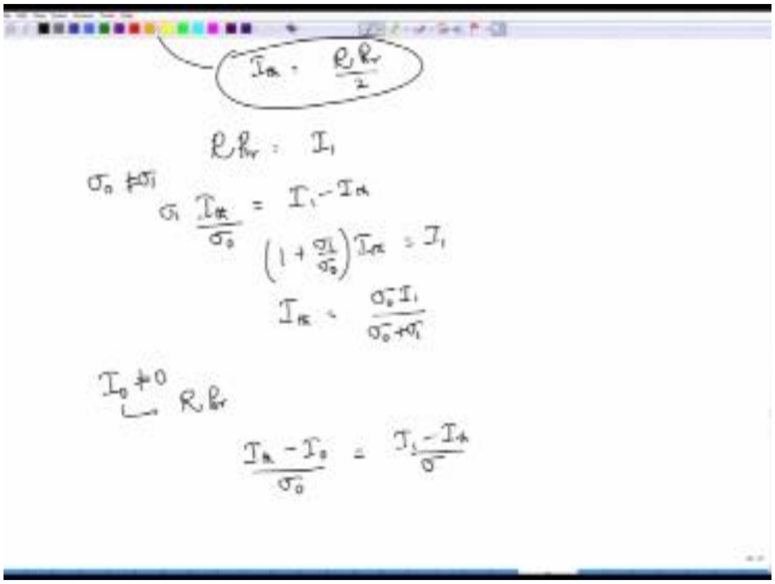

And instead of writing this as Ith you then you can show that it should be I_{th} –I_0 / σ 0, which should be equal to I_1-I_{th}/ σ 1.

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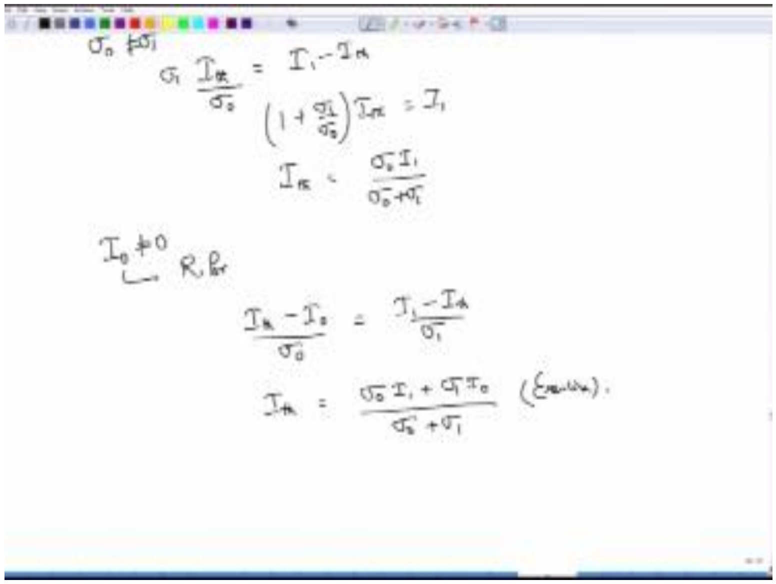

Solving this equation will give you the threshold current as $\sigma_0 I_1 + \sigma_1 I_0 / \sigma_0 + \sigma_1$, i will leave this as an extremely simple exercise for you to work out, okay. So what we have looked at? Is this is a kind of a hand waving argument we have not really derived that these are the optimum values infect the expression for the optimum values quite complicated, and in the assignment you will see how to differentiate the expression for the probability of error, in order to find out what would be the optimum value of the threshold.

Bu what is interesting in all this, weather it is hand waving or something ids that unlike a radio receiver where σ_0 is always by default equal to σ_1, for ON, OFF keying optical communication it is not, the short noise that you get is dependent on the input optical power or the received optical power, and different receivers actually respond in different ways, right.

Suppose you consider the simplest case of the PIN and APD receiver, okay you don't have an optical pre amplifier when the loop you simply consider the PIN or an APD receiver, first consider a PIN receiver, in the PIN receiver signs 0will be purely based on the thermal noise, and σ_1 will have both thermal as well as the short noise, okay.

And for a given value of this q factor which we have defined, and new here is a small problem, this is the q factor which we defines as the I_1-I_0/ σ 1+ σ 0, this is not to be confused, so not to be confused with q function, it's unfortunate that we have used the same expressions for both, but that's how life goes, okay, so you have the same notation, the context actually tell you the difference.

So if you look at this Q, value, okay or if you want you can call this as some γ value, dosent really matter, okay so this value is that one that determine what would go into thye probability of error, correct? So this is what determines what would go into the probability of error, and this factor will be different for different receivers, for the PIN receiver okay.

For a! So suppose you want to obtain let say BER of 10^{-9-}, okay , this means that in the expression for the probability of error for your chosen the optimum value of the threshold correctly then you need to have a certain value of γ , such that Q(γ) is the bit error rate, right or this is the probability of error.

And for whatever γ that you choose right, so this γ should be around 6db, for the case where you are considering 10^{-9} bit error, okay, and this γ is the one that goes into this particular expression, okay so if you have received a certain power then q factor that you said is the expression that goes into the q function.

This ratio the current to the σ is the one that actually determined, with the q factor or that goes into to q function, which then determine what would be the probability of error

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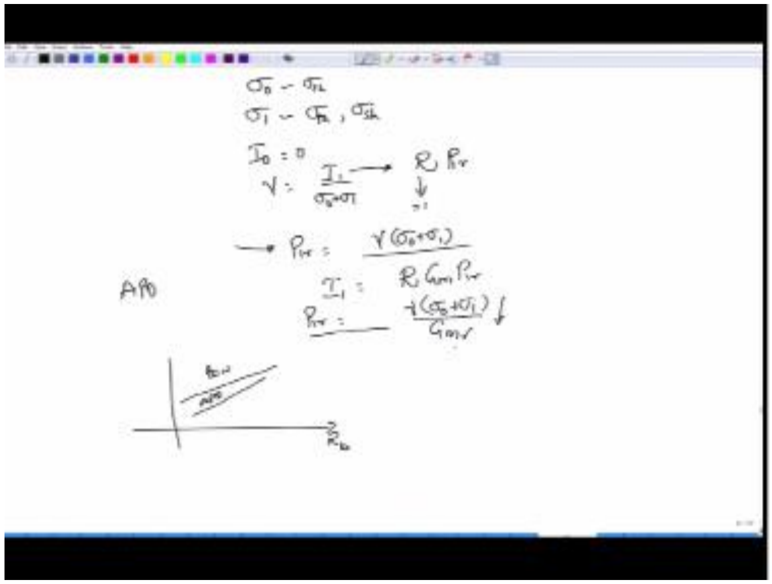

Now what happens is that? For the case of PIN diode σ 0 is σ thermal, σ 1 includes σ thermal as well as σ short noise, and if you look at resulting values that you want assume that also I_0=0, just for purposes, then the γ factor which is you know I_1/ σ 0 $_+$ σ 1, and I_1 is nothing but RP_{1R}, okay, and you can go back and assume that R=1, this is kind of an ideal scenario, then the required received power be γ times, σ 0 + σ 1 okay.

This is what you require, on the other hand you are looking at, an APD receiver, because the current I_1 will be multiplied by the multiplication factor of the APD, so it could be RG_M P_1R, you would see that required P_1R, is given by σ 0+ σ 1/ G_m,.

Already without doing any calculation, you can see that this is APD which requires this amount of the received power where as P1R requirement has gone down by a factor of G_M, what it means is that if you look at a function of the bit rate right, the sensitivity of a pin diode is the worst okay, whereas because there is a large gain factor G_M, the sensitivity of the APDs is much better, this fellow is the APD and you see that received power requirement for the same BER is actually much lower for a given bit rate, in the case of an APD receiver, okay.

And this is the internal gain, this is because the internal gain of the APD structure itself. However with APD what would be the noise source, well there I thermal noise, which will you anyway get when you have σ_0 but when you have $\sigma_{1,}$ that is when you transmit bit 1, thermal noise as well as short noise, but this short noise gets multiplied by a certain factor right?

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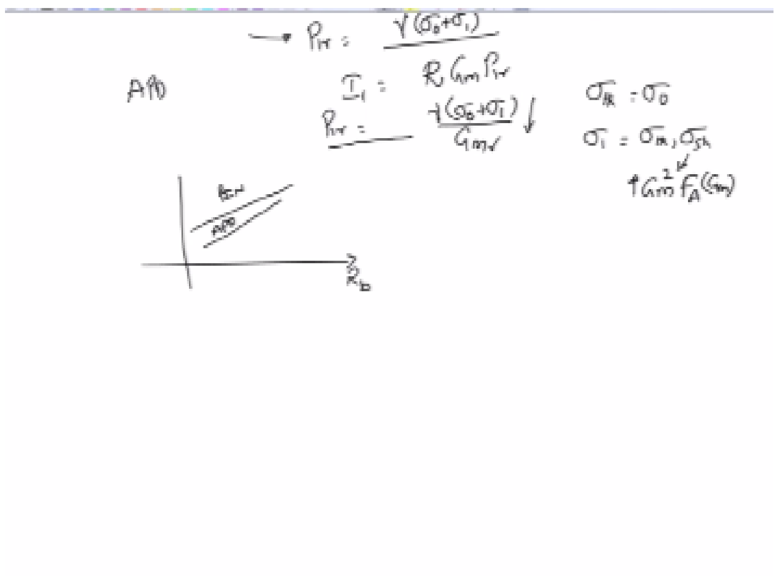

So where is the Gm^2, F_A (Gm) where is F_A is the noise factor or the excess noise factor of the photo current, right? Because of this multiplication which happens, the short noise component actually increases with σ_1, okay. The short noise that you get with the APD is much larger compared to the short noise that you get with the pin diode receiver. However you can show later that the required optical power actually goes down when you have this APD receiver.

Moving on to next case where you have optical P amplifier, in the optical P amplifier case, what would be the major noise factor? Well the signal beating with the spontaneous emission playing the major noise factor. So because of this, this could be the case when you transmitted bit 1, for bit 0 you have spontaneous noise, you have thermal noise, you have spontaneous noise as well a some amount of short noise, short noise is actually equal to 0, because you transmitted bit 0 here, okay.

What you get is only thermal noise and spontaneous emission noise, but these values are so small compared to the signal spontaneous noise P, that one kind of neglect this in the expression for the received power of the sensitivity of the receivers, with the result that the sensitivity actually goes better with respect to optical pre amplifier. So for the same bit error rate and for the same BER, the power requirement for the pre amplifier goes even further down.

And that I all happening because the signal and the spontaneous emission noise process. We have already looked at this one, let go back to the previous modules to exactly see the expression for the signal spontaneous noise source. Finally when you have coherent receivers we have also looked at what happens with the coherent receivers, the short noise components for then you transmit the bit 0, is the same as the short noise component when you transmitted bit 1.

And this happens because both are Lo limited. It is the local oscillator power which determines the short noise, so this overwhelms all the other noised in the circuit and because of this you actually gain even more, actually gin 3Db sensitivity improvement with coherent receivers, okay. So the last one is the coherent receivers, if you see the bit rate, you see that the worse is pin receiver which require high power at the receiver to detect. The next is the APD receiver which already gives you better sensitivity but that is not completely sufficient, that is not completely good, compared to a pre amplifier which increases the sensitivity even better.

Finally you have a coherent receiver which actually performs even more wonderfully than pre amplifier. This line also indicates the increasing complexity, you have pin receiver followed by trans impedance amplifier that is a complex kind of amplifier you can think of, APD’s they are little more expensive, have to make with increased complexities in the circuit because power requirement for the reverse biased APD is higher, much higher than the pin receivers.

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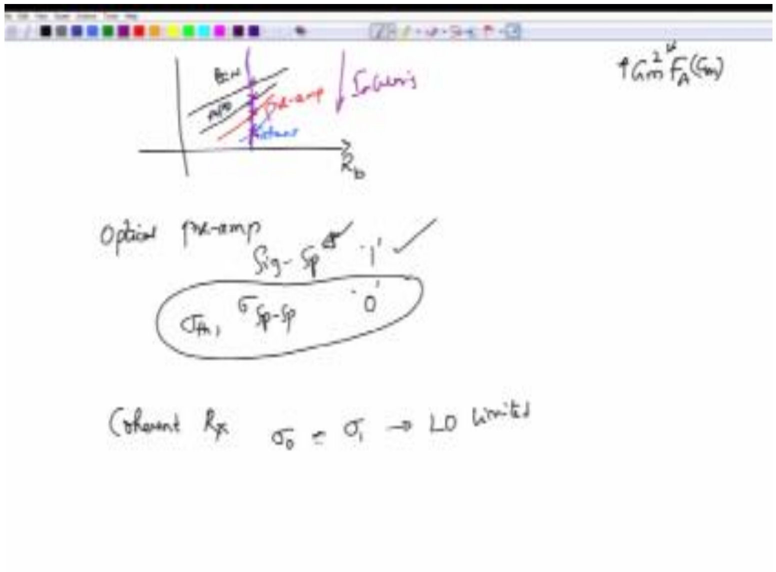

O it is slightly more complex, so pre amplifiers require you actually install a optical amplifier which is even more complex and coherent receivers in the complex of all because it requires polarization tracking, phase tracking as well as the frequency off set being in constant or equal to 0. So complexity increases, but what ways of increased sensitivity, so larger the source you put in, the larger the in this particular case would be the sensitivity of the receivers.

So this is what we wanted to talk about the on off keying receivers and we will end our module today with this one, we will later look at other modulation source and the residual effects in the fiber that we have not considered in the next module. Thank you very much.

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