Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Optical Communications

Week – V
Module – III
Modes in Optical fiber-I(contd.)

by
Prof. Pradeep Kumar K
Dept. of Electrical Engineering
IIT Kanpur

Hello and welcome to the course and optical communications in this module we will continue the discuss of the previous module we will apply the steps that we are outlined in the previous module to discuss and derive the optical modes inside an optical fiber okay we have spelled.

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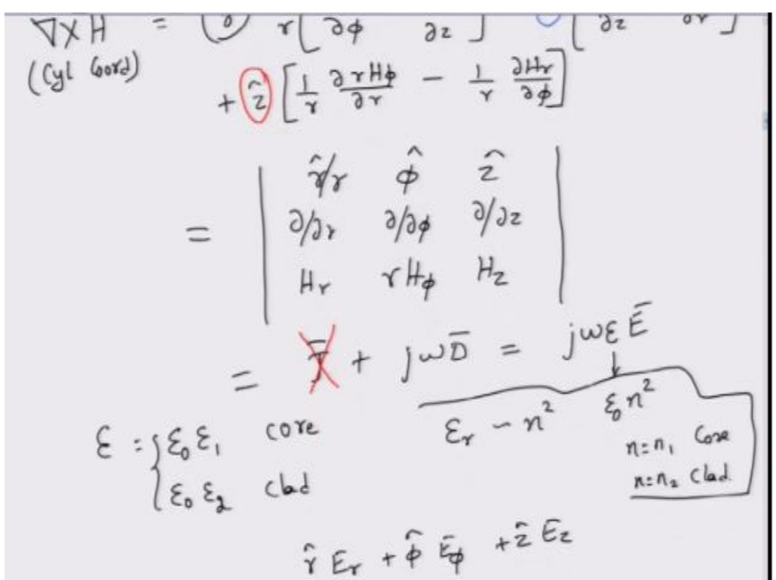

Out all the assumptions in the previous module so please go back and refer to those assumptions so the objective in the previous module at the end of the previous module was that we where about calculate the electric field and magnetic field pattern and we would be starting with step one which corresponds to having starting and noting done the Maxwell's equations in the appropriate coordinate system because this is like an optical fiber which is cylindrical in nature it is easy for me to talk about the are easy for me to consider the cylindrical coordinates system cylindrical coordinate system is characterized by $r_{Ø}$ and z coordinates.

And the unit vectors along $r_{Ø}$ and z in terms of those the curl equation is given by this particular expression so you have to you do not have to remember this you can look at it from any text book are any hand book of mathematics the curl of H I have used H in this example but you can you know that E also needs a curl equation so in that case substitute H with E okay so curl of H as three components $r_{Ø}$ and z components and for your bad luck none of these H_Z $H_Ø$ H_R components are 0.

So every exits so welcome to complications simple way of remembering this if you where to Keene if you are Keene on remembering this is to think of this as the determinant which is given by r^ / r Ø^ and z^ where r^ is the unit vector along the radial direction Ø^ is the unit vector along the Azimuthal direction and z is the radial z^ is a unit vector along the z axis that is axis of the cylinder so you have those r^ Ø^ and z^ and this the expression γ/ γ_r stands for the partial derivative of with respect to r $\gamma/ \gamma_Ø$ is the partial derivative with respect to Ø and γ/ γ_z is the partial derivative with respect to z.

So this is the curl of H step 1 right down Maxwell's equations so I have written down curl of H having all these components but I know that curl of H on the right hand side is given J+ δD/ δt or because time derivative have been through out by converting into the phase form δ/ δt becomes JωxD luckily for us J will be equal to 0 because there is no current inside an optical fiber because optical fiber is made out of dielectric region so the dielectric we are assuming it to be perfect dielectric no losses in the dielectric therefore this term will be equal to 0.

So lucky for us J is not there but you still have JωD there is one further simplification that you can make with this equation you can say that this is JωεE and the value of ε will be different for the core and it would be different for the clad so this is for the core and this is for the clad where

ε_1 and ε_2 are the relative permittivity's but I also know that relative permittivity is nothing but refractive index square so I can write down this ε as ε_0 n^2 will substitute $n = n_1$ for the core region $n = n_2$ for the cladding region, okay.

So anyway we will not make this substitution at this point but I just said that you can do this one slightly later we will do that one slightly later so I have on the left hand side of this expression given this complicated looking expression but the right hand side is jωεE but if we have to you know expand this E itself E will be r E_r + Ø $E_Ø$ + z E_z So you will actually have E_r component $E_Ø$ component and Ez component, right.

Then you can equate the r component on the left hand side which is this expression to the r component on the right hand which is jω so if you multiply this whole thing by jω, jωε then jωε E_r must be equal to this expression, okay. The one that I am showing here so this expression, similarly there will be a Ø expression or you know the component for the Ø that should be equal to jωε this should be equal to jωεE Ø, Ø^ of course, right. Because is the vector equivalence and this last one should be equal to jωεEz okay again ε should be different for coherent cladding that you have to keep in mind, okay.

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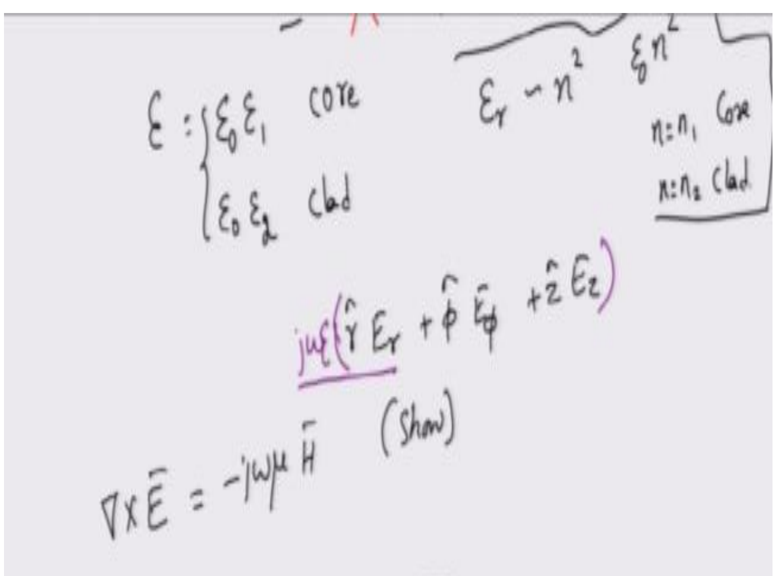

So we have written down $\nabla \times H$ equal to this equation this is called as ampere Maxwell law then you also behave Faradays Law which is $\nabla \times E = - j\omega\mu H$ I leave this as an exercise to show, okay. So $\nabla \times E$ again will have this same you know three variables in place of H you can simply replace E and you will have to equate this one to the right hand side in the right hand side you will have $– j\omega\mu H$, okay. So let me just write down it will take some time but I would like to just write down this one. So that you see all the equations at one side, okay.

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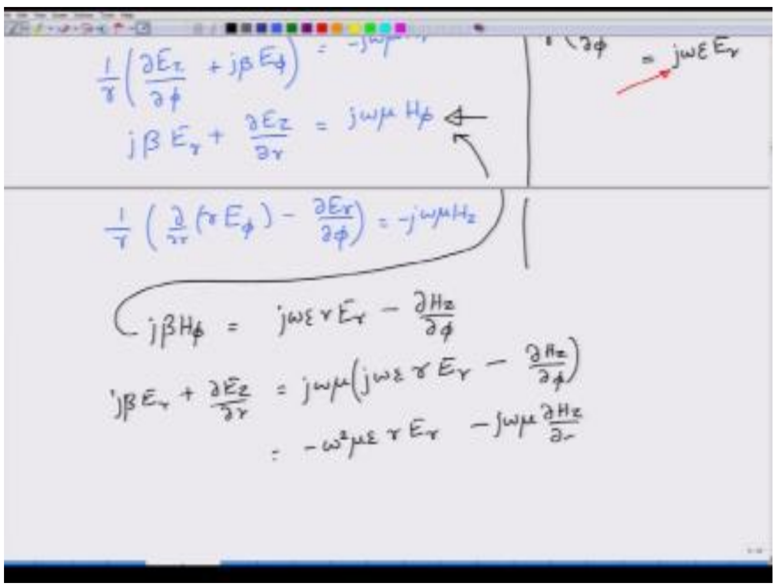

So let me just write down this equations, so equating the r components I get $1/r\ \partial E_z/\partial Ø - \partial E_Ø / \partial z$ this should be equal to $–jj\omega\mu H_r$ now we introduce one more assumption, we will assume that no matter what the electric field or the magnetic field components are as a function of z they are given by $e^{-j\beta z}$ they are characterized by a certain propagation constant $e^{-j\beta z}$ therefore when you take $\partial/\partial z$ here right.

On the z dependence what you are doing is you are simply pulling out this $–j\beta$ just as for the phase $e^{-j\omega t}$ you look deal by ∂t and pulled out $j\omega$ here you are pulling out $–j\beta$ so in that in the place of this $-\partial E_Ø/\partial z$ I can write $– j\beta$ times $E_Ø$ so this part can be re-written so I can re-write this equation I will write down here in the glue may be just to show the difference you get $\partial E_z/\partial Ø$ + $j\beta E_Ø$ this whole this is equal to $-j\omega\mu H_r$.

We do this simplification in other places as well, so the next equation that I get will be the Ø term which is $j\beta E_r + \partial E_z/\partial r$ should be equal to $j\omega\mu H_Ø$ okay and then you have the third equation which is the z component you get $1/r\ \partial/\partial r\ r\ E_Ø - \partial E_r/\partial Ø$ this should be equal to $-j\omega\mu H_z$ you are not lucky enough to replace $\partial E_z/\partial r$ or $\partial E_z/\partial Ø$.

This should be equal to – jωμ Hz you are not lucky in off to replace $\delta z/\delta r$ or $\delta ez/\delta_Ø$ well that is the life you do not get everything that you want so these are the curl equation for the electric field or the faradize law there must be equivalent or there must be analogs equations for δ cross H as well which are any obtained from the top one right so what I will do is I will also leave this as an exercise to write down the corresponding terms to the H δ cross H = j $\delta d/\delta t$ I will start off with one expression you can fill in the remaining two so you will have $1/r\ \delta Hz/\delta_Ø + j\beta H5$ right so replacing e/H this should be equal to jω epsilon E_r.

So you can fill in these two what you do in the next step is that you need to express Er, $E_Ø$ Hr $H_Ø$ in terms of Ez and Hz how do I do that well I have to try and you know equate couple of this equations right for example if I look at this one I know that E_r is given here so if I look at this equation to the left hand side I have jβ Er there is $\delta Ez/\delta r$ right and then jωμH5 there is also one more equation here which is jω epsilon Er equation which involves at j5 term here and a Hz term here so if I can combine these two equations and substitute one in place of the other and simplify them then maybe I will be able to obtained a relationship for Er.

Okay so I mean I can try doing this you can eliminate one with respect to the other so if I will just try it out now to see whether I can still do this analysis otherwise as usual I will leave this as an exercise to you it is not complicated you just have to look at one equation substitute for the other okay so let us, let us substitute for $H_Ø$ from this equation right, so from this equation let us substitute for $H_Ø$ let me re write that equations so I get $j\beta H_Ø$ is = jω epsilon r Er - $\delta Hz/\delta_Ø$ so I can substitute for $H_Ø$ in this expression right so in this expression I can substitute so I get jβ Er + $\delta Ez/\delta r$ = jωμ for $H_Ø$ I can substitute jω epsilon r Er - $\delta Hz/\delta_Ø$ okay.

So if you expand this out you get – $\omega^2\mu$ epsilon there is r Er right and then you have – $j\omega\mu$ δHz/ δ5.

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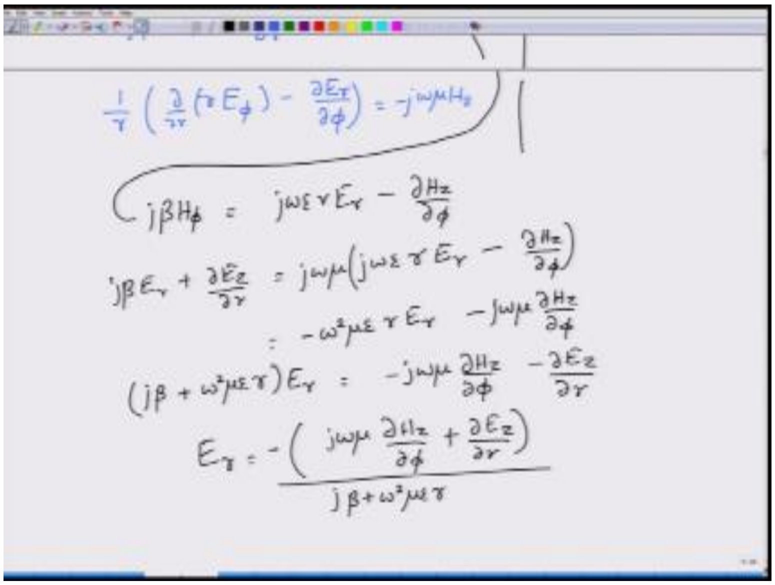

I can pull this E_r on to the left hand side so I get $j\beta + \omega^2\mu$ epsilon okay there is an r Er here I hope that I have got the r E_r components correctly so I do get r this whole thing times Er must be equal to – $j\omega\mu$ δHz/ $\delta_{\varnothing}$ there is also - δEz/ δr okay so you get - δEz/ δr if I take –j as a common factor or no maybe I do not want to take them as a common factor at this point, but I can write down this Er as $-j\omega\mu\delta Hz/\delta\phi-\delta Ez/\delta r$ this whole thing so minus I can take out as a common factor so this becomes plus inside the brackets, so there is a –sign sitting out divided by $j\beta+\omega^2\mu\varepsilon r$, they are seem to be a small problem here because I am not expecting this r here, so I was not really expecting this one in that case maybe where I have made a mistake is to write down the simplification somewhere.

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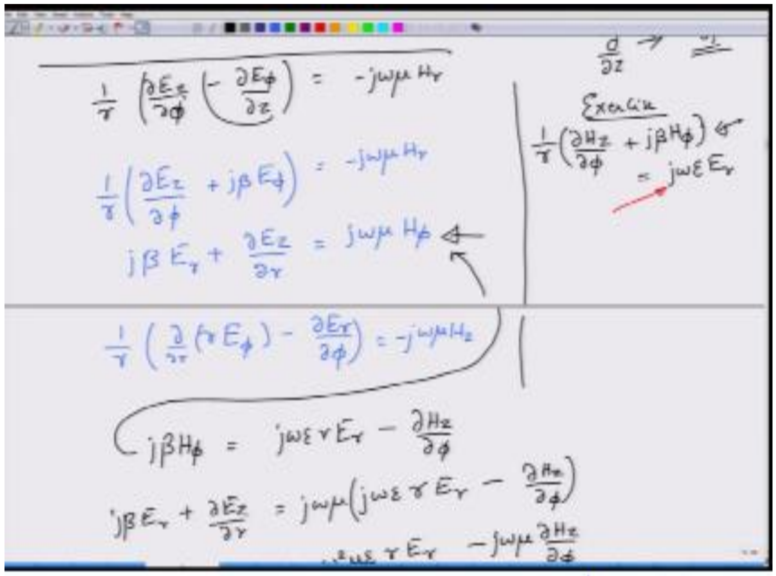

So you can actually go back to this equation there was $j\omega\varepsilon$ times Er here, and there was a 1/r term sitting there and this was $\delta H_z/\delta\phi + j\beta\delta H_\phi$, so maybe when we substituted this equation we gor $j\beta H_\phi = j\omega\varepsilon r E_r - \delta H_z/\delta\phi$.

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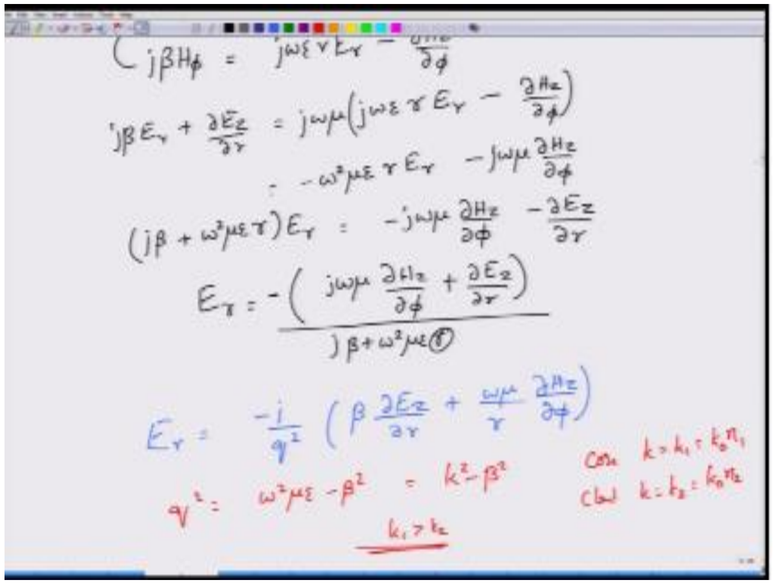

So we can look at that equation and then maybe you know you will forgive me for this one. What I would like to give you is the final expression so and I would like to give this an exercise to you, the result for Er in terms of Ez and Hz is that you get $-j/q^2$ there is $\beta\delta Ez/\delta r+\omega\mu/r\ \delta Hz/\delta\phi$ where q^2 is nothing but or q^2 is given by $\omega^2\ \mu\varepsilon-\beta^2$, okay and we know that $\omega^2\varepsilon$ defines the propagation constant k^2 so this can be written as $k^2-\beta^2$ obviously in the core region $k=k_1$ because this is nothing but the free space wavelength k_0 times n_1, right whereas in the clad region $k=k_2$ which is k_0 times n_2 and clearly k_1 is larger than k_2 because n_2 is smaller than n_1, okay.

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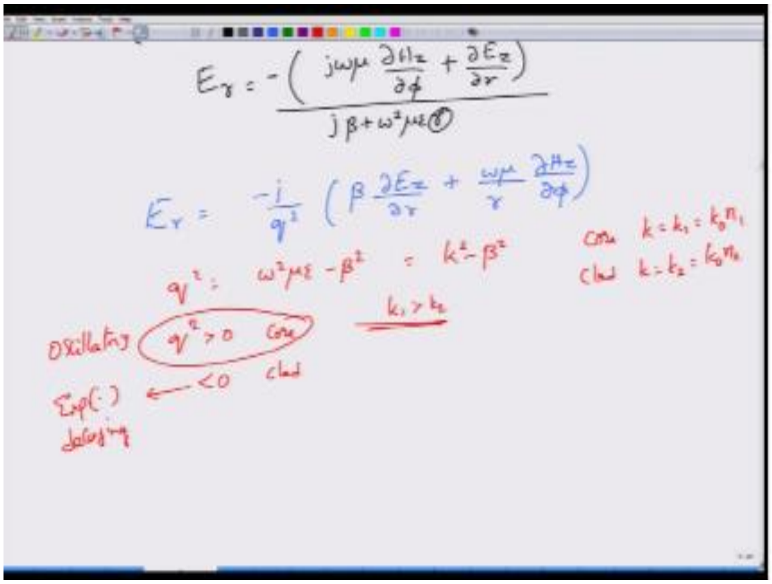

So these are some results that you will know and if you look at the way q should behave q would actually be positive in the core region and q^2 would be negative in the clad region, because of this there will be an oscillatory solution here and because of this nature there would be an exponentially decaying solution as we will see later, okay. So this is just for Er where again I think I have made a mistake in multiplying the terms somewhere.

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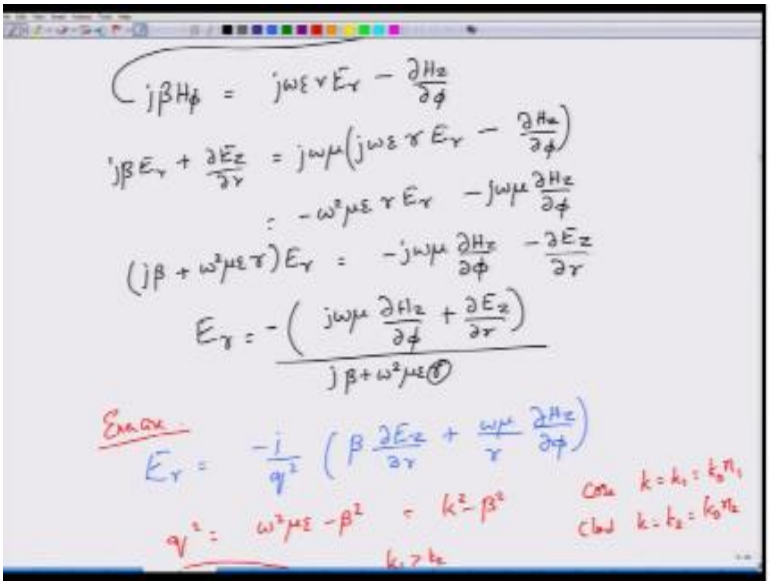

You can just figure it out where I have made a mistake that might be a very good exercise for you, so leave this as an exercise to show that Er is given by this.

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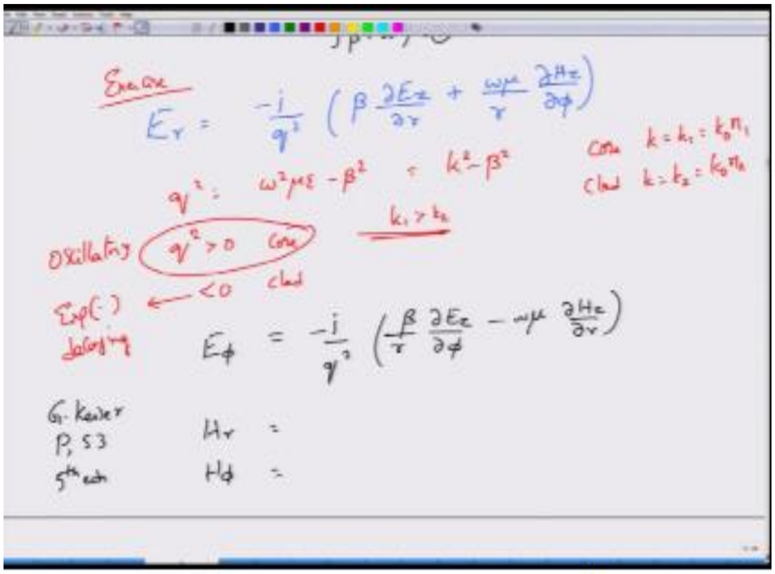

Similarly, you can look up the text book and find out expressions for E_ϕ in terms of these components so I will give you the answer you can sit and verify this, okay so this would be $\beta/r\delta E_z/\delta\phi-\omega\mu\delta H_z/\delta r$) similarly there will be H_r and H_ϕ I am not going to write them you can look at the text book write which is the text book that we are following for the course, and this is page number 53 of the fifth edition. So all these formulas are taken from there, so you can look at the text book when you have some time, okay this completes our step 2.

But what about step 3, well I need to know what is, the step 3 step 3 comes from solving hellhole equation that is.

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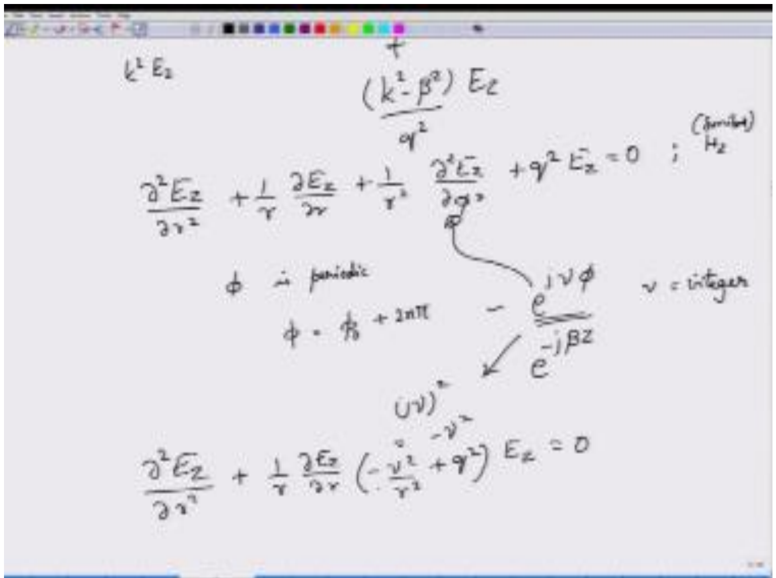

(∇^2+k^2)Ez or Hz=0, except that you now need to write down this ∇^2 in the cylindrical coordinate system, okay which again is a little bit of a problem so you will have to get to the expressions for the cylindrical coordinate system I will just give the solution here, okay. Since we are applying ∇^2 to Ez and in this particular case it is not k^2 which is, which needs to be evaluated I mean included but rather q^2 because that would correspond to $\omega^2\mu\varepsilon-\beta^2$, okay so because of that this would be q^2, okay.

In the case when you are assuming that Ez goes as $e^{-j\beta z}$ okay, so alright let me just prove this one if you are little bit confused I can prove this one so let me write down what is ∇^2 applied to E_z in the cylindrical coordinate system so ∇^2 apply to Ez in the cylindrical coordinate system is another complicated expression bu this is $1/r\ \nabla^{/}\nabla r\ E_z$ so this is actually r E_z /∇ r so this is the r expression + $1/r^2\ \nabla^2\ E_z/\nabla^2 + \nabla^2\ E_z/\nabla z^{2.}$

So this is the expression for $\nabla^{2\text{ now}}$ to this if you add K^2E_z I can add k^2Ez and I know that $\nabla^{2/}\nabla^{z2}$ is nothing but $-j\beta^2$ because $\nabla/\nabla z$ pulls out $-j\beta$ again you pull out $-j\beta$ – and – is + so jx j is – so this is actually $-\beta^2$right so to this if you add $k^2\ E_z$ what you get is when you group the terms for K_z

you are going to get k^2-β^2 x E_z that dependents on r and five still remains the same so this plus this equation and this is the reason why I said k^2-β^2.

Because that is defend as Q^2 I can replace this in to this expression okay so if I do this and then write down what is the resulting expression also after taking this double differential thing you know after differentiating this one partially with respect to r twice and you know simplifying this equation you get $\nabla E_z / \nabla r^2 + 1/ r\nabla E_z / \nabla r + 1/ r^2 \nabla^2 E_z / \nabla \varphi^2 + Q^2 E_z = 0$ and we will get a similar expression for Hz as well so similar for Hz component.

So this is the complicate equation that you are about to solve however the solution can be written in the form of well known functions if I recognize that φ is actually a periodic function right my solution is should periodic in five because if I change the azimuthally angle from say some value φ0 to φ0 + 2n π where n is an integer I should basically get back to the same point right it is like going around the circle and I am back on to the same point when I change this an indolent by change the angle by 2πradium thing.

So I am so my solution so also should be periodic they have to reflect this periodicity around the cylinder when you go in the azimuthally direction so that kind of a constrained allows us to choose the solutions as some e^j so let me chose this one as some μφ okay where μ is an integer okay μ is an integer so my solutions is in terms of φ will be along e^j μφ okay, my solution in terms of z I know they are of the form $e^{-j \beta z}$ the only thing which I do not know is what is the solution along r.

Substitute this kind of a solution in to this expression so when you take $\nabla^2 / \nabla \varphi^2$ this e^j μφ will pull out j φ^2 right which is nothing but $-\mu^2$ so I can go back and rewrite the expression or the wave equations here I can say $\nabla^2 E_z / \nabla_r^2 + 1/r \nabla E_{z/} \nabla_{r\text{-}} \mu^2 /r^2 + q^2$ times $E_{z\,=}\,0$ so this is all E_z so this is + so you get a slightly simplified equation okay.

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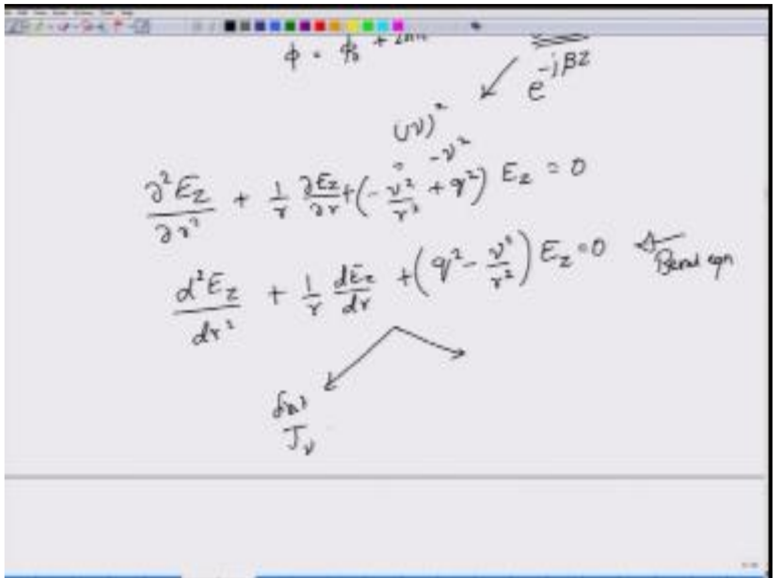

And since this is only r function of r because for new constant value of new this entire term q^2 – μ^2 /r^2 is just a function of r so I can remove this ∇^2 /∇ kind of a thing and go to total divert so I get d^2 E_z / d_r^2 + 1/rdE_z /Dr + q^2 – μ^2/r^2 x E_z = 0 it turns out that the solution of this equation this equation is known as Bessel equation Bessel differential equation the solutions of this r of two kinds this is the first kind and there are two types of first kind solutions which are given as j and let us say some this one that one you need so this μ is the integer here j ν of some argument.

So if I call these argument as u^2 inside the core and w^2 in the cladding, so u^2 is the core and w^2 in the cladding the solution will be of the form j ν (ur) and then there will be one more solution which is Nν (ur), okay, for the core solutions. And because in the cladding region this is defend as, rather I should define as the -w^2. Because I would like to define i am anticipating the result that the solutions have to decay, for the solutions to decay out inside the cladding this argument q^2-ν^2/r^2 should be negative, okay, so that is given by negative w^2 , okay, so because of that i get the solutions of the second kind. The second kind solutions are y ν (wr), and let say k ν (wr), okay. And if you look at how these equations behave, you will see that.

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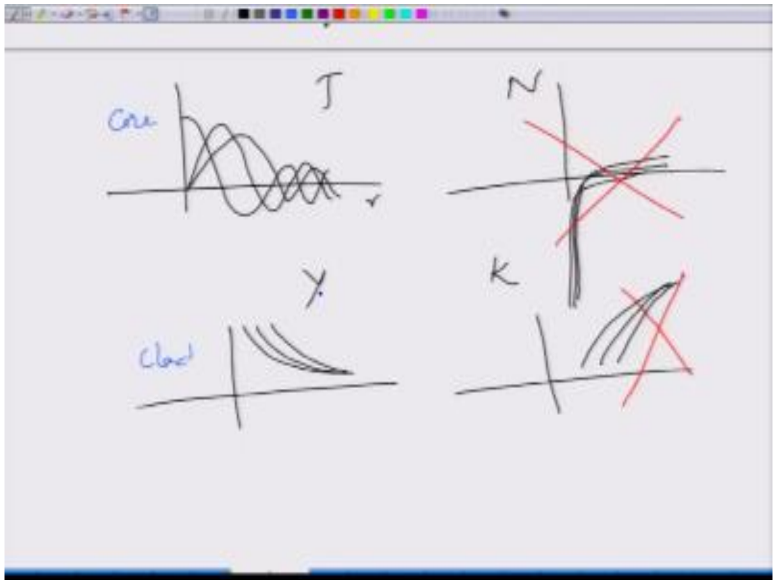

If you sketch them as a function of the radial distance, this is how the solutions could behave. This is your J0 some arc, this is your J1 then you have J2, okay these are like damped sinusoidal excitations, and then if you look at "n" that is the cell function of the first kind but with "n" you will see that these are all going towards infinity or rather minus infinity at R=0, then the other solutions, namely Y and K the solutions for Y will go as exponentially decaying solutions.

So these are the solutions and then for K you have exponentially increasing solution, okay, based on this nature it's obvious, that if I want some guiding inside I have to choose these as the core notes, and I have to choose these as the clad modes and reject these because at R=0, I don't want my field quantity to go to infinity, similarly my field quantity inside outside the cladding are not going to increase exponentially rather they would actually decay exponentially.

So because of that reason we will choose in the core region oscillator solutions, in the clad region exponentially decaying solutions, okay,

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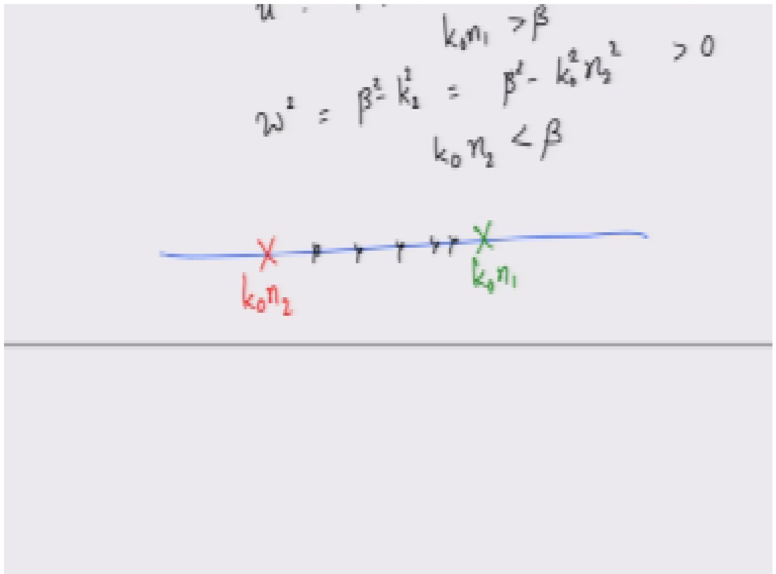

So my expression for Ez=(r>a) that is to say, in the core will be given by some constant A similarly Hz(R>a) that is inside the core region, so this entire thing corresponds to core region will be another constant B in terms of their behavior inside the core there have to be special function of order ν ,first kind vessel functions ,order ν and the argument for that inside will be ν r and there will be e^{j} ν φ ν is the integer $e^{-j\beta z}$,okay.

For the clad mode solutions the solutions must be E_z (r>a) and $H_{z\,(,}$ r>a) ant this have to be some constants C and D, the constant C goes for Ez constant D goes for Hz, the argument here will be the same, it would be Yν (wr) because these are the one which have to be decaying, $e^{j\,\nu\,\varphi}$ $e^{-j\beta z}$ so okay, so these are the constants AB and CD which you need to determine.

One very important thing before we entering this module is that the value of "u" which is, you know? $K1^2$-β^2 that is given by $k0^{2-n1}2$-β^2 has to be positive, which means that k0n1>β, W^2=β^2-k_2^2 or β^2-$k0^2n2^2$ has to be positive or rather the $k2^2$ β^2 has to be negative or w^2 has to be positive, that is the solution. So for this reason be k0n2 should be less than β, so you will see that actually end up with the range of β, on the lower side you have k0n2 on the upper side you have k0n1 and the value of β will be any discrete fellow that would be lying between k0n2 and k0n1.

You can actually substitute apply boundary condition, I will not talk about the boundary condition here, the boundary conditions are for the tangential components at the boundary interface r=a that is Ez and Eϕ, please remember Er is not the tangential component at r=a, similarly you Have the tangential component Hz and Hϕ for the magnetic fields and all your saying is that these values in the core must be equal to these value, that is to say, the values Ez in the core and Eϕ in the core must be equal to Ez and Eϕ in the clad. I already have Ez, but then it is your job to find out Eϕ.

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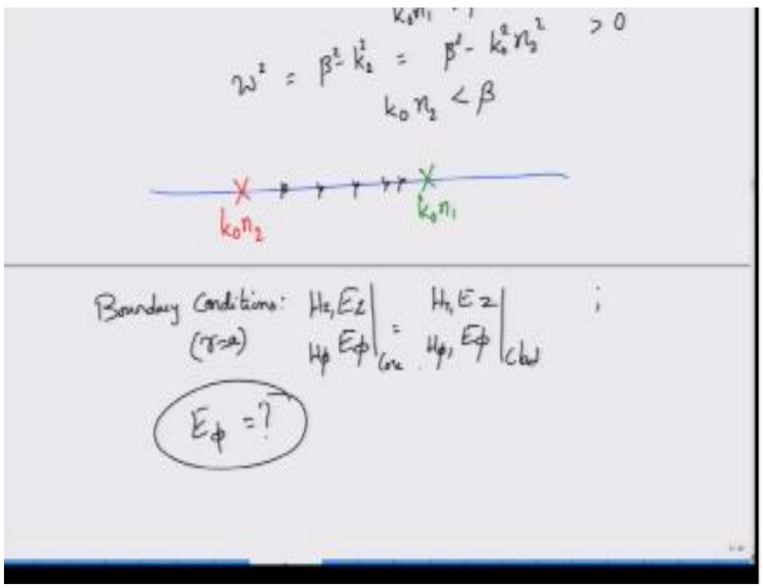

What is Eϕ find out that, similarly there will be equivalent results for Hz and Hϕ as well, the core Hz and Hϕ must be equal to clad Hz and Hϕ at the boundary. So you can apply the boundary conditions and to end up with the complicated equation for finding β, that equation is called as the characteristic equation which gives you the roots of β or also if you solve the characteristic equation, you are going to find the values of β. In the next module we will take up more properties. Thank you very much.

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