Principles of Modern CDMA/MIMO/OFDM Wireless Communications
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Lecture – 07
Bit Error Rate (BER) of AWGN Channels

Hello welcome to another module in this massive open online course on principles of MIMO CDMA and OFDM Wireless Communication Systems and as we have seen in previous module, what we are looking at the performance analysis for a basic wire line communication system which can be can be modelled as an additive white Gaussian noise channel.

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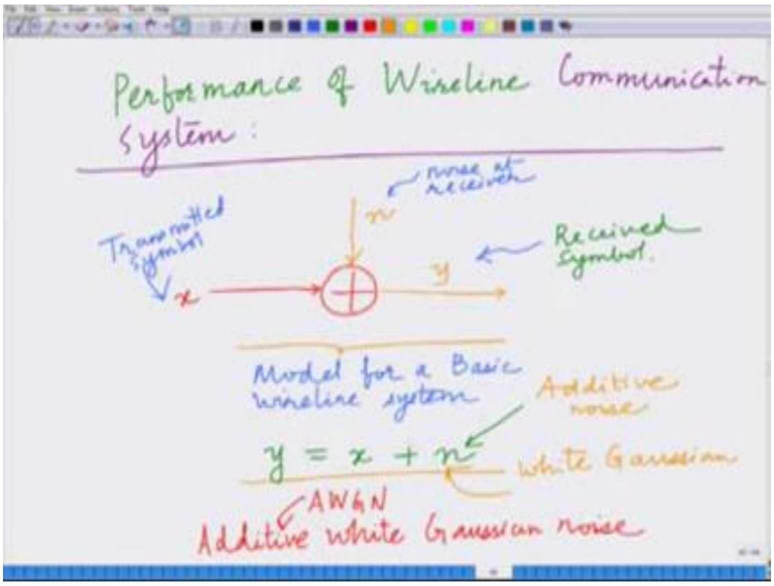

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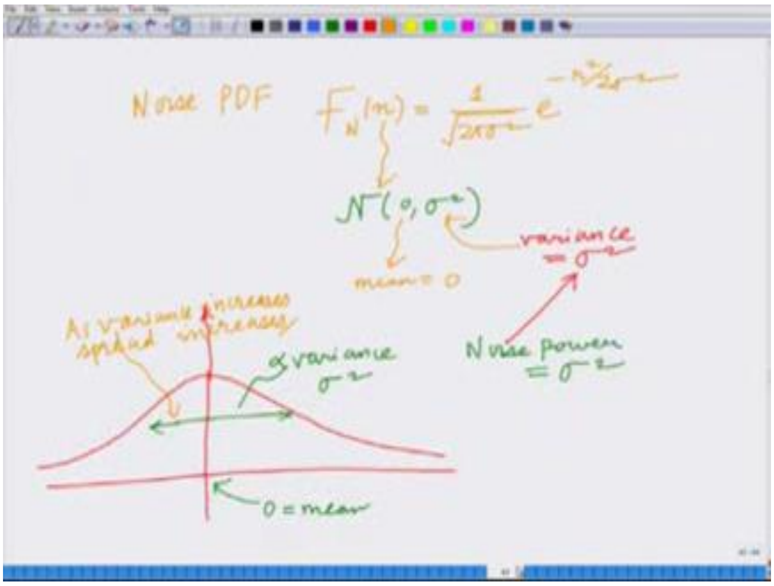

That is $y = x + n$

where x is the transmitted symbol and y is the received symbol the noise n is additive and white Gaussian in nature, with the following probability density function that is $F_N(n)$ is:

$$F_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}$$

where σ^2 is the variance or the noise power and this noise has a 0 mean. We are also considering BPSK modulated symbols where 0 is mapped to the voltage level $\sqrt{P}$ and 1 is mapped to the voltage level $-\sqrt{P}$.

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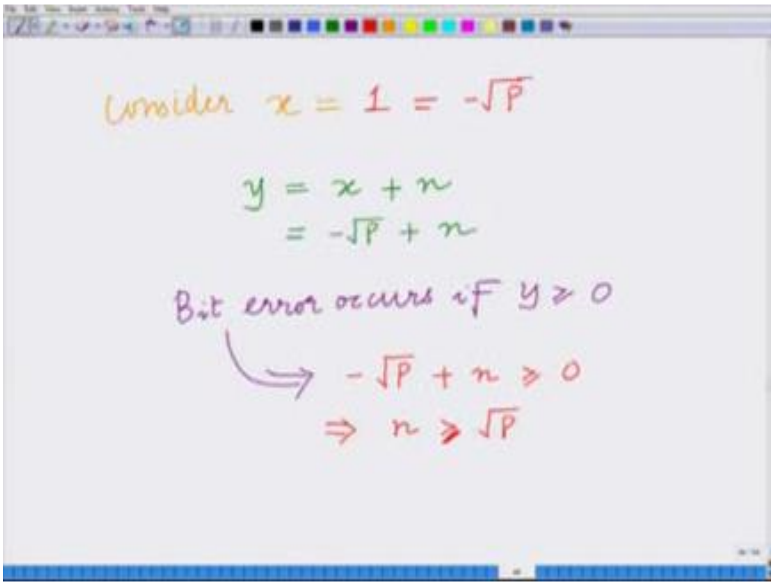

Let us now consider that transmission of the information symbol 1. So, now, let us say consider x, that is transmitted symbol x equals information symbol 1, which is corresponds to the voltage level $-\sqrt{P}$. So, let us have a transmitted symbol is 1, corresponding to the voltage level $-\sqrt{P}$, at the receiver we have the received symbol y which is equal to x plus n. Since x is equal to $-\sqrt{P}$ in this case this is

$$y = -\sqrt{P} + n$$

and as we had seen before since we are transmitting the voltage symbol 1.

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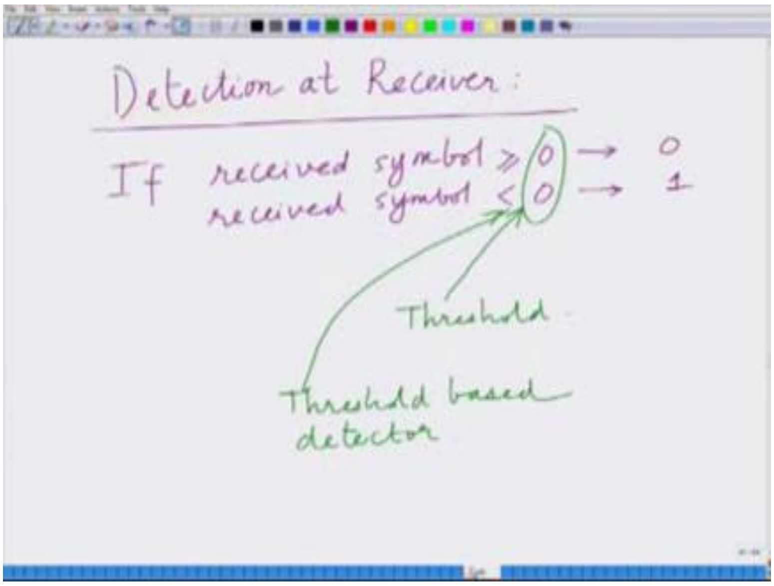

Therefore, there will be an error is the received symbol is decoded as an information symbol 0 and that happens when the received symbol is greater than or equal to 0. So, Error occurs or a bit error occurs; if y is greater than or equal to 0. Because if y is greater than or equal to 0 it is mapped to the information symbol 0; however, we are transmitting the information symbol 1 therefore, this corresponds to the bit error rate 0, bit error rate if y is greater than equal to 0. And y is greater than equal to 0, this implies naturally

$$-\sqrt{P} + n \geq 0$$

$$\Rightarrow \quad n \geq \sqrt{P}$$

That is the Gaussian noise n, the additive white Gaussian noise n is greater than or equal to $\sqrt{P}$.

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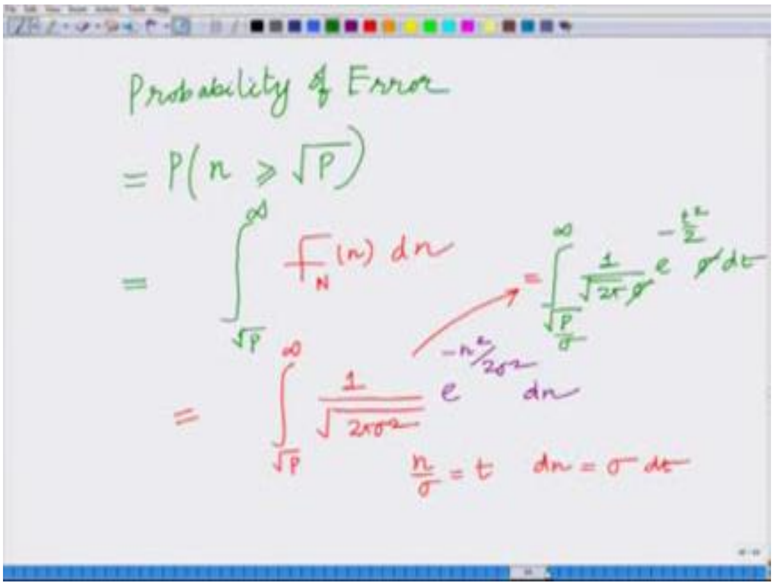

And therefore, naturally

Probability of error, $= P(n \geq \sqrt{P})$

$$= \int_{\sqrt{P}}^{\infty} F_N(n)\, dn$$

Now, what I am going to do is, I am going to substitute the probability density function of the noise. We know that the noise follows the Gaussian probability density function; which we had mentioned earlier. So, this is

$$= \int_{\sqrt{P}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}\, dn$$

And now what I am going to do is I am going to use this substitution

$$\frac{n}{\sigma} = t$$

which means $dn = \sigma\, dt$ and therefore, this integral can be written as

$$= \int_{\sqrt{P}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2}}\, \sigma\, dt$$

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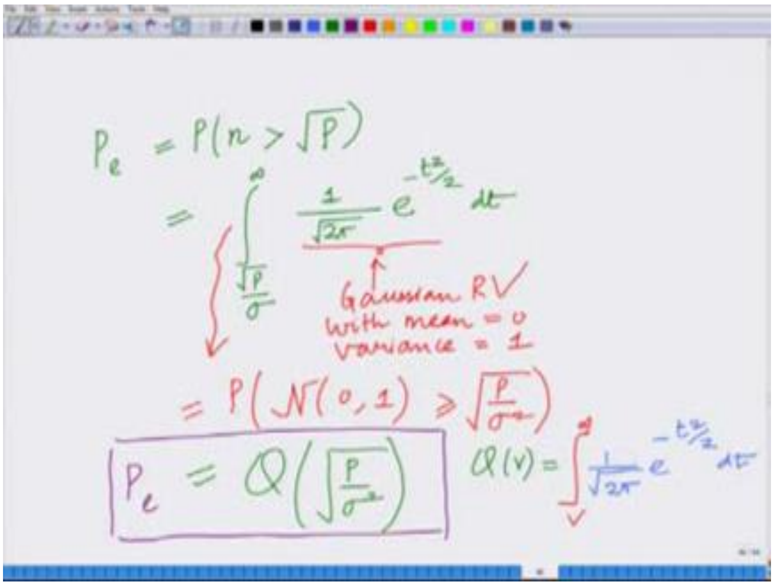

That is the probability of error equals to probability, that this noise $n \geq \sqrt{P}$, this is equal to the integral as we have seen before $\int_{\sqrt{P}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2}} \sigma \, dt$. And now you can easily see that this expression corresponds to a Gaussian distribution of a Gaussian random variable with mean 0 and variance 1 right. This corresponds to a Gaussian random variable this corresponds to a Gaussian RV with mean equal to 0 and variance equal to 1.

Therefore, what we are asking is what is the probability that this Gaussian random variable with mean 0 and variance 1 is greater than or equal to this threshold P over σ^2 and this is basically given as

$$P_e = Q\left(\sqrt{P}/\sigma\right) = Q\left(\sqrt{\frac{P}{\sigma^2}}\right) = \int_{\sqrt{P}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2}} \sigma \, dt$$

this quantity is the Gaussian random variable with mean 0 and variance 1 and hence this function is the Q function and therefore, the probability of bit error is given as

$$Q\left(\sqrt{\frac{P}{\sigma^2}}\right) = \int_{\sqrt{P}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2}} \sigma \, dt$$

So, the bit error rate of the wire line communication system that is without multi path proposition with only additive white Gaussian noise without any fading is given as the probability of bit error is equal to $Q\left(\sqrt{\frac{P}{\sigma^2}}\right)$.

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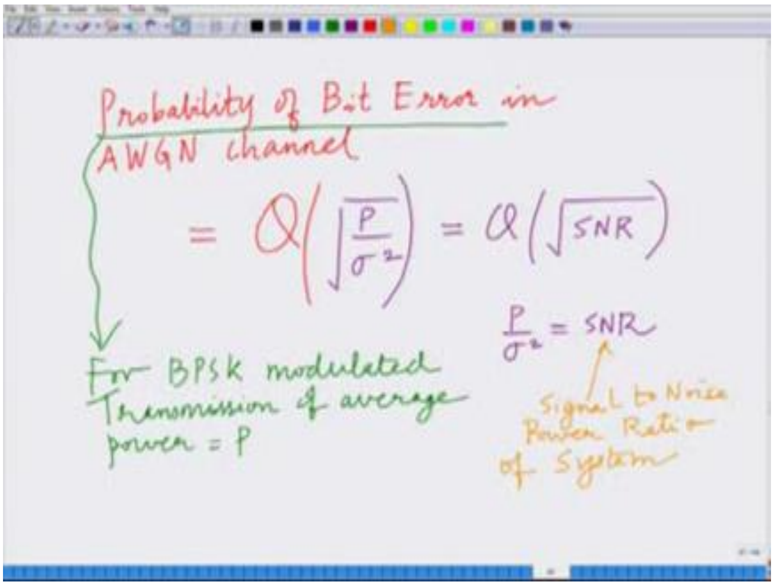

The probability, let me just write this down probability of bit error in an AWGN channel equals $Q\left(\sqrt{\frac{P}{\sigma^2}}\right)$. And this is the probability of bit error for a BPSK modulated system the probability of that error for BPSK modulated transmission for BPSK modulated transmission with average power P, further you can also see as $\frac{P}{\sigma^2}$ equals the SNR and therefore, this bit error rate is also

$$Q\left(\sqrt{\frac{P}{\sigma^2}}\right) = Q\left(\sqrt{SNR}\right)$$

Further you can see since P is the transmitted power σ^2 is the noise power, $\frac{P}{\sigma^2}$ is the SNR or Signal to Noise Power Ratio of the communication system. So, we can say

$\frac{P}{\sigma^2}$ equals the SNR where SNR denotes Signal to Noise Power Ratio of the communication system therefore, we have an elegant expression for the bit error rate in terms of the SNR. Simply says the bit error rate for BPSK modulated transmission over an additive White Gaussian noise channel is simply $Q(\sqrt{SNR})$ that is the Q function of the $\sqrt{SNR}$ Signal to Noise Power Ratio of the communication system.

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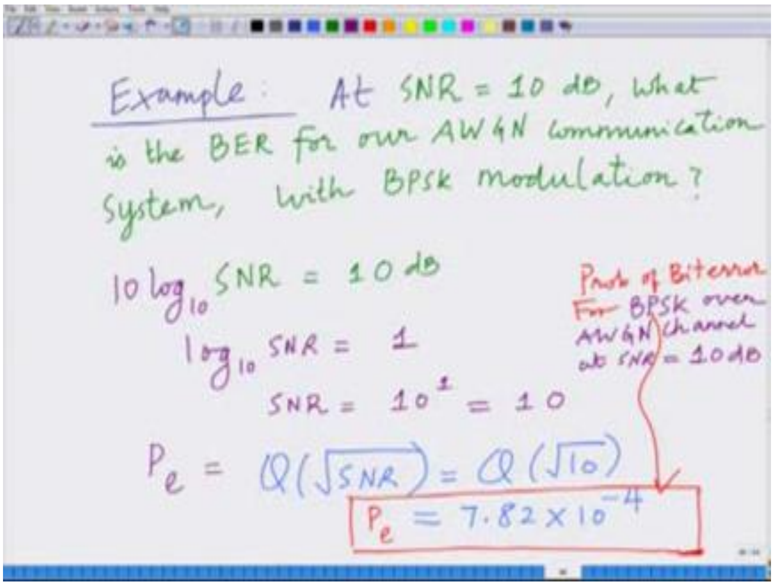

Let us do a couple of examples to understand this better. The first example let us do a couple of examples. So, in the first example what we want to do is we want to compute at SNR equals 10 dB. We want to ask the question at SNR at a Signal to Noise Power Ratio of 10 dB. What is the bit error rate? What is the BER for our AWGN communication system with BPSK modulation system and what we have seen is at SNR equals 10 dB. Which means that our 10 log 10, this is the dB SNR therefore,

$$10 \log_{10} SNR = 10 \text{ dB}$$

$$\log_{10} SNR = 1$$

$$SNR = 10$$

$$P_e = Q(\sqrt{SNR}) = Q(\sqrt{10}) = 7.82 \text{ X } 10^{-4}$$

There is no close form equation to compute the Q function.

So, I have to compute it from the tables and this value is equal to 7.82×10^{-4}. This is the probability of bit error, what is this is the probability of bit error of BPSK modulated transmission over an AWGN channel at SNR of 10 dB. So, this is probability of bit error for BPSK over AWGN channel at SNR equals 10 dB. As we have said there is no close formed equation for this Q function correct. Therefore, this has to be evaluated using some software or online tables that are available for this Q function and this gives us the value of 7.82×10^{-4} which is the bit error rate.

Another convenient way to evaluate this is if you are not using for looking for an accurate or the exact to evaluate the exact value or if we are rather looking for a convenient approximation for this Q function, then we can use the approximation Q(x).

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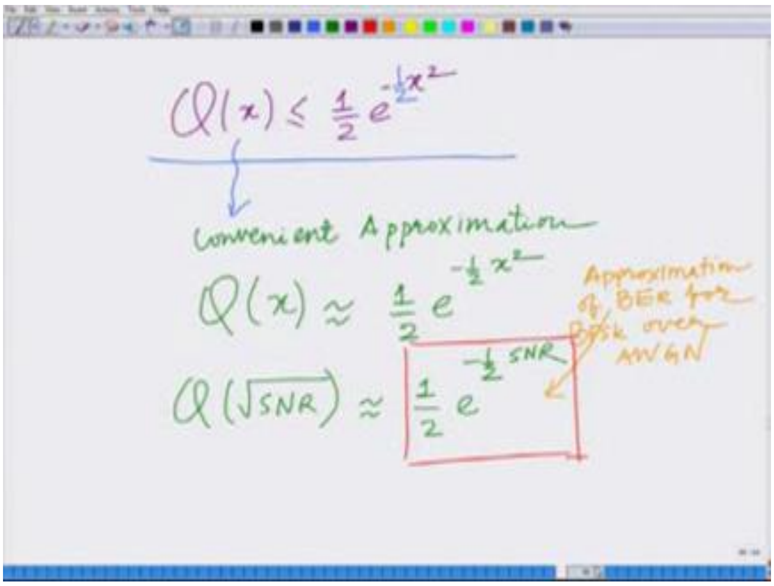

$$Q(x) \leq \frac{1}{2} e^{-\frac{1}{2}x^2}$$

This bound I can use it as a convenient approximation we can use it as a convenient approximation for the Q function.

And therefore, what we will have is we will use frequently approximation Q(x) is :

$Q(x) \approx \frac{1}{2} e^{-\frac{1}{2}x^2}$

$Q(\sqrt{SNR}) = \frac{1}{2} e^{-\frac{1}{2}SNR}$

So, our approximate bit error rate for BPSK modulated transmission over the AWGN channel is $\frac{1}{2} e^{-\frac{1}{2}SNR}$. This is our approximation of BER for BPSK over the AWGN channel. That is $\frac{1}{2} e^{-\frac{1}{2}SNR}$. So, we can use this approximation and using this approximation we can re calculate the SNR.

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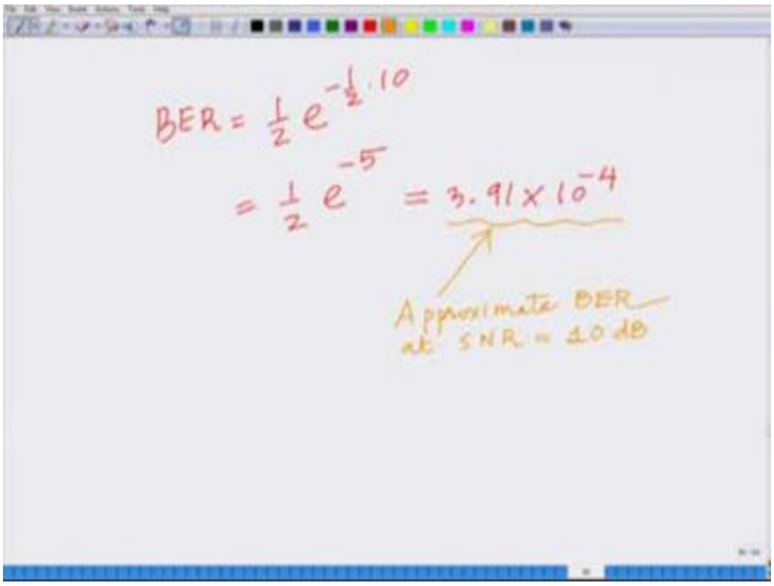

Bit error rate for the previous example as

$BER = \frac{1}{2} e^{-\frac{1}{2}.10}$

$= \frac{1}{2} e^{-5} = 3.91 \text{ X } 10^{-4}$

This is the approximate bit error rate at 10 dB SNR. Which we have and you can see this is fairly close that is 3.91×10^{-4} is fairly close to the exact value that we calculated which is 7.82×10^{-4}. This is the approximate bit error rate at SNR equals 10 dB.

We can look at another example in which we are asking the alternative kind of question. We are asking it in the reverse form that is given a certain bit error rate 1 can ask what is the minimum SNR what is the SNR required for transmission.

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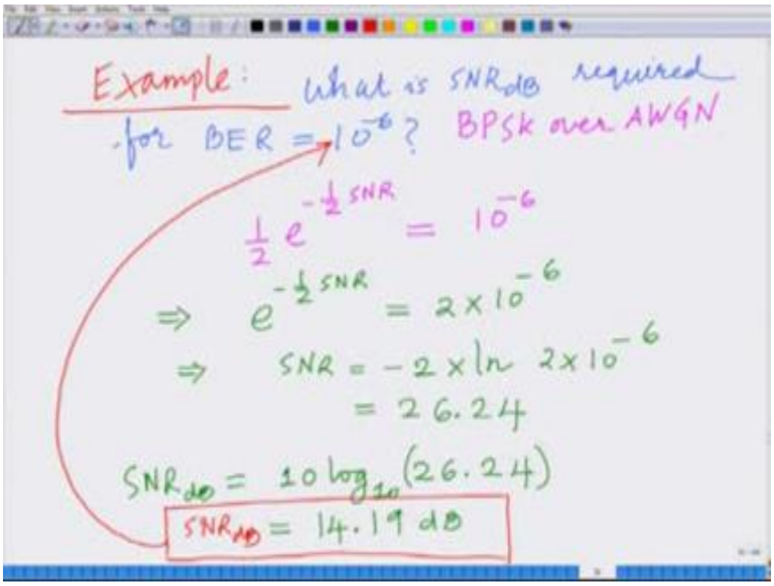

So, let us look at that example also which. So, another example is the following thing which is the same question in reverse that is, what is the SNR dB required for bit error rate equals 10^{-6} again for BPSK over AWGN channel. This SNR required can be calculated as follows we have $\frac{1}{2}e^{-\frac{1}{2}SNR}$, using our approximation is equal to

$$\frac{1}{2}e^{-\frac{1}{2}SNR} = 10^{-6}$$

$$e^{-\frac{1}{2}SNR} = 2 . 10^{-6}$$

$$SNR = -2 \times \ln 2 . 10^{-6}$$

$$= 26.24$$

$SNR_{dB} = 10 \log_{10} 26.24$

$= 14.19$

This is approximate, this is the SNR required for BPSK transmission over the AWGN channel in order to achieve at bit error rate of 10^{-6}. Our question remember our question was what is the SNR in dB required to achieve bit error rate of 10^{-6} for BPSK transmission over the AWGN communication channel and our answer is that, this SNR is approximately equal to 14.19 dB. These are the 2 example which illustrate, how do we analyse the performance of a of a communication system a digital communication system bit digital modulation that is BPSK Binary Phase Shift Keying over a channel with no fading that is simply additive white Gaussian noise and no fading, that is this correspond to wire line channel where there is no multi path propagation.

There is a single wire path between the transmitter and the receiver and we have seen that the bit error rate for this simplistic system is given by the elegant expression Q function or $Q(\sqrt{SNR})$ and we have seen some examples where we calculated the bit error rate for a given SNR and we also calculated the calculated the SNR required for a given bit error rate. So, this complete with analysis for the performance of an AWGN communication system.

Next we are going to look at the performance analysis for a wireless communication system with fading between the transmitter and the receiver and that is going to show us some interesting results and some interesting insights into the performance of these wireless communication systems. We will end this module here.

Thank you very much.