Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Optical Communications

Week – II
Module – V
Review of Signals and Representations – IV

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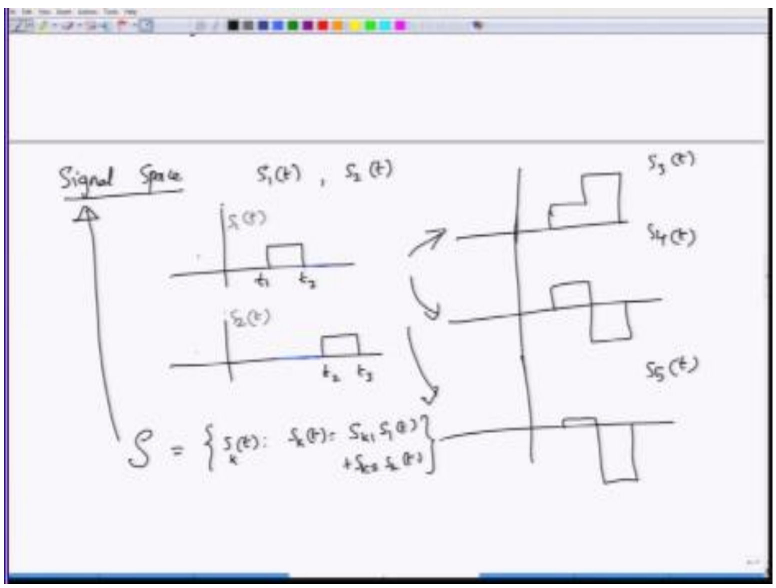

Let us consider a very special space called as the signal space which we will be encountering quite often in our study this signals space is actually formed by signals okay let us consider two signals $s_1(t)$ $s_2(t)$ may be they are of this particular shape okay they are of duration t1 and t2 okay that's I one signal then I have 1 more signal here which goes you know on this particular case set 2 and t3 so I have to revise my statement what I want to say is that each of these vectors are from duration t1 to t3.

Okay this vector is 0 from t2 to t3 where as this vector is 0 from 0 to t1 to t2 how ever these are the two signals that I am considering $s_1(t)$ and $s_2(t)$ okay I might have used the same words as signal and a vector but I hope by now that we have learned to recognize a signal as a vector so I will use this words interchangeably so hopefully there should not be much of a confusion okay so you consider two signals $s_1(t)$ $s_2(t)$ of this particular fashion and then I can add them to form these signals okay.

So I can add them to form this signal I can also add them to from this signal I can add them to from this signal okay so these are all different signals $s_3(t)$ $s_4(t)$ and s_5 (t) which I have optioned by simply adding $s_1(t)$ and $s_2(t)$ in this case you can see that $s_2(t)$ has been multiplied by some larger number and then added s1(t) here you can see that $s_2(t)$ has been multiplied by -1 and added to $s_1(t)$ here you can see that $s_1(t)$ has been shrunk where as s2(t) has been expanded in the reverse direction that is by multiplying it by minus of a large quantity right but in all these vectors and there could be of course be an infinite number of such vectors right.

If I call all these possible combinations as a span s right this span or the sub space which is now span by $s_1(t)$ and $s_2(t)$ right which will be acting as though they are unit vectors or the bases vectors they are not yet unit vectors they are the basic vectors for us so as though s_1 and s_2 are the basis signals or basis vectors for us if I take any linear combination of there could an infinite number of such linear combinations resulting a infinite number of signals but each of those signals can be built up just by stretching and expanding each of those $s_1(t)$ and $s_2(t)$.

Okay we are not allowed to move it outside the domain but we are allowed to multiply it by any real number okay so such space is called as the signal space. Signal space consists of all those s(t) okay or all those $s_k(t)$ such that each $s_k(t)$ can be built up as linear combination of these two vectors or these two signals $s_1(t)$ and $s_2(t)$ okay.

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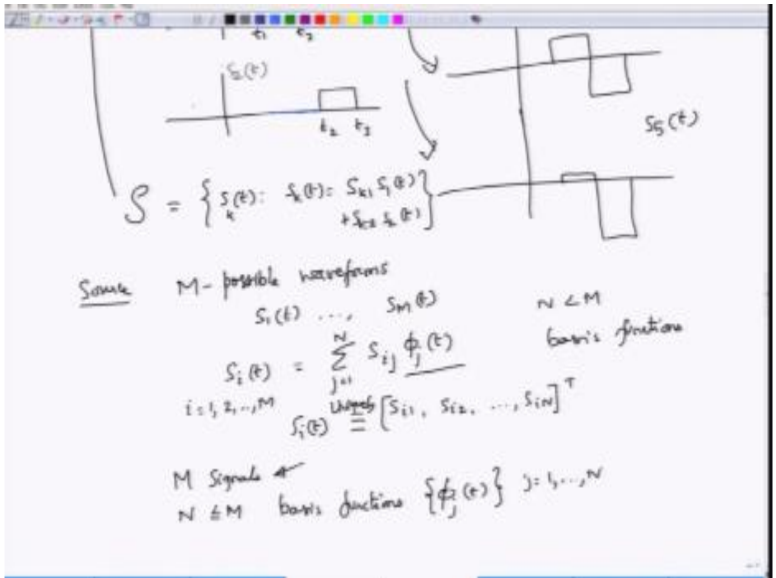

If you consider a source a digital communication source this source generates M possible wave forms okay each of those wave forms would be some binary symbol or it could be a symbol data symbol associated with that each of wave forms are nothing but signals as functions of time okay wave forms are functions of time or they are essentially signals so each of those M possible wave forms called as s1(t) to sM(t) okay we should be able to express each of those M possible wave forms as some linear combination.

Okay if I am looking at the i^{th} wave from $s_i(t)$ where i =1, 2 … M if I can write this $s_i(t)$ in terms of any i^{th} signal as say s_{ij} $Ø_j(t)$ right as a linear combination of n signals n less than M right then this $s_i(t)$ can be very unequally specified by s_{ij} components right s_i (t) can be specified by writing this as s_{i1} s_{i2} and so on up to s_{in} the transpose just to show the column vector so this particular thing uniquely specifies the signal $s_i(t)$ of course you might want to ask what is this $Ø_j(t)$ these $Ø_j(t)$ are called as basic functions and we should be able to find n such basic functions that is given a set of M signals.

Okay we can find n which could be less than or equal to M in the worst case n will be equal to M we should normally be able to find n such basic functions okay thorough this basic functions I am able to expand or write down any or represent any M possible wave forms okay so these are the basic functions these basic functions let us call this as $\O_j(t)$ okay j is equal to 1 through n now we will not.

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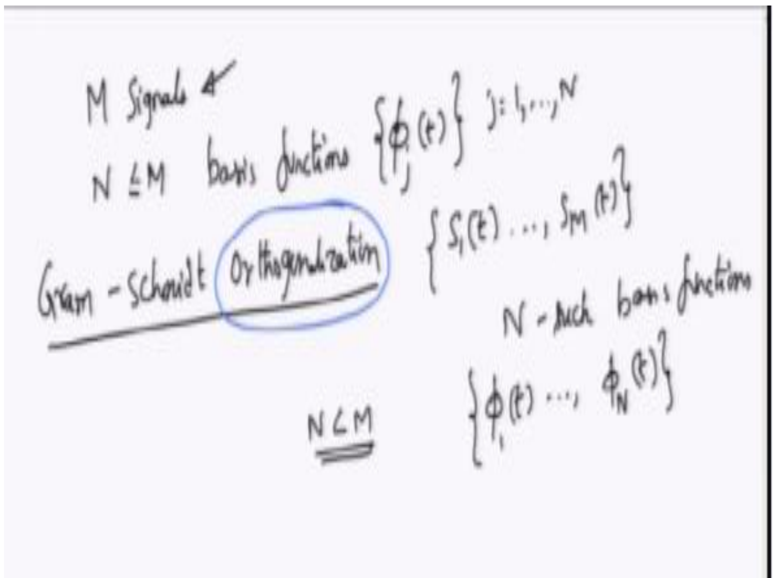

Go in to the details of how to obtain the basic functions okay I will simply assume that process is covered elsewhere you can actually generate them by going through a process called as Gram-Schmitt orthogonalization, okay. So in this procedure or this procedure allows you to get n basic functions from a given m waveforms, okay. If I have given you a 16 waveforms I should be able to find n which is say may be 2, 8, 10 depending on the waveforms would have been given.

I should be able to find n such bases function, okay. N such basic functions or n basic functions which can be represented as Ø1 (t2) Øn(t) this would be an efficient representation n if n, m, okay. So this word orthogonalization might tell you or might remind you of another work orthogonal you know in fact orthogonalization would be a process of orthogonalizing something, right. So what is this orthogonal word?

Well we go back to the vectors vector v and vector w okay we call these two vectors as orthogonal to each other if their inner product is 0 or if their dot product is 0, so this brings up an important notion which is geometric in notion called the dot product or the inner product or sometimes called as the scalar product, we define this dot product or inner product or the scalar product as.

Or we denote it by v. w for a 3 dimensional Cartesian coordinate space we are familiar with the physical interpretation of this right, I take the vector w and then I have the vector v the dot product basically gives me the length of v along w, m right. So this might be the definition that you might have seen v. w is actually length of v along the vector w of course both v have to belong to the same vector space w also have to belong to the same vector space.

So if they belong to the same vector space then v. w will give me the length of v along w, right? This is in a physics text books or in electromagnetic text book you would find this as v magnitude of v x magnitude of w Cos θ_{vw} where $\theta_{v\,w}$ will be the angle between the vectors v and w, the geometric notion is that by performing the dot product you are able to find out the component of v or the length of the component of v along w.

But this is just the length, if I want to find out the vector component of v along w then I have to take this dot product which is this number and then multiply with the unit vector along the w axis right, so I have this orange color one is my unit vector along the w axis of course this is symmetric geometrically right, so if v the component of v along w is exactly equal to the component of w along v, right.

So that would essentially be the physical interpretation of this inner product, but as you might suspect we might want to generalize or we will be generalizing this notion of inner product we will also write down the inner product with a different notation.

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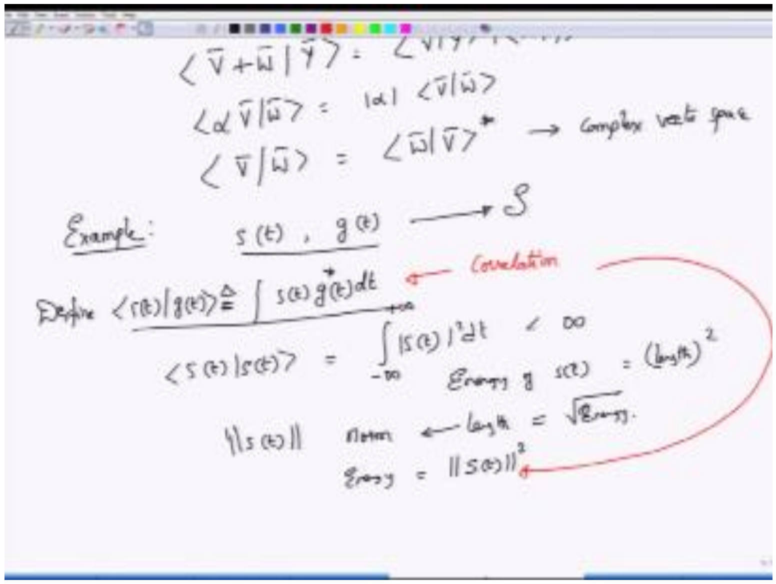

We write down this as v w, we put a bar here okay. So I define the inner product by certain rules I first of all claim that this is the notation for that inner product.

I claim that v upon v itself must be greater than or equal to 0 this is of course true unless v happens to be itself equal to 0 unless I take 0 this is true, geometrically what does the inner product of a vector with itself give you, you back to the Cartesian coordinate system V_x x^ + V_y y^ I am just restricted to 2 dimensional then the inner product of v with itself is given by vx^2 + vy^2 which is actually length of the vector v^{-2}, correct?

This is giving me the length of the vector v^2 in other words if I want to find out the length of the vector v I should take the inner product of v with itself and then take the square root of it, okay. And then take the square root of it assuming that these vectors are all real vectors, okay. Now the other rule that we want to say is that i9f I take the vector v or if I take the sum of two vectors, okay.

Call this the sum as v + w and then I take one more vector which is say y this should be equal to vy+ wy, okay. Another criteria for me would be if I take the scaled vector and then take the inner product this should be equal to magnitude of α times v w, right? The inner product of a scaled vector with vector w will be magnitude of α times v w this magnitude I am using because is α could be real or complex, okay.

So this would be the notation, there are also couple of other points about the inner product that if I take the vector v with the vector w and I say that vector v and vector w are complex vector that is they belong to complex vector space then this should be equal to w and vector v complex conjugate that is if I interchange the order then I should get the complex conjugate provide this is for the complex vector space.

Okay, for a real vector space that is a vector space that is consisting of only real vectors real valued vectors you can drop this conjugate right it would not matter because it would give writes to a scalar or a real nu8mber and real numbers will do not have this conjugate property okay now these are the rules by which we can construct the inner product right now let us take an example of our signals and the see if we can define the inner product right.

So I take s of t one signal and say g of t as an another signal both belonging to a certain signal space s okay both belonging to a certain signal space s and if I define the inner product as s of t x g complex conjugate of t I can assume that this signals are complex valued signals I am assuming them to be complex valued for real valued signal so simply I have to drop this conjugate term okay so this would be s of t multiplied by g complex conjugate of t and then integrated over the duration whatever the duration that you have considered originally for the two signal you have to integrate them over the duration right.

And now let us see whether this particular definition this would be the inner product of the vector s of t and g of t this is the definition if I define this way will I be able to satisfy all the rules of inner product certainly s if I take interchange g of t and s of t order in place of g of t getting complex conjugated I will have s of t getting complex conjugated which will satisfy is last criteria for right if I multiply s of t by number α then α will go inside if α happens to be a real number then α will simply come and multiply s of t so I can pull α out of the integral as long as it is constant.

So the entire inner product is simple scaled up the third going in the reverse way this one is If I consider the sum of two signals s of t + g if t and then I want to find out the inner product of this some with another signal say y of t it is straight forward that this equation is also satisfied one last thing would be if I take s of t and then take the inner product of that with itself this will be s of t magnitude square of dt and if I consider this to be an infinite dimensional signal that is it is going all that from – ∞ + ∞

Then it is clear that this integral does not always exist okay if we demand this integral must exist and must be finite that is the reason why w had earlier also impose this condition okay it is not necessary that this condition will be imposed okay but this be imposed in this way there are other ways of imposing similar conditions but for as it works out very nicely because we know that this integral of s of t magnitude square of dt will actually give me the energy of the signal s of t right and this signal if also the length square right length of s of t2 in some sense so you can make a connection that length is equal to square root of energy okay and this length is called as norm in vector space theory or the signal space theory okay.

And this norm is denoted by writing two lines like this okay so of t and therefore energy is basically norma2 okay, energy is norm2 norm itself doing the length I hope this connections are clear what you have to remember is that we started off with signals of the form s of t and g of t of over ascertain duration this duration could be finite or this duration could be infinite but all such signals were belonging to this space s and then we have shown that if we define the inner product by this particular way right we will be able to talk about the inner products of s of t.

In fact in signal theory this particular integral as a name this is called as correlation right this is the correlation of s of t with g of t if they are perfectly correlated like if they are the same then you get to the energy of the signal if there completely uncorrelated you get to the inner product will be equal to 0 or there essentially uncorrelated right.

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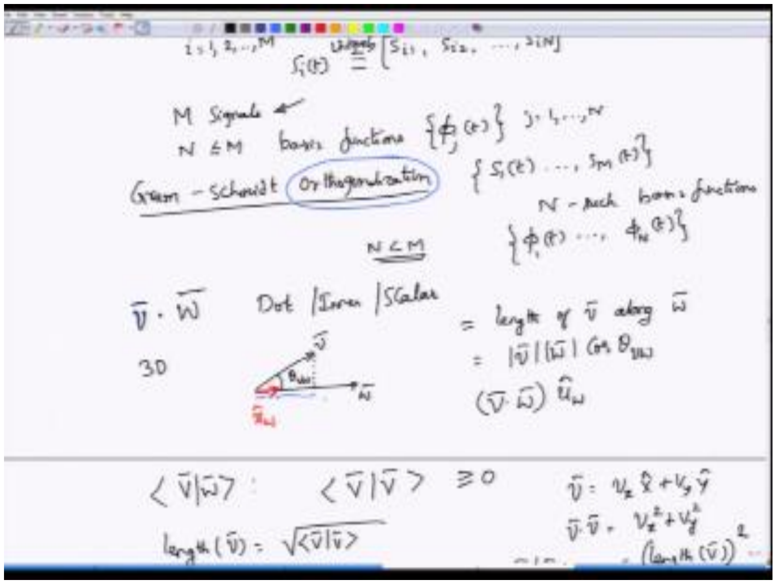

So when they are uncorrelated we call them as orthogonal signals we some time denote this by writing this perpendicular and say f of t and g of t are orthogonal if their inner product vanishes right this is for the signal if you go back to this Cartesian coordinate example they will be V and W vectors will be orthogonal when they are perpendicular to each other right so this perpendicularity is nothing but orthogonally.

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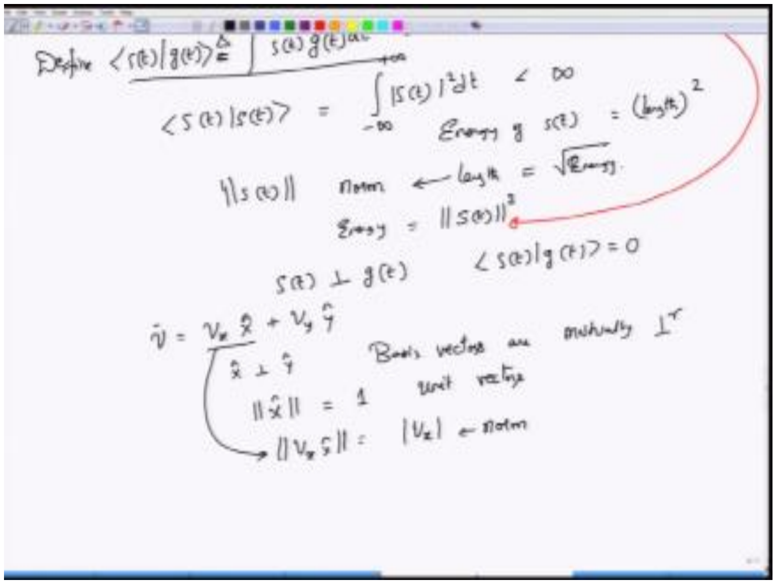

Now why is this orthogonality important for me because if you go back to the Cartesian coordinate case of decomposing a vector into it is components you will immediately notice that the vector x is perpendicular to vector y correct so this additional property that the basis vectors are mutually perpendicular if you where to consider three dimensional case then x is perpendicular to y x is perpendicular to z y is perpendicular to z right.

So there all mutually perpendicular to each other moreover what is the length of vector x or then norm of the vector x is one correct that is why these are called as unit vectors, vectors having magnitude 1 which can be stretched or shrunk to form additional components, so what would be the norm of this v_x.x^ vector what is the norm of this vector well it is nothing but magnitude of waves, because it has to be positive, right. So this is the norm of this particular vector, okay.

Remember v_x is just a number it could be positive, negative it could be complex, when you multiply that one with $\hat{x}$ assuming in the Cartesian coordinate system then v_x can only be real, it could be positive or negative or it could be 0, but when you take a number and multiply it to a vector you get a vector, and norm is specified for vectors not for scalars, okay. So this is the notion of norm.

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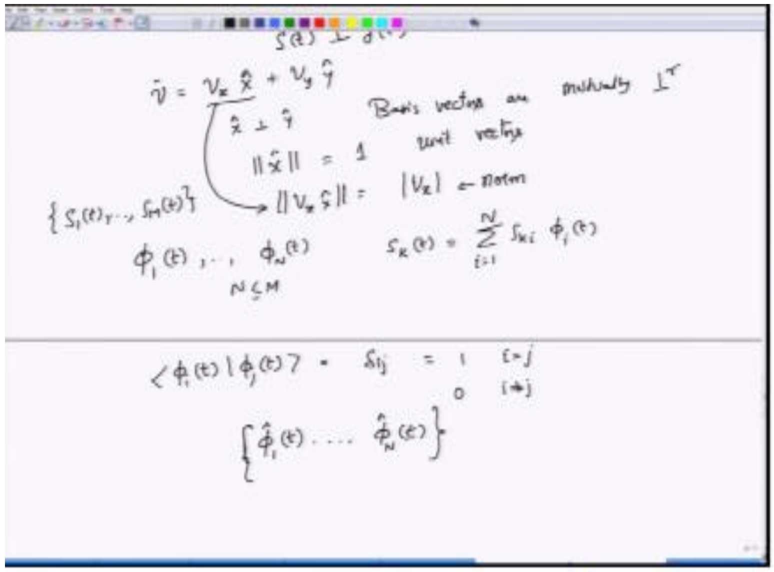

Now you back to the signal space that we talked about, right we had signals $S_1(t)$ through $S_m(t)$ there where m signals and we said that if we would find n basis functions, right we could find n basis functions $\phi_1(t)$ to $\phi_n(t)$ with n being less than or equal to m, right then I can expand any vector $S_k(t)$ or any signal $S_k(t)$ as the sum of these vectors, right $\phi_i(t)$ I can expand this as a combination of these basis functions $\phi_i(t)$, further if I choose these basis functions in such a way that if I take two basis functions and find out what is there inner product and say that this inner product is equal to one in case the vector happen to be the same or it would be equal to 0.

When these two vectors or these two signals are basis functions are different then what I obtain is a set of ortho normal basis functions, okay these are called as ortho normal basis function ortho because they are orthogonal to each other so this is the set I have orthogonal to each other, you can take any signals they will be mutually the inner product will be 0 that is they will be completely uncorrelated, right these are signals therefore they will be uncorrelated.

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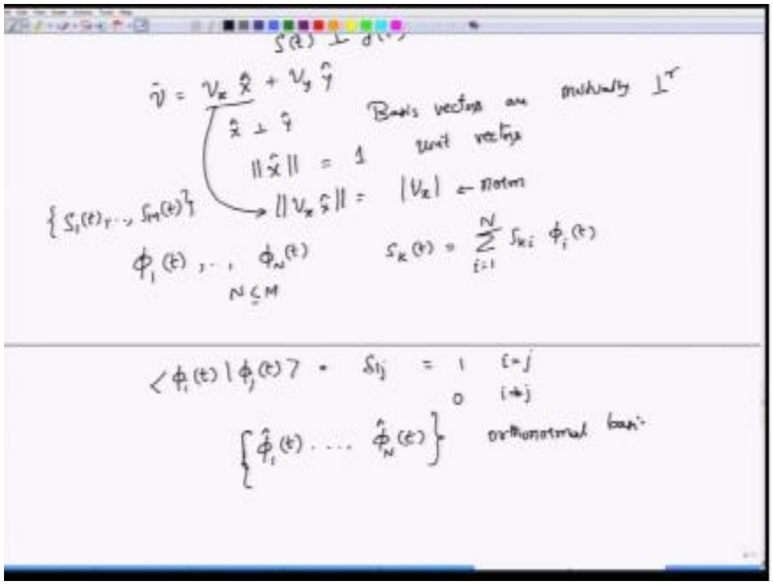

However, if you consider the inner product of the signal with itself you will end up having the inner product=1, just as that would be the norm for a unit vector, right. So that is why they are called as normalized their lengths in some sense has been normalized, okay. So this forms the so called ortho normal basis.

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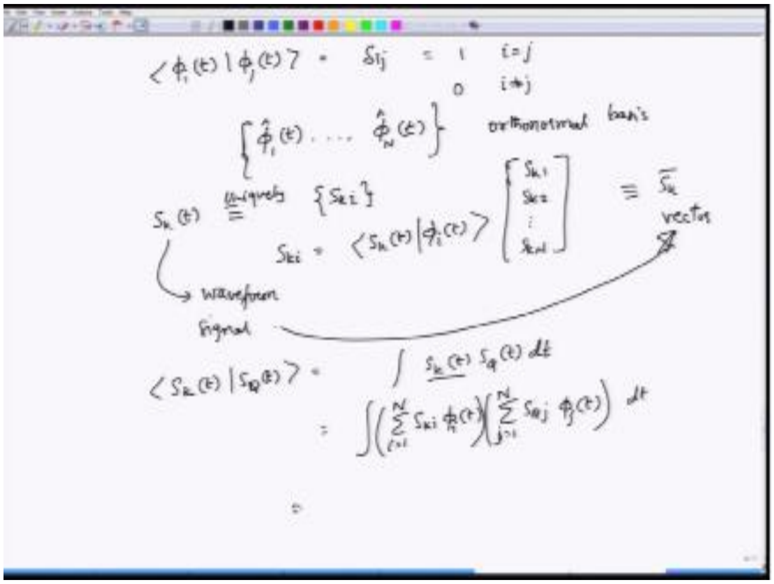

Now we are ready to close out discussion on the signal space, what we have found is that any signal $S_k(t)$ which is a time duration wave form can be uniquely represented or specified by giving its components S_ki, right. These where, what are this $S_k$is, $S_k$is are obtained by taking the signal $S_k(t)$ and then forming the inner product of this with the ith basis function, okay. So this would be the S_ki and the set of S_ki how many S_ki should I have for each S_ki this would be S_k1, S_k2 all the way up to S_kn, right.

So if I arrange them in the form of a column vector I can write down this in a short hand notation and write this as $\bar{S}_k$ which will now be a vector, okay. $S_k(t)$ is a wave form or a signal which has now been represented as a vector, okay can I talk about inner product of two vectors $S_k(t)$ and $S_Q(t)$, yes what would be this inner product go back to the definition this would be $S_k(t)$ assume that these are all real so this would be $S_k(t)$ $S_Q(t)$ the complex conjugate thing will go away multiply by dt, right this is the inner product definition over whatever the duration these two signals have been specified as, right. Now in place of $S_Q(t)$ you can write down its representation in terms of S_ki, right this would be S_ki $\phi_i(t)\sum$i= 1 to n and for $S_Q(t)$ I can do the same thing this would be S_Qj $\phi_j(t)$ where j=1 to n, there is an integral that is still going on, right.

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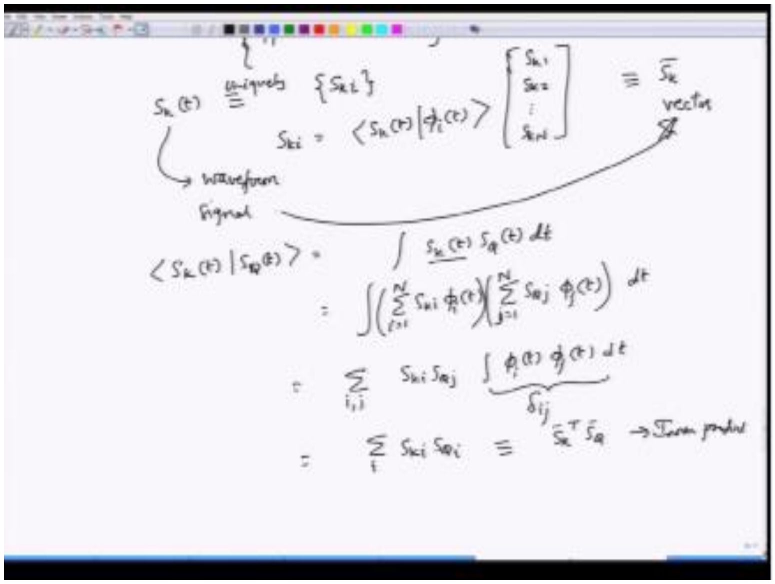

Now I can interchange the order of $\sum$ and integration, okay so I can write this as I and j this is a double sum I can write this as S_ki, S_Qj and I have the integral of $\phi_i(t)$ $\phi_j(t)dt$ but I know that these two this is nothing but the inner product of ϕ_i and ϕ_j which would be equal to δij that is they would be equal to 1 when i=j and they would be 0 otherwise, allowing me to collapse this double sum into a single sum and you get S_ki S_Qi, this is nothing but taking the vector S_k^T it multiplying it by S_Q vector, okay. Or you know component by component multiplication this is the inner product, okay.

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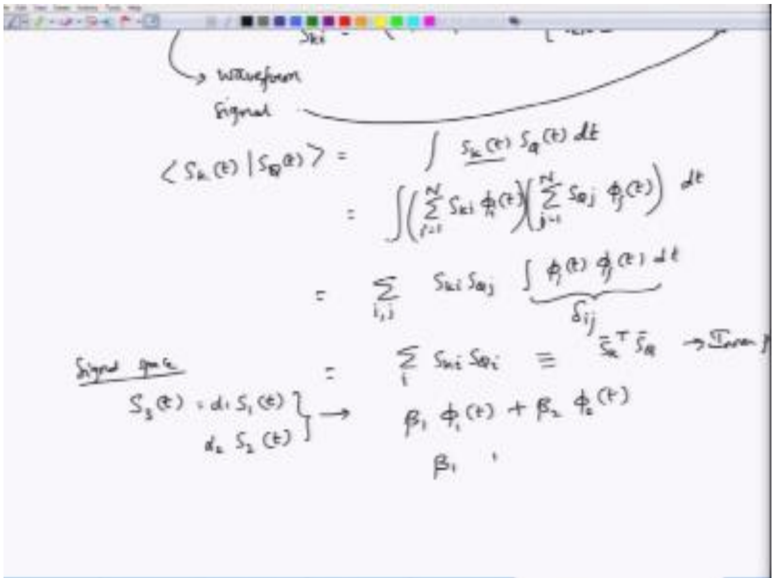

I will leave an exercise to find out S_3(t) and S_n(t) but I would like to very quickly give a geometric representation, right since I can write down S_3(t) you know for the signal space example that we considered I can write down S_3(t) as S_1(t) and s_2 (t) from this $s_1$of t and linear combination of s_1 and s_2 I will obtain a different combination let us call this as β_1 and β_2 space of s1 I will get 5_1 of t and in place s_2 I will say I will use 5_2(t) where 5_1 and 5_2 are the ortho normal basis function which have been obtain from s1(t) and s_2(t) if you go back to s_1 and $s_1$2 there already orthogonal therefore the orthogonal functions would also be 5_1(t) and 5_2(t) which have been scaled up to get the energy = 1.

So I can write down this s_3 (t) in terms of these vectors or equivalently simply specify this in terms of this β_1 and β_2 right.

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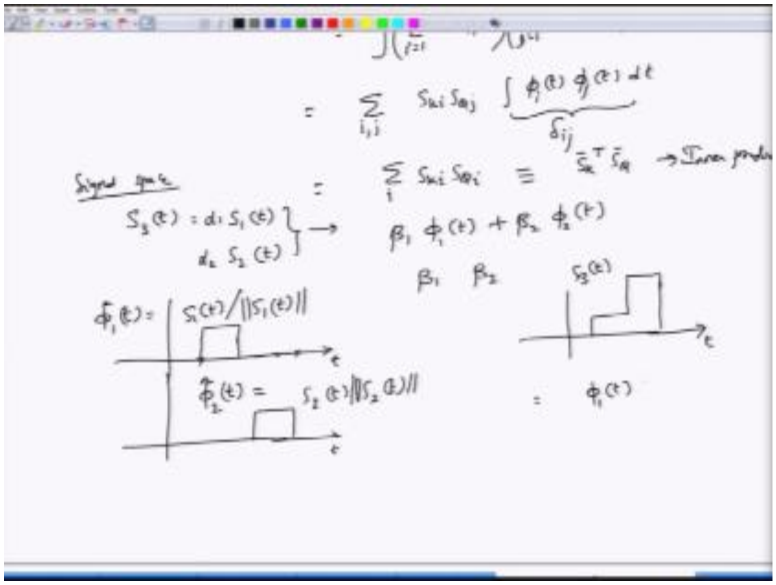

Then geometrically I can represent this as you know this is $5_1$2 axis 5_2 axis I can represent this one as this vector whose components are β_1 and β_2 we will talk about it later so if you go back to the signal space s_3(t) can be written has some linear combination of s_1(t) and s_2(t) if you recall what s_1(t) and s_2(t) look like this is how s_1(t) looked and this is s_2(t) looked right over the duration t_1 to t_3 right so what is the duration t_1 to t_3 these 2 function are clearly the inner product of these two is clearly is equal to 0 meaning that these are orthogonal signals already.

If you take this signal s1 of t and divide this one by it is norm or the energy of this signal or square root of the energy I will be able to obtain the corresponding orthogonal or ortho normal basis functions right so how do I generate the basic functions take the signal and then scale it up such that it is energy is = 1 so I get 2 basis functions which I am using a caret to indicate there, there is a hat on top of it okay.

These two are the basis functions right form these 2 basis functions I can write down this vector right I can I mean I can represent this signal s_3 (t) as say 1 unit of 5_1 (t) + 2 unit of 5_2(t) for example. Right I can write down s_3 (t) as one unit of t as one unit of $5_1(10)$ 5_1 (t) and 2 units of 5_2 hat of t or equivalently I can simply give coordinates one and two to this right.

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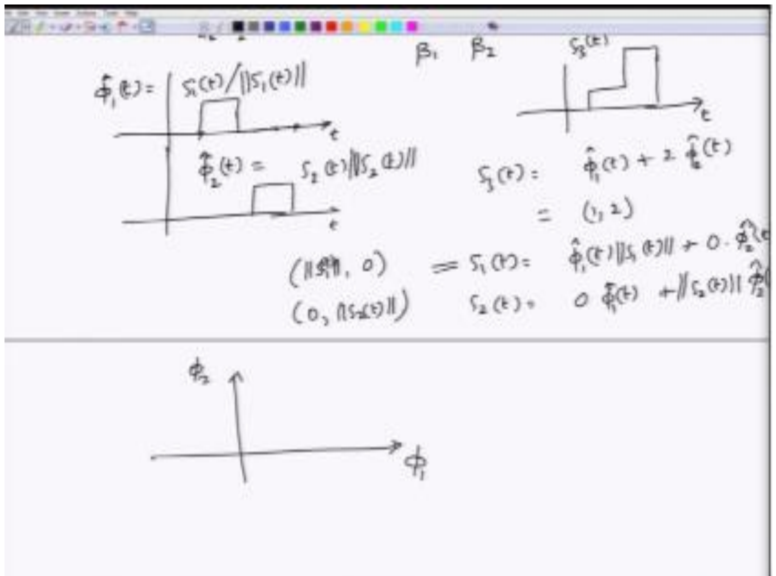

So in the 5_1 and 5_2 plan right if you can imagine this as 5_1 plan and 5_2 plan will drop the time notations because 5_1 (t) itself can be written as I time 5_1(t) + 0 times 5_2 (t) right so this can be written as well let us not go to this way this is not the correct one to write. We can simply considered this 5_1 and 5_2 as vectors or the as 2 mutually perpendicular vectors right and then any vector s_3(t) can be written in terms of this 1 and I mean by giving them the combinations $\alpha_1$and α_2 right.

So in this case what would be s_1(t) itself will be 5_1(t) times s_1(t) is norm okay and s_2(t) will be +0 time 5_2(t) in this particular case for s_1(t) and it would be 0 times 5_1(t) for s_2(t) = s_2(t) in norm times 52(t) so this can also be equivalently represented by two numbers which one s1 norm, which would be a number right and 0 this one can be written as 0 s_2(t) η, I can now represent all these three waveforms in terms of their corresponding vectors, this is S3 vector, this is S1 vector, this is S2 vector, so S1 vector may be here, S2 vector is this one, because this length would be η and this would be the length S2(t) η.

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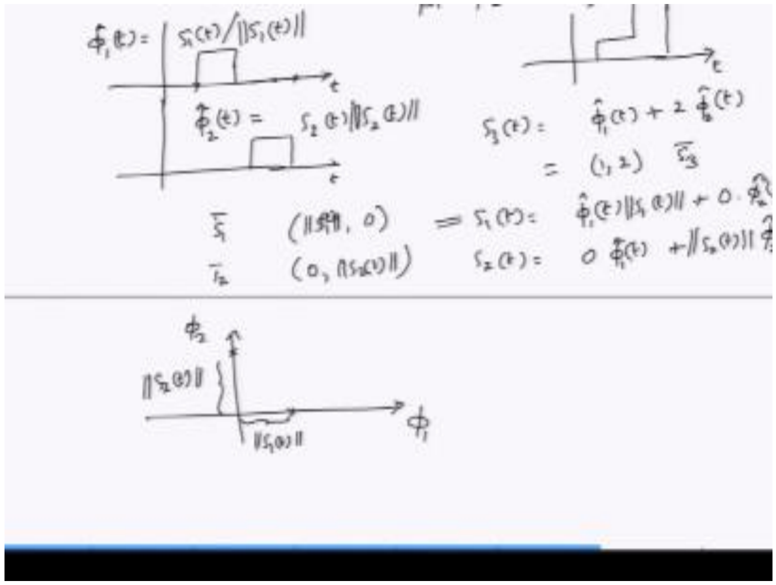

Then S3(t) can be written as units of say 1 and 2, so let's call this as on and this as two, so this would be the, S3 vector. So this is S1 vector, this is S2 vector and this is S3 vector. Now I can very clearly see that, if I want to project or if I want to take the inner product of the S3 with S1 I will get a certain number, so it has component along ϕ 1 and ϕ 2, and I have used ϕ 1 and ϕ 2 here, but you could have used any other number, you could have call this as x, you could have call this as y, the basic idea is that every signal space you should be able to write it as linear combination of the basis function, from the basis function that we have written, whatever the numbers that we are multiplying, those numbers you can collect to form a column vector and that would b represented in the corresponding plane.

On this same representation what can you say about the distance between two vector say S(t) and G(t), what would be the distance between two vectors? I know that S(t), if I expand to terms of the orthogonal base functions, can be equivalently represented a vector S, this would be represented equivalently as a vector G. The distance between these two vectors is nothing but S-G η, this is the distance or distance square, this is the distance between S vector and G vector, square is nothing but this particular thing.

So on some space, this case it was a two dimensional space may be in this particular waveform, maybe it's a three dimensional space or four dimensional space, I will be able to write down S(t) vector by giving it's vector representation. I also have the vector G and this would be the difference vector S-G. And this distance or the difference between the two will be given by thee length or thee η square of this particular vector.

So this I how we can write down, you know or we can give representation of vector 2 signal, so we can think of signal as a vector and this decomposition of any of the source waveform, say Si(t) or Sk(t) doesn't matter. So this decomposition of Si(t) into the J unit or the orthonormal vectors, orthonormal basis functions would give rise to representation of the waveform as a vector on the appropriate signal space ort appropriate signal band by thee basis functions ϕ(t), is that ok? This is ϕj(t), so that the equivalent way in which I can write down this is by giving it as a vector and call it as vector Si.

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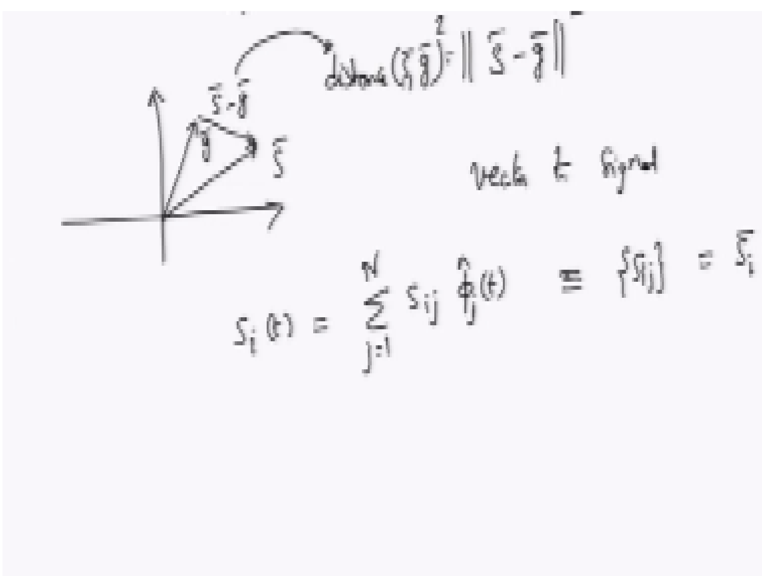

These concepts of thee distance between two vectors and how to represent this or how to utilize this vectors, the signals as vectors will be very important when we discuss digital modulation techniques, where we see that ASK, BPSK,QPSK, QWAM all these waveform can be thought of as vectors in a geometrical space, even noise can be thought of as vector space and then it will be developing lot of intuition, about how to go about calculating the probability of error and whether, I mean all other different metrics are involved in this.

So in the next module we will be utilizing this concept, so this module was largely a background material, from next module we will be utilizing these concepts to discuss the modulation methods. Thank you.

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