Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Optical Communications

Week – II
Module – IV
Review of Signals and Representations – III

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Hello and welcome to the fourth module on optical communications, in this module we will discuss very briefly discuss the idea of representing a signal as a vector and then we will connect these signal theory or the signal as the vector theory we will discuss, into discussing some fundamental modulation techniques. We will also give the physical way of realizing those modulation techniques all, we will concentrate today on digital modulation techniques, we will come back to analog modulation techniques sometimes later in a different module.

To begin with we are all quiet familiar with a notion of a vector, a vector is something that have been taught to us as having a, in a physical quantity that has both magnitude as well as direction. An example of a vector would be a car moving along a particular highway, because it has both magnitude, in this case it has magnitude, it is called a the speed and it is moving along the certain direction, may be from south to north or from east to west.

There is a specified direction also associated with this moment, so any physical quantity has both magnitude and direction is a vector, and when we talk about vector, we also kind of want to represent them mathematically. How do represent vector mathematically? Is mathematically the only representation? No we all are familiar with the graphical representation of a vector. How do I represent graphically a vector? I will simply draw an arrow and I know the arrow has a certain origin and it has a certain end.

The length of the arrow determine the magnitude of the vector and the direction in which this arrow points, would be the direction of the vector, say something that we know from the earlier courses in physics. Now mathematically we represent, well it is possible to actually just talk of vectors graphically and then add two vectors, multiply two vectors in all graphical ways, but if you want to you know give some number to that, then you have to find a way of representing them mathematically.

The way we normally represent a vector mathematically would be that, we also associate with this because we want to give some numbers to the vector right? So we want to associate numbers to the vector, so what we do is we also associate a coordinate system to this. A simple example that might be familiar to you would be gain the velocity vector expressed in terms of the three dimensional space. So this tree dimensional space can come from any variance coordinate system, most common one that is familiar to you would be the rectangular Cartesian coordinate system, so you have an x axis and z axis, any vector that is there on this particular coordinate system, we will conveniently locate the origin of the coordinate system to go inside the origin of the vector itself.

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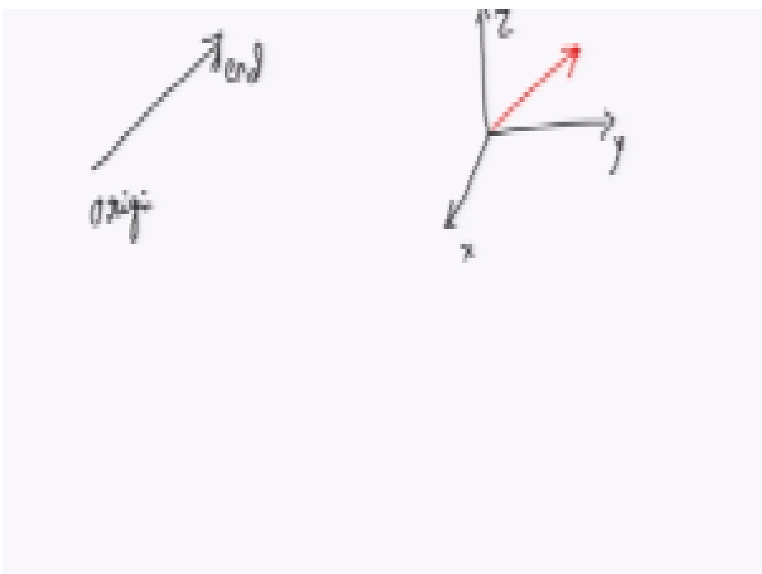

And then we, so called this origin as O and then end point of the vector as AP, then we have a vector which can be represented as OP, we can also give vector by a different name, so let us say this vector Op is actually a velocity vector V. The way we distinguish vector and scalar is that, we write a bar on top of it, several ways in which you can write a vector would include writing an arrow over the variable or you can sometimes find in certain text books that people have written a under bar, rather than arrow or over bar.

I will be using this over bar to represent or to denote a vector, so this vector be in this which particular case the vector OP is situated at the origin of the co ordinate system and it has certain length. Now we also specify this in terms of what are called as coordinates, so we specify this in terms of three values with which this particular vector may mix with the respect to x and z axis. So we write down this V as some component and Vx which would be a number and multiplied by x hat +Vy y hat+Vz z hat.

What are these numbers Vx /Vz they re these lengths, so If I'm lightly erase these position, then just how you, what Vy and Vz are, the length of this vector P, OP or the vector V along the y axis, that is this particular length, is the component Vy, similarly this length I the component Vz and to obtain the component Vx you have to first drop this vector P onto the XY plane and then connect it to the x axis, and then you will be able to get what is Vx, this is how we normally write down.

An example would be a vector V given a X hat +Y ht +2Yhat+3Zhat, these X hat, Y hat and Z hat are called s the unit vectors, these vectors allow me to represent any other vector V, ny vector V in these three dimensional space can be expressed as, in the form of Vx+Vy yht+Vz zhat, in this case, this will actually have addition of three vectors, each of those vectors are Vx xhat is actually a vector which is directed in this Cartesian coordination space, called this as X, Y and z axis.

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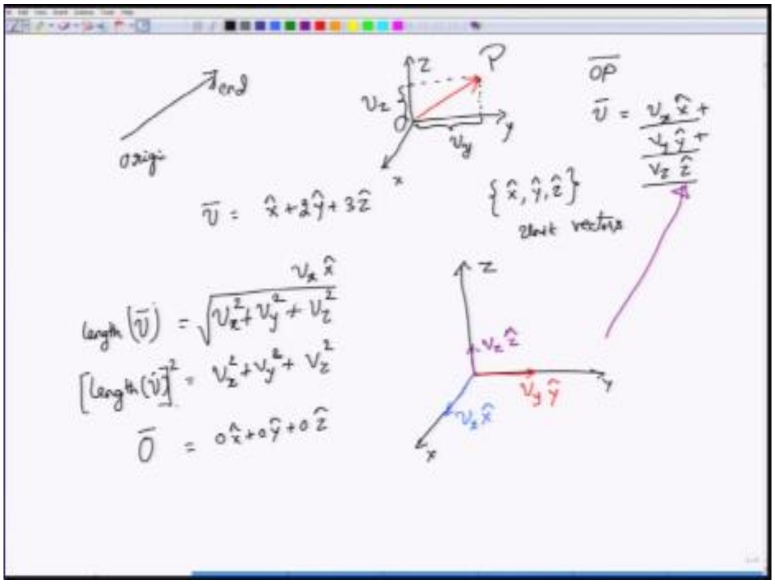

These vectors are Vx xhat so this is Vx xhat, it is a vector along the X direction and it's magnitude or the length is Vx, similarly you have a vector Vy, along y hat and finally you vector Vz along z hat, so what this particular expression for V is telling you is that you have to take three vectors and then form a sum of these vectors, so if you form sum of these vectors, you will end up in the original vector V or this original vector OP, OP is V in our case.

These are the concepts that you already know, associated with is would also be a couple of additional things that you know, If I want to find out what is the length of this vector, so if I want to find what is the length of this vector, I know that from its components which are given as Vx, Vy and Vz, I can write this as Vx^2+Vy^2+Vz^2 under root.

Okay the square of the length itself will be simply vx^2 +vy^2 + vz^2 we also are familiar with the concept of a null vector okay null vector will have 0 length along x Axis 0 length along y axis and o length along z axis it is length will be also equal to 0 so these are some concepts that we know of vectors now we can actually.

Generalize all these concepts there is no reason why we have to stick to 3 dimensions or 2 dimensions we can generalize this to n dimensional vectors and in some cases we can also generalize this to an infinite dimensional cases or infinite dimensions okay so we will be little more abstract than we have been so far that is we know we and then this Cartesian coordinate system and we have defined certain compounds or certain concepts over that vector but we can choose to be little more abstract and when we be little more abstract we end up into a theory called as vector space theory.

Okay we of course not be able to do full justice to this theory but some basic ideas of vector space theory is very important for us to further talk about the modulation technologies okay so with that reason why we are going to vector space theory let us be little more abstract and say what exactly is a vector or let us defined mathematically in s slightly more abstract sense what is vector okay we call a collection.

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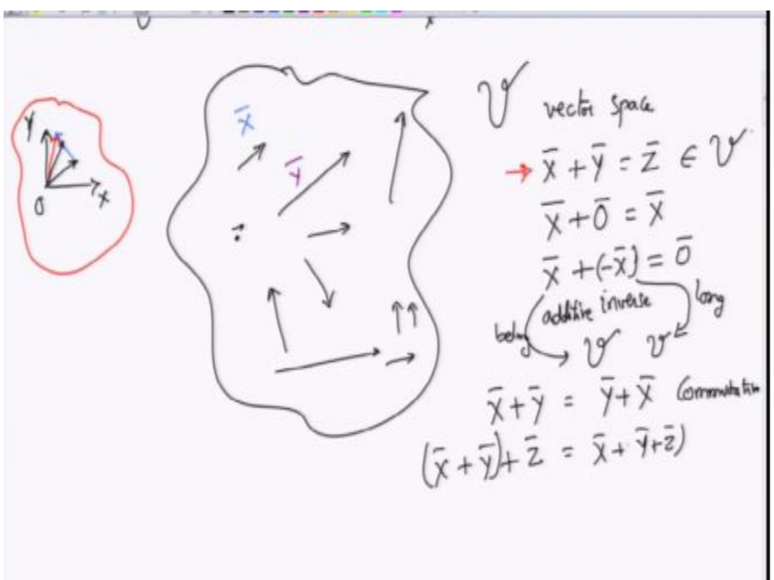

Of objects okay we have a collection of objects you can think of this as kind of bag which consists of different kinds of vectors okay they are all oriented at a different way there is also a dot vector okay you have another vector this way you have this vector you have this vector you have this vector okay so you actually have a collection of such vectors okay let us call this collection as v and then each of these will be a vector itself.

Let us call this vector as x vector and call this vector as y vector of course I could have called any vector in this collection as x vector any vector in this collection as y vector then we say that this vector v is actually a vector space okay we call this as a linear vector space which simply means that it is a collection of objects know as vectors just take a bag but all the vectors in that okay some times that bag as to be so large to encompass an infinite number of such vector but theoretically let us say we have put everything into a bag and we call that bag we just give the name of that bag as a vector space okay what are the characteristics of those object that are content in that vector space.

What can we say about vector x , vector y is there something that we have to talk about them so that the concepts that we know in terms of a magnitude and direction can be applied to these vectors right so the basic elements of these vectors space are all these vectors and they satisfy certain rules right so you have two vectors say x and y the sum of these two vectors will be a new vector call this as z vector but this z vector must also be contained in the same bag that okay this is kind of familiar to us.

I take in the 2 dimensional Cartesian coordinate system if I take the x, y plane or you know x, y coordinate system and then I consider two vectors so this vector 1 and this vector 2 the sum of these two vectors which would be obtained graphically by putting this vector on top of other vector and then completing the triangle this red color is the sum of these two vectors okay hopefully that kind of visible to you this red vector is essentially one more vector which is still lying in the same plane correct.

So it is a same vector which is lying in the same plane or it is in habiting the same space as the two earlier vectors so this idea of sum of two vectors resulting in a new vector which is lying in the same space is captured by this equation I have x vector + y vector which results in z vector x lies in vector space v y lies in vector space v z also lies in the vector space v okay.

Some additional properties that we have to satisfy is that it should be possible that the bag must contain a 0 vector or a null vector okay such that if you take any vector x and add that 0 vector or the null vector to it we should get back to the same vector x clearly x belongs to v 0 belongs to v x+0 which is x also belongs to v for every x that you consider in this vector space you should also be able to find–x vector or this is called as the inverse of the vector, okay. This is called as additive inverse because it is getting added in the sense so this additive inverse vector must also exist such that there some must be the null vector, so x belongs to v – x which is the vector inverse must also belong to v itself, okay. And this must be true for every vector then we say that this vector space is a linear vector space it is closed, okay.

Couple of additional things that you still need to that are still valid here are that the sum of these two vectors does not matter which way you add them, right you can take x + y or y + x they both should be equal this rule is sometimes called as commutative rule, okay. So this is something that you have seen from algebra take two matter it does not matter in which order you add them , so it is a same case with vectors also.

It does not matter whether you put vector x and y or vector y and x together you should get the same vector in addition there is also what is called as a associative property to take two vector x and y and then add the resultant to the third vector it does not matter in which order you have added them , okay this is the associative property of the vector, okay. Now these are the vectors life would be very boring if you simply add his vectors and not.

I am simply adding them or finding the inverse of each vectors we are unable to generate any new vector from this right I mean given this set of vectors we can keep adding them the resulting new vector will all be in the space, okay. We also want to get back to the idea that I have a vector and I can alter its magnitude, how do I alter the magnitude of a vector? I if I say vector is v, right?

If I want to shrink the vector if I want to shrink the vector then I will be multiplying that vector by a number that is less than 1 if I want to expand the vector then I will take that vector and multiply it by a number greater than 1 but in this set of rules that we have discussed so far we do not have that opportunity for us, right? We have not considered multiplying a vector by a scalar which means I do not know how to shrink or expand the vector.

If I multiply a vector by a complex number it will actually result in rotation of a vector, okay. We will come to those things later but first I need to be able to specify the specify a way in which I can shrink or expand a vector that can be done.

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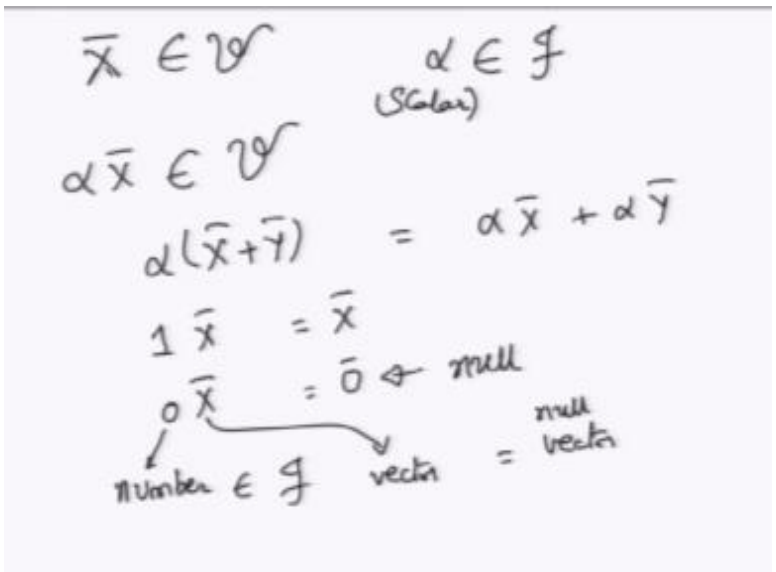

By scalar multiplication of a vector, okay. Suppose this is a vector x which belongs to the vector space v then if α is a scalar that belongs to a certain number system okay if this α happens to be real then it belongs to a real number system if it if complex then it happens to belong to a complex number system, okay. So for whatever it is this particular number is actually called as a scalar in this vector space theory.

So this x being a vector if I take α and multiply it by x, okay I will generate a new vector now because this vector space has to be closed right I also demand that this vector B in the same space as the vector space V, okay. Clearly I can take this two vectors x and y form their sum first and then scale them or I can first scale the individual; vectors x and y by multiplying it by the factor α.

And add them together so I should get the same result in both cases, there should also be a scalar called 1 such that if I multiply 1 with vector x I should get vector x itself there should also be a scalar called 0 if I multiply 0 with x I should get to 0 vector, please not the difference between these two things, okay. If this is little unclear to you, you can call this 0 vector as the null vector, okay. This is a number 0 is a number which belongs to this f.

And then this is multiplying the vector so vector multiplied by a number the result is actually a vector, in this case it happens to be a null vector, so these are some basic generalization of the abstract concept or known as vector space let us give some example s, okay.

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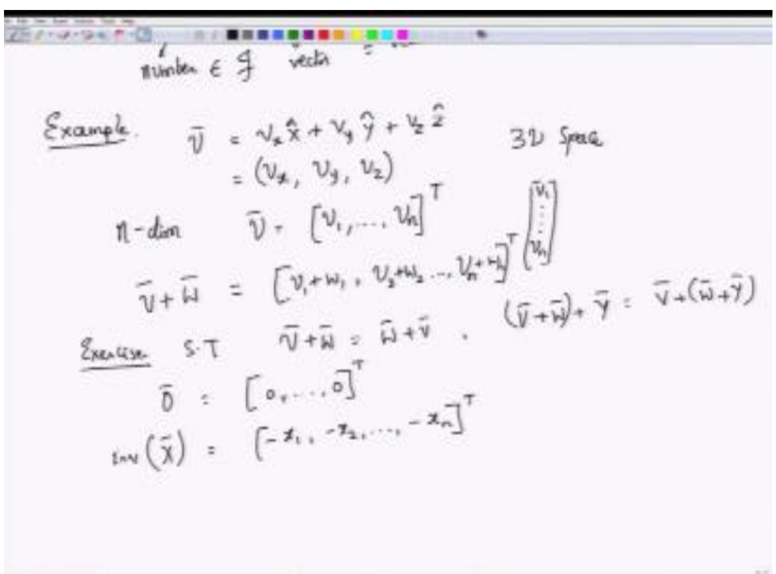

Let us look at as some familiar and little un familiar examples, as for the example that I will consider this is the generalization of the concept of a vector into n dimensions, right? Going back first to the 3 dimensional example for vector v can be specified you know this is the way we had written out the vector we will come back to this notion later but we can write a vector in terms of this particular decomposition of vectors into xy and z components alternatively I know that x^ y^ and z^ are kind of fixed they are not changing if I have the coordinate system fixed then I can write down in place of x6 y^ and z^ I can simply give these numbers V_x V_y and Vz and I will be able to uniquely specify the vector we this is in the three dimensional space why is it three dimensional because there are three numbers which are require to specify the vector v

Now I can go to n dimensional space by saying that every vector in this n dimensional space must be specified by n numbers okay and I want these numbers in a particular fashion which is called as a column vector so I take a row vector and then put a transpose this T denotes the transpose operation so physically you are thinking of a vector as being a column vector with the values V_1 through V_n this would be an n dimensional vector because I have n components to specify okay you can ask what are these components these components will be associated with the unit vectors along those n directions.

Okay I have associate with the unit vectors along those dimensions we will talk about it is shortly okay how do I add two vectors in this vector space let say V and W are two vectors okay I can add them component twice that is the new vector will be given by $V_1 + W_1$ that is the first component $V_2 + W_2$ and so on up to $V_n + W_n$ and clearly both V and W must be of the same size for me to add them and there is a transpose operation sitting on top of it okay from this you can very easily show that two competitive and associative properties right.

Because it does not matter whether I add $V_1+ W_1$ two numbers in this way or $W_1+ V_1$ so you can show I will give this as a simple exercise for you to show this show that ST stands for me show that associative, associative and competitive operations both are valid if the define and n dimensional vectors so this is the competitive rule and V+ W+Y = V+ W+Y is the associative rule okay you can show this by going back the basic definition what would be the null vector in this case and null vector will be a vector which consist of zeros n0s will be the null vector and what would be the additive inverse of a vector x the additive inverse which we can call as inverse of x.

Will be – x1 – x2 and so on all the way up to – xn sorry there is transpose here for all these vector because I am assuming all vectors as column vectors okay so this is the example of a straight forward generalization of three dimension to n dimension.

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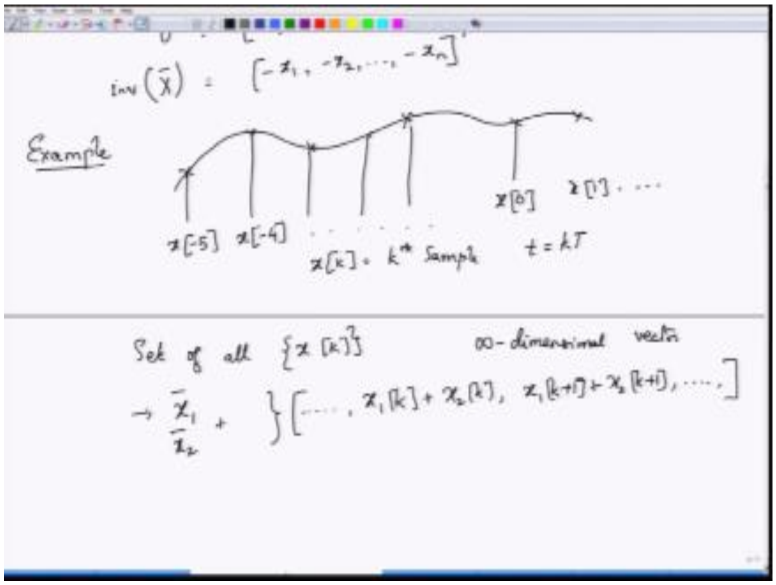

Now is it possible for me to generalize this two an infinite dimensional case yes suppose I have a discreet signal okay which may be obtained by sampling a continuous time signal okay have sampled these signals and I have generated the sample sequences so let say this is coming of as x of – 5 x of – 4 and so on and let say this is the point which is x of 0 okay this is x of 1 and so on where these numbers in the square bracket denote the time that is the k^{th} sample x of k was actually taken is the k^{th} sample.

Okay it was taken at a time t = k x t where T is the sampling period that I have used to obtained the discrete signal of course it is simply possible to have discrete sequences okay it is possible to have a sequence that consist of an infinite number of points each sequence consist or each sequence consisting of an infinite number of points is possible for me to have that but this is a physical way of generating an infinite dimensional were assuming that you have turned on the analog to digital converter at time t = - infinity and you will stop it only at equal to + infinity you can obtained a discrete signal x of k which I have written over here and if I consider a class of all such continuous time signals.

And then sample all of them to obtain different discrete time signal so I am basically going to consider a class or rather a set of all x of case okay this would be a set of all these discrete time signals, okay then I can consider this as a infinite dimensional vectors, because I need an infinite number of points to specify. How would I do vector addition and vector multiplication, sorry vector addition and null vector and additive inverse for this case, well I considered two such vectors say $x_1[k]$ and $x_2[k]$ these two may have come from two different continuous time signals which have been sampled or they have been given as two sequences as to you.

The vector addition to this will be adding them component wise, right so at each kth time you take the value of $x_1[k]$ and then add it to $x_2[k]$ at the kth, [k+1]th time this would be the sum and at k-1 this would be x1[k]-1 and x-k-2, so this would be the vector, okay. So there is a slight abuse of the notation here, we are using $x_1[k]$ to talk about the vector itself as well as the kth sample, hopefully this should be little clear to you, this is a vector, okay maybe we should put our arrows here where to define that this is the vector.

But this vector is an infinite dimensional vector because it has the kth sample as given by $x_1[k]$, so maybe we can, not abuse the notation and this just write it as x_1 vector and x_2 vectors, okay and the sum of these two vectors that is x_1+x_2 is given by adding each kth sample of the x_1 vector to k sample of the x_2 vector, okay.

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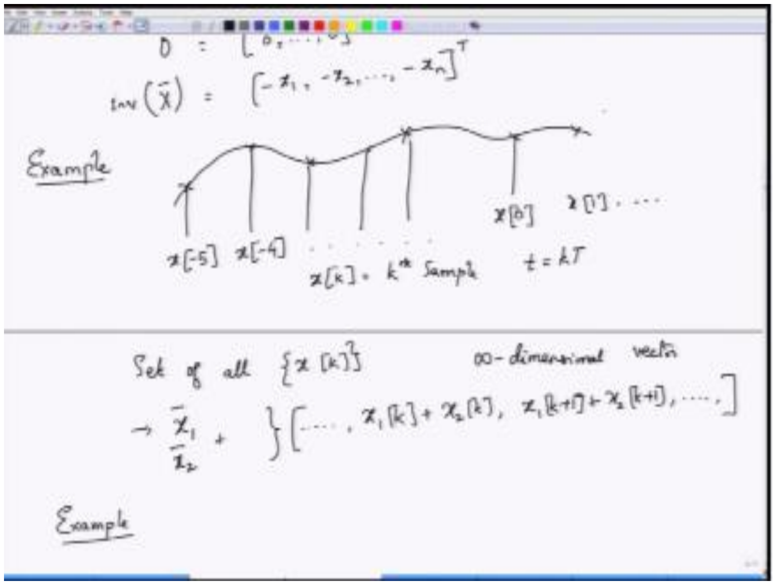

A 0 vector would be a vector whose samples are all zeros, right so that is the 0 vector or the null vector. Now let us go into a slightly more abstract thing where in we want to consider not the samples wave forms x[k] but rather a set of all continuous time signals, okay.

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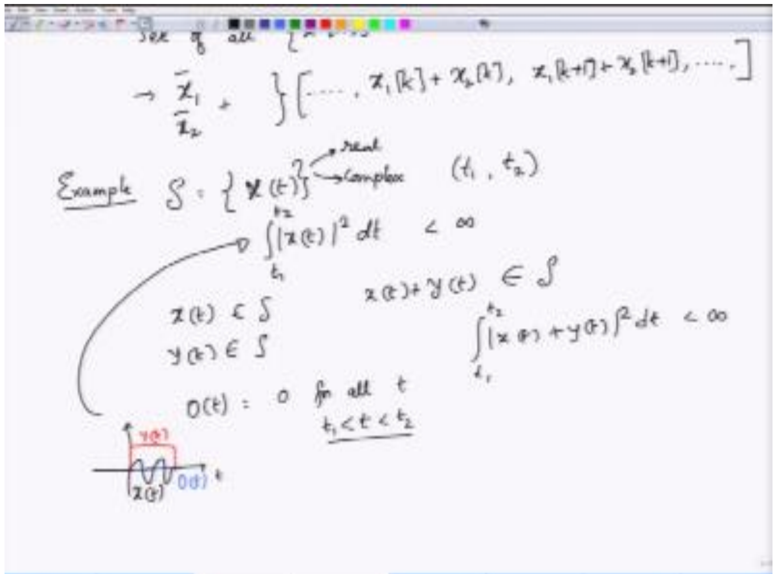

We want to consider the set of continuous time signals, okay such that each signal say x(t) which goes over the time duration say t_1 and t_2 I consider all continuous time signals, okay my bag essentially consists of all continuous time signals x(t) which you know are over the duration t_1 and t_2 such that they also satisfy this additional criteria that their integration from t_1 to t_2 of x(t) magnitude square should be finite, okay I am using magnitude because these signals can be either real valued or they can be complex valued, okay.

So in this case, will this set of x(t) signals which all satisfy this particular criteria form a vector space, yes because if x(t) call this set as S, so if x(t) belongs to S, y(t) belong to S then their addition vector x(t)+y(t) must belong to the same set S, correct because even this x(t)+y(t) signal will satisfy this particular criteria, right. So if x(t) belongs to S y(t) belongs to S the sum will also belong to S, more over what would be the null vector here, null vector will be 0(t) which is defined as 0 for all time t such that time t is between t_1 and t_2, right.

So if in graphical ways if I call this as my time access then this would be the 0 vector, this for example would be 1 vector say x(t), okay this would be another vector y(t), okay so this could be y(t) this would be the 0 vector and this fellow will be the signal x(t) of course this particular set S consists of all such vector that is it is actually infinite number of such signals all those signals satisfying this particular criteria and if I want to specify this as a point, right I have drawn a graph here but how many points I have to specify how many samples of x(t) I have to specify I should actually specify an infinite number of values, right.

Because t is varying continuously over t_1 and t_2 there are an infinite number of points, so each vector or each signal which is now a vector, right will be specified by an infinite number of points and there are of course an infinite number of such vectors it is possible for us to also considered of finite set of such signals okay not an infinite set you can just considered a finite set of signals but in each of that finite signal of that finite set that we are considering each of those signals itself will be infinite dimensional because you have to put an infinite number of points okay to fully specify that x(t).

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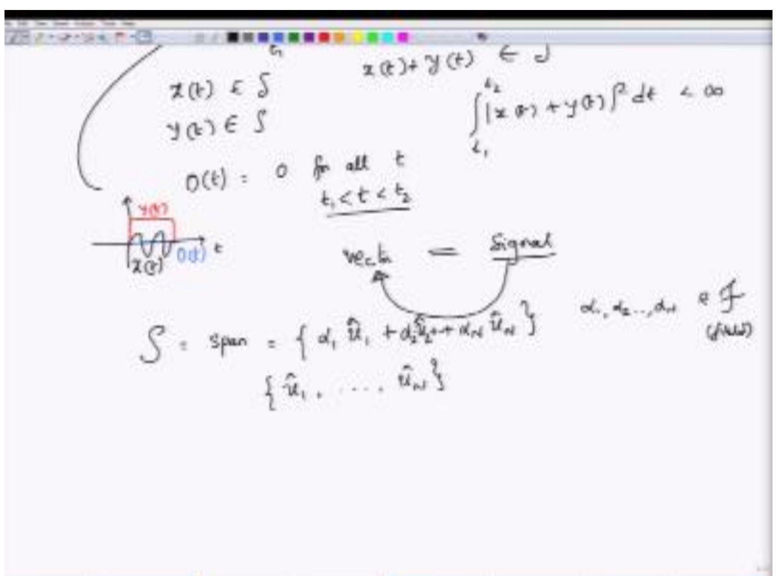

So we have kind of made jump from vector which is a physical quantity and then said vector is nothing but signal or rather which should, way the other way round we said that signals can be considered as vectors okay this is the base connection that we wanted to establish. now let us consider a set of vectors okay let us call the set of vectors as s and we specify this or we give this name of the set s as span and what is this? This is basically any linear combination of the set of vectors okay.

There are n vectors her which I have considered and then any linear combination or rather all vectors which are linear combination of these vectors u1 through u_n okay you have to consider the set u1 through u_n there are n such vectors here and if I take each of these vectors and multiplied by α_1 α_2 α_3 that u_1 multiplied by α_1 α_2 u_2 and so on right up to α_n u_n for α_1 α_2 α_n all coming from some space F okay all coming from some number system F in vector space theory this is actually called as a the field, okay.

Something that we do not really want to bother about but if I take a set of vectors u1 through u_n and then form linear combinations that is I have multiply each of those vectors physically what is happening is I have taken this vectors and I am basically expanding shrinking each of those vectors and then adding them all together, okay. So I shrink and then I expand some vector I shrink some vector I reverse the direction by multiplying it by -1 reverse it and then shrink it and then ad all these vectors to form the composite vector.

And now any such vector right which can be expressed in this passion will be contain in this bag which we are calling as span we say that all those vectors okay which satisfy or which are build up as the linear combination of this so id I call this vector as V were in each of those vectors v is given by a linear combination of these vectors for different value of u1 through $u_{n.}$

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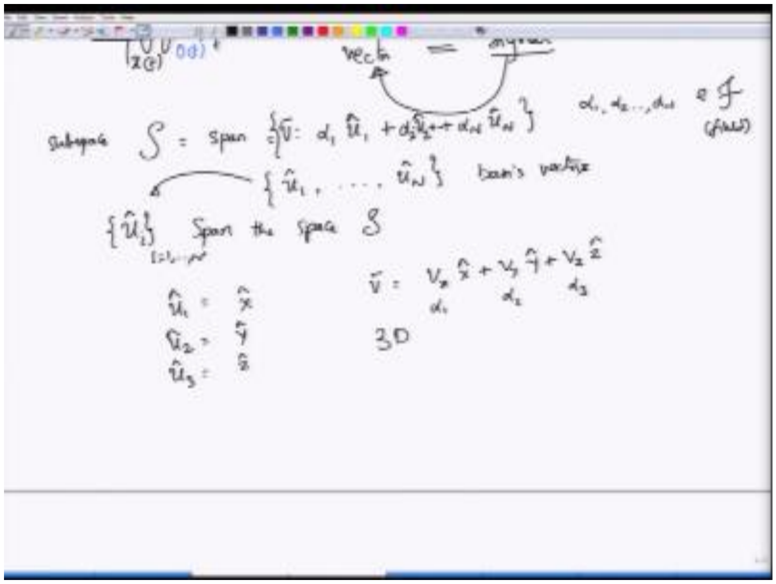

So the set of all though vectors we which satisfy with this particular condition or which have built up as linear combination of vectors u1 through u_n are set to be span or set or belong to the span and we say that u1 through u_n span the space s okay we say that vectors u1 through u_n span s so another convenient short hand notation for me would be to write this has u_i and then say I is equal to 1 to end okay.

So the set of vectors u1 through u_n span the space s or span what is called as the sub space okay this is normally called as the sub space okay so this span the sub space s if there is no confusion I will be using sub space but for out course sub space is actually equal to s space we do not want to distinguish anything there are technically small differential but we do not want to distinguish any of them at this point.

Okay so these vectors u1 span the space s or this vectors ui span the sub space s poor statements mean the same to me, these vectors which are spanning the space s are called as basis vectors okay simple example of the would be to go back to our citation coordinate system take identify u1 with x u2 with y and then u3 with z vectors and any vector v can be written as $v_x\ \hat{x}+v_y\ \hat{y}+v_z\ \hat{z}$.

Here is v_x is α_1 v_y is α_2 and v_z is α_3 okay and a collection of such vector V forms the 3 dimensional coordinate space or 3 dimensional space right the rectangular 3 dimensional space is formed by or this is span by this basis vector $\hat{x}$ $\hat{y}$ and $\hat{z}$ become the basis vectors because any other vector can be built up on the basis of these vectors. Thank you.

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