Principles of Modern CDMA/MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 09
Exact BER Expression for Rayleigh Fading Wireless Channel

Hello, welcome to another module in this massive open online course Principles of CDMA, MIMO, and OFDM Wireless Communication Systems. And what we are going to do today in this module is to basically go through the evaluation of the exact bit error rate expression for wireless communication system.

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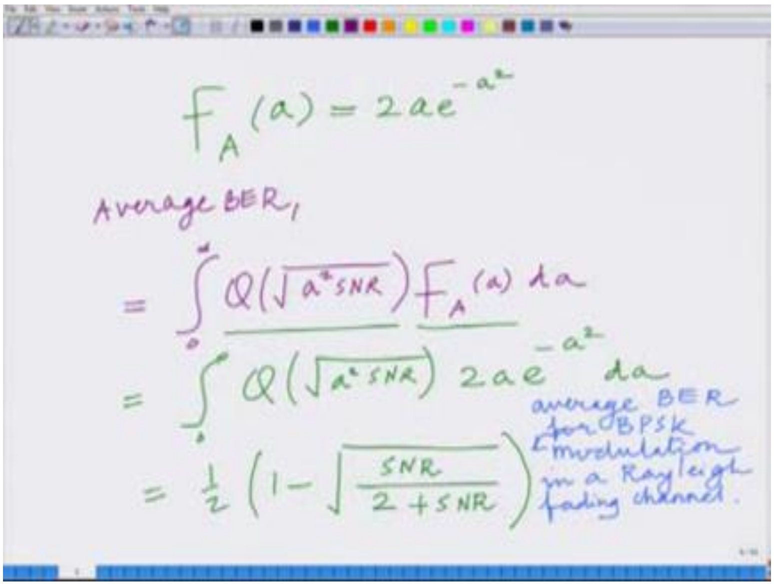

Remember, in the previous module we are stop short of the complete derivation what we said was, we have to evaluate the average this bit error rate expression $Q(\sqrt{a^2 SNR})$ with respect to the distribution of the amplitude of the fading channel coefficient a that is given by the Rayleigh density

$$F_A(a) = 2a\, e^{-a^2}$$

So, this integral that is average bit error rate this and we have gone through the derivation is to derive the final expression in the previous module. So, that is something that we are going through in the module today that is to evaluate the exact expression for the bit error rate.

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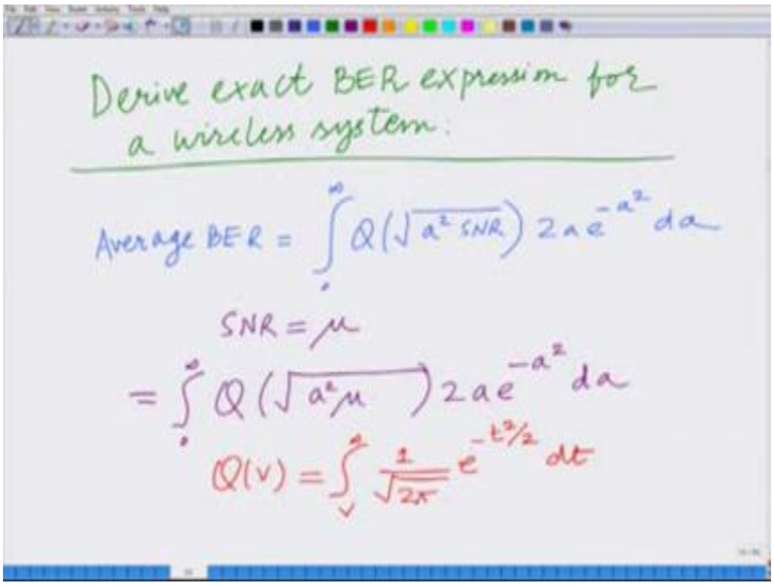

So, we are going to derive exact bit error rate expression for a wireless communication system. And we already seen that the average bit error rate is given as,

$$\text{Average BER} = \int_0^{\infty} Q\left(\sqrt{a^2 SNR}\right) 2a\, e^{-a^2}\, da$$

Now, what we are going to do is we are going to substitute this SNR by the symbol μ and therefore, I can write this quantity as

$$\text{BER} = \int_0^{\infty} Q\left(\sqrt{a^2 \mu}\right) 2a\, e^{-a^2}\, da$$

we know that the expression for the Q function we know

$$Q(v) = \int_v^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

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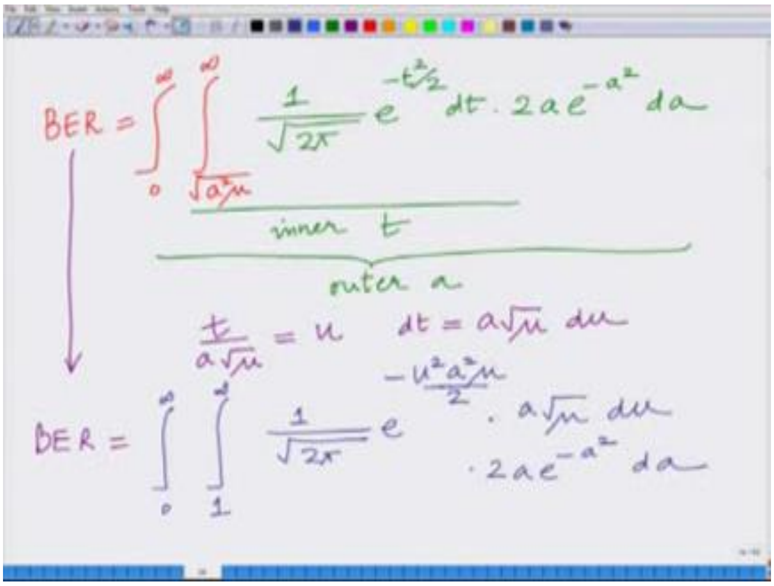

If I substitute that Q function here. I will get the bit error rate equals

$$\text{BER} = \int_0^{\infty} \int_{\sqrt{a^2\mu}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \; 2a \, e^{-a^2} \, da$$

Where this inner integral is with respect to t and the outer integral is with respect to with the quantity a right. So, this is a double integral the inner integral is with respect to the quantity t outer integral is respect to amplitude to the fading channel coefficient a.

Now, I am going to the substitution $\frac{t}{a\sqrt{\mu}} = u$ which also gives me $dt = a\sqrt{\mu}\, du$ therefore, substituting this my bit error rate expression now reduces to

$$\text{BER} = \int_0^{\infty} \int_1^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2 a^2 \mu}{2}} \, a\sqrt{\mu} \, d\mu \; 2a \, e^{-a^2} \, da$$

So, now I have this modification using the substitution $\frac{t}{a\sqrt{\mu}} = u$, I have this modified double integral.

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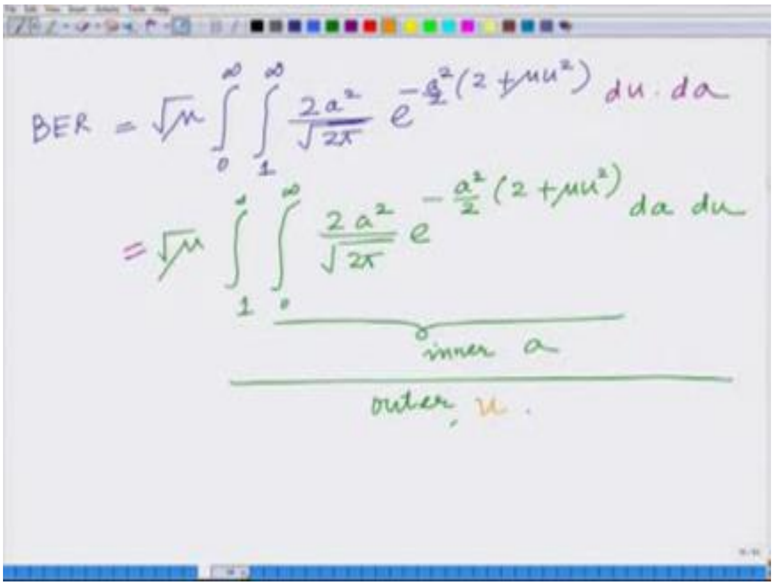

Now I am going to have 2 terms with minus with a square in the exponent.

$$\text{BER} = \sqrt{\mu}\int_0^\infty \int_1^\infty \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+u^2\mu)}\, du \,.\, da$$

$$= \sqrt{\mu}\int_1^\infty \int_0^\infty \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+u^2\mu)}\, da \,.\, du$$

Therefore, I have the inner integral with respect to a and I have the outer integral with respect to u right.

So, I have double integral I have changed the order of the integral, reverse the order of the integration. So, I have the inner integral with respect to a and outer integral with respect to u. And now what I am going to do is I am going to first evaluate the inner integral which is basically $\int_1^\infty \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+u^2\mu)}$.

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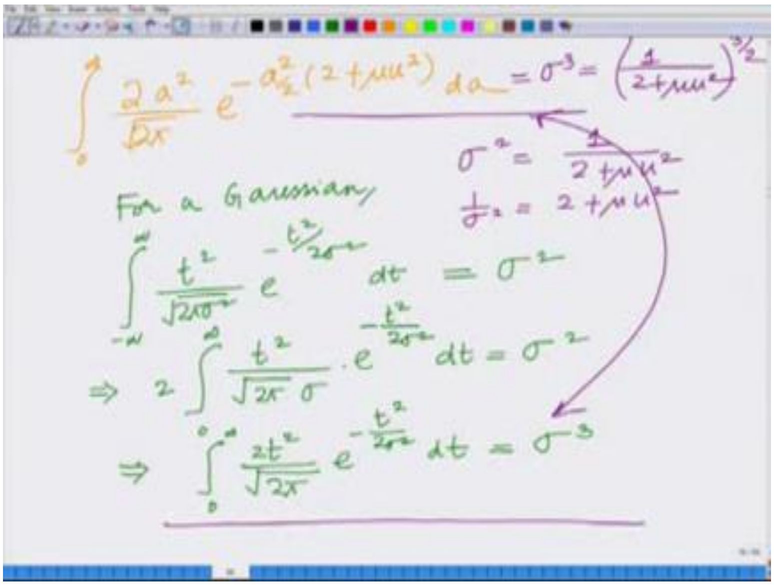

Now, if you look at this, let us look at the following result for Gaussian we have the following result. For a Gaussian distribution we have

$$\int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}\, dt = \sigma^2$$

Now, this quantity this is simply an evaluation of the variance of the Gaussian distribution of the 0 mean, Gaussian random variable with mean 0 and variance σ^2. So, therefore, this is equal to σ^2 and now you look at this is integral involves t^2. So, it is symmetric for $-t$ and that is for negative values and positive values. So, this implies

$$2 \times \int_{0}^{\infty} \frac{t^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}\, dt = \sigma^2$$

$$\int_{0}^{\infty} \frac{2t^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}\, \sigma\, dt = \sigma^3 = \int_{1}^{\infty} \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+u^2\mu)} = \left(\frac{1}{2+u^2\mu}\right)^{3/2}$$

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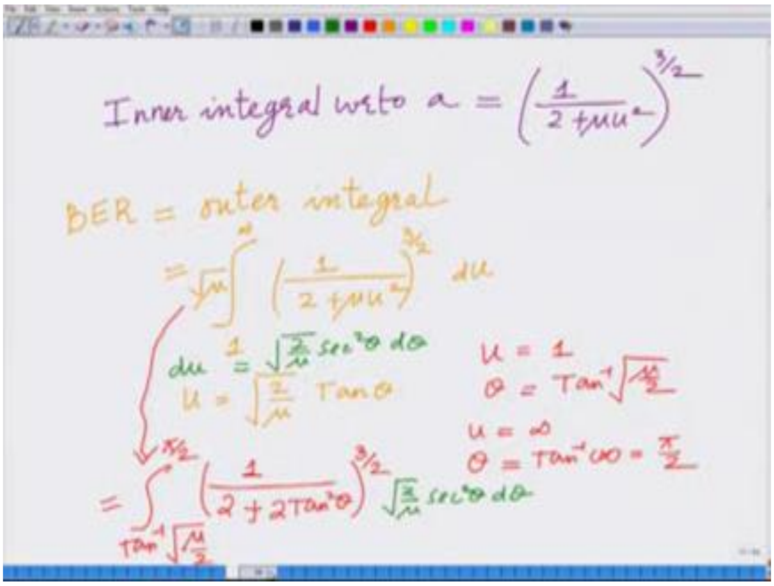

So, therefore, inner integral with respect to a is equal to $\left(\frac{1}{2+u^2\mu}\right)^{3/2}$ that is inner integral with respect to a.

Now, let us evaluate the outer integral with respect to u. So, now, I have bit error rate is given by the outer integral, which is

$$= \sqrt{\mu}\int_1^{\infty}\left(\frac{1}{2+u^2\mu}\right)^{3/2} . du$$

$$u = \sqrt{\frac{2}{\mu}}\tan\theta$$

Now, therefore, when u = 1 , I have $\theta = \tan^{-1}\frac{\mu}{2}$;

when $u = \infty$, $\theta = \tan^{-1}\infty = \frac{\pi}{2}$.

So, the limits of integral become

$$= \sqrt{\mu}\int_{\tan^{-1}\frac{\mu}{2}}^{\frac{\pi}{2}}\left(\frac{1}{2+\tan^2\theta}\right)^{3/2} . \sqrt{\frac{2}{\mu}}\sec^2\theta\, d\theta$$

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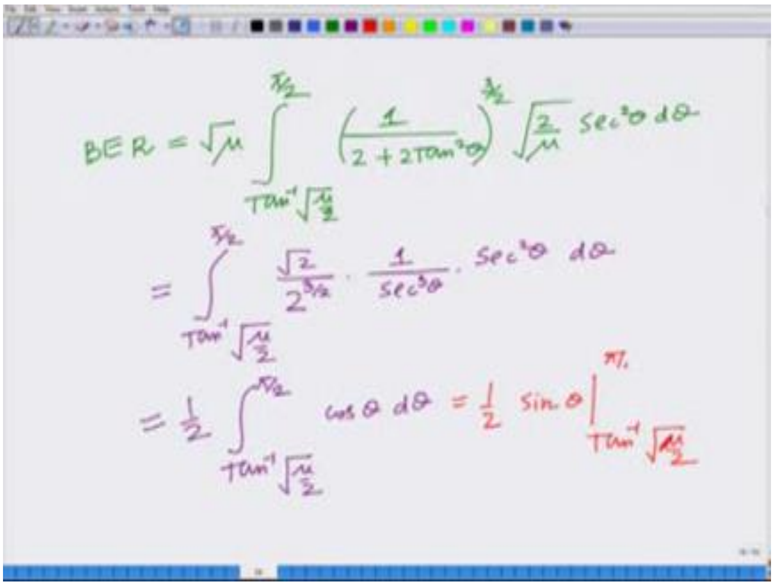

And now when I simplify this integral I have

$$= \int_{tan^{-1}\frac{\mu}{2}}^{\frac{\pi}{2}} \frac{\sqrt{2}}{2^{\frac{3}{2}}} \frac{1}{sec^3\theta} \; . \; sec^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{tan^{-1}\frac{\mu}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta$$

$$= \frac{1}{2} \; sin\theta \, |_{tan^{-1}\frac{\mu}{2}}^{\frac{\pi}{2}}$$

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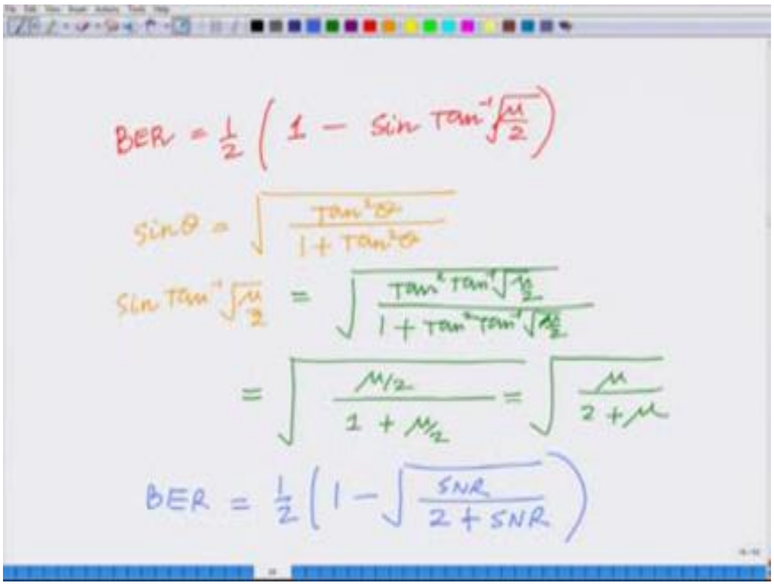

Now, we are going to use the identity, we are going to use trigonometric identity that is

$$\text{BER} = \frac{1}{2}\left(1 - \sin tan^{-1} \frac{\mu}{2}\right)$$

$$\sin \theta = \sqrt{\frac{tan^2\theta}{1 + tan^2\theta}}$$

$$\sin tan^{-1} \frac{\mu}{2} = \sqrt{\frac{tan^2 tan^{-1} \frac{\mu}{2}}{1 + tan^2 tan^{-1} \frac{\mu}{2}}}$$

$$= \sqrt{\frac{\frac{\mu}{2}}{1 + \frac{\mu}{2}}}$$

$$= \sqrt{\frac{\mu}{2 + \mu}}$$

$$\text{BER} = \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{2 + \text{SNR}}}\right)$$

Therefore, what we have done is now we have evaluated this integral, we have gone through the detailed derivation of this bit error rate expression to derive the relation that the bit error rate in a wireless communications system. The average bit error rate which is average with respect to the distribution of the amplitude of the fading channel coefficient a is given as $\frac{1}{2}\left(1-\sqrt{\frac{\mathrm{SNR}}{2+\mathrm{SNR}}}\right)$.

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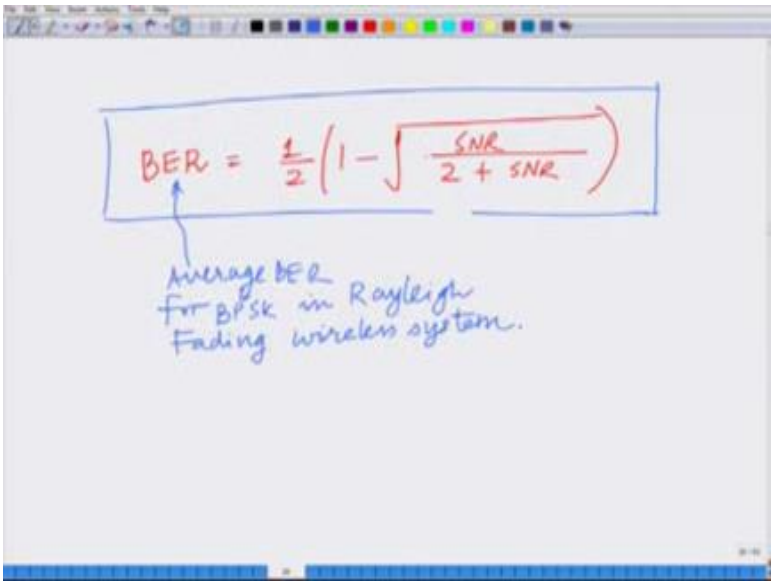

Therefore, our bit error rate or bit error rate in the wireless communication system is

$$\mathrm{BER} = \frac{1}{2}\left(1-\sqrt{\frac{\mathrm{SNR}}{2+\mathrm{SNR}}}\right)$$

This is the expression for the bit error rate, this is the expression for the average bit error rate; mind you this is the average bit error rate for BPSK transmission in Rayleigh fading. The nature of the fading is also important, Rayleigh fading wireless this is the average bit error rate. This is the formula of average bit error rate for BPSK modulation in a Rayleigh fading wireless channel and what we have done is we have derived the exact expression for this bit error rate.

So, this module, we will stop this module here and we will continue it other topics in the subsequent module.

Thank you very much.