Principles of Modern CDMA/MIMO/OFDM Wireless Communications
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Lecture – 04
Fading Channel Distribution

Hello. Welcome to another module in this MOOC on Principles of CDMA, MIMO, and OFDM Wireless Communication Systems.

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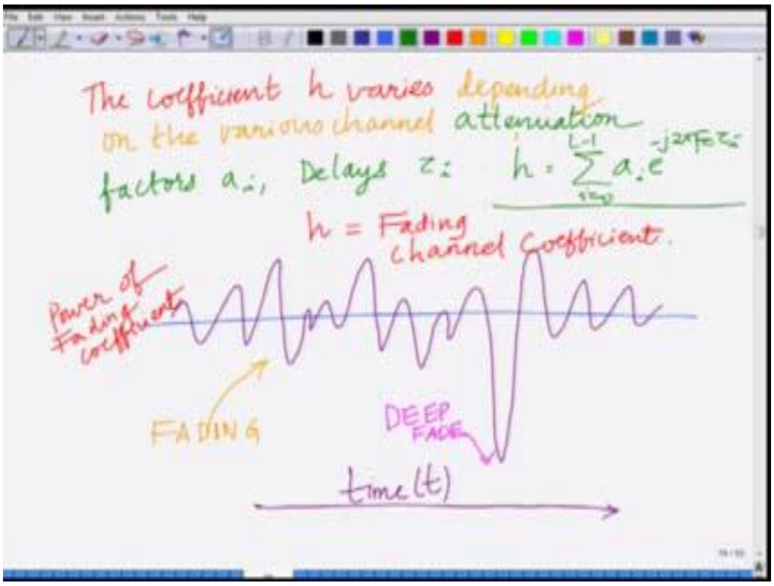

In the last module we had seen this Fading Channel Coefficient h which depends on the attenuations and the delays of the different paths and we said that this Fading Channel Coefficient has an important role to play in the wireless communication system.

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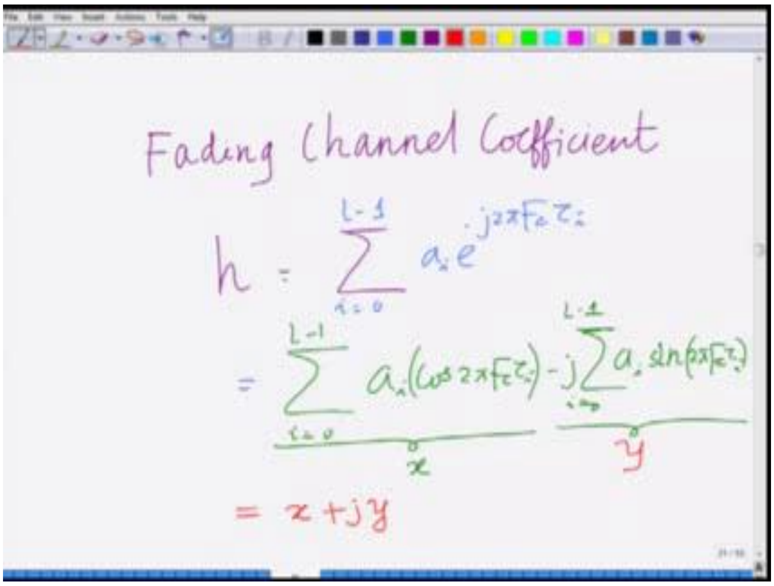

Therefore, today we are going to develop models for this Fading Channel Coefficient. So, our Fading Channel Coefficient h which is given as

$$h = \sum_{i=0}^{L-1} a_i \; e^{-j2\pi f_c \tau_i}$$

This is depends on that attenuations a_i and the delays τ_i. I am now going to expand this as

$$= \sum_{i=0}^{L-1} a_i (\cos 2\pi f_c \tau_i) - j \sum_{i=0}^{L-1} a_i (\sin 2\pi f_c \tau_i)$$

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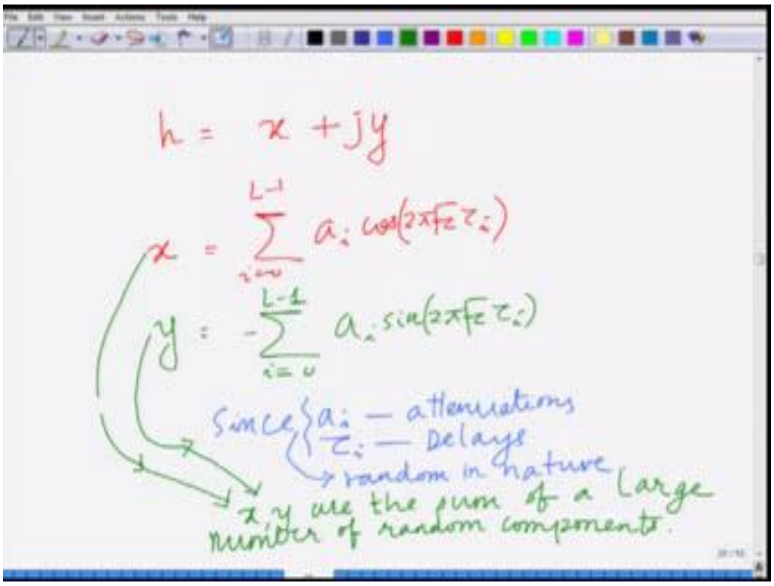

I can write my Fading Channel Coefficient as $h = x + jy$.

where $x = \sum_{i=0}^{L-1} a_i (\cos 2\pi f_c \tau_i)$

and $y = -\sum_{i=0}^{L-1} a_i (\sin 2\pi f_c \tau_i)$

Now you can clearly see both these components x and y are the sums of a large number of random components involving the a_i's which are the attenuations and the τ_i's which are the delays.

Depending on the wireless communication scenario these attenuations a_i and these delays τ_i are random in nature right and when you add a large number of these different random components what results from the central limit theorem is Gaussian random variables. Hence these x and y which are the sum of a large number of random components; so first we realized that since a_i which are the attenuations and τ_i which are the delays these are random in nature. Hence this x, y are the sum of a large number of random components. Hence by the central limit theorem x and y can be assumed to be Gaussian distributed random variables.

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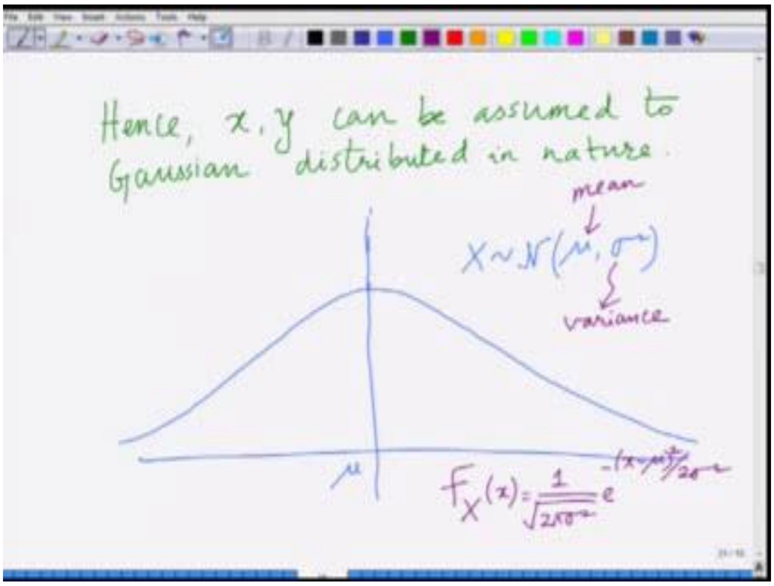

Hence, x y can be assumed to be Gaussian, this has to be Gaussian in nature. So, we are assuming x and y to be Gaussian random variables and just to briefly tell you about a Gaussian random variable; a Gaussian random variable has a PDF which looks as a bell shaped curve which is a probability density function that is a Gaussian random variable which is centred at the mean μ of the Gaussian random variable. So, x is a Gaussian random variable which is denoted as N that is with mean μ and variance σ^2 that is the spread of this Gaussian random variable; that is the width of this bell curve is related to which variance σ^2, the mean is μ and the PDF of this Gaussian random variable this is given as

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This is the PDF of this Gaussian random variable.

Further, we are going to assume that this x and y are independent Gaussian random variables with mean 0 and normalized to variance $\frac{1}{2}$ each. So, x is a Gaussian random variable which is distributed with mean 0 and variance $\frac{1}{2}$.

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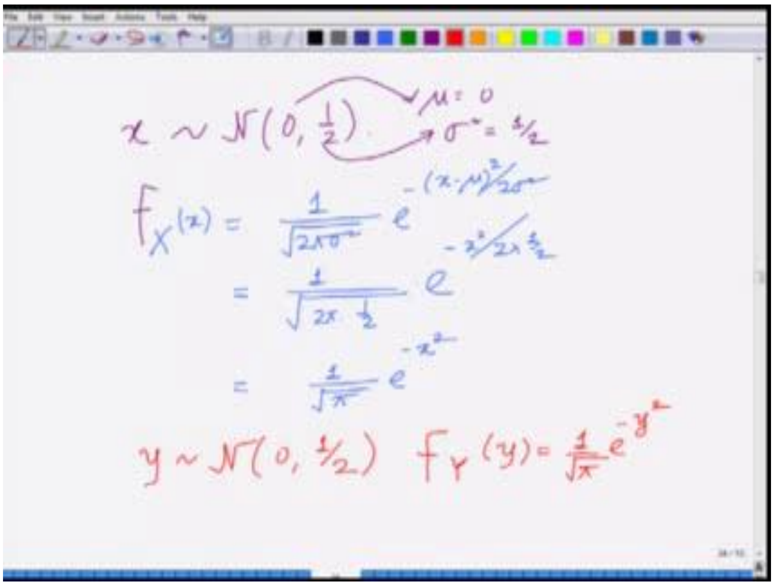

As we had seen therefore, if I substitute in this expression $\mu = 0$ and $\sigma^2 = \frac{1}{2}$; we have

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Substituting $\mu = 0$ and $\sigma^2 = \frac{1}{2}$,

I have , $$F_X(x) = \frac{1}{\sqrt{2\pi\, 1/2}} e^{-\frac{(x)^2}{2 \times 1/2}}$$

$$F_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

This is the distribution of the real component x. Similarly also assuming y to be a Gaussian random variable which is distributed with mean 0 and variance $\frac{1}{2}$; we have

$$F_Y(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}$$

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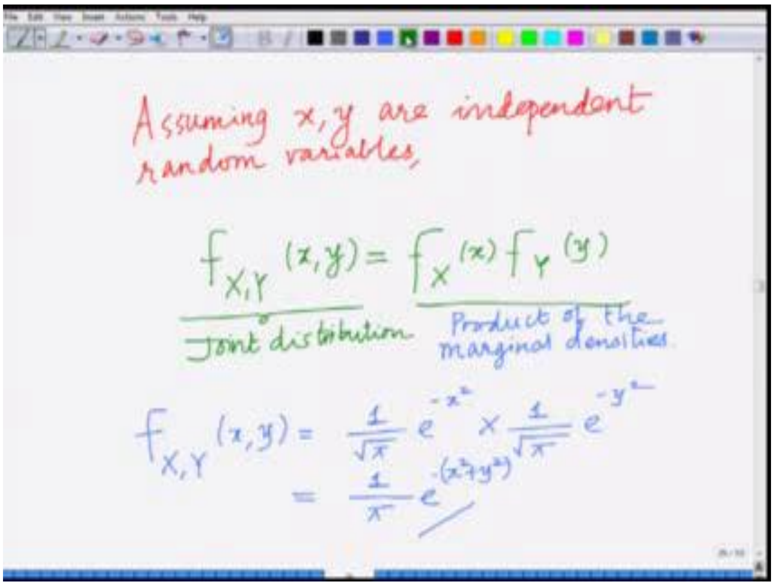

Further assuming that this x and y are independent random variables what we are further going to assume that; assuming x, y are independent random variables. We have the Joint distribution that is $F_{X,Y}(x,y)$, the Joint distribution is given as the product of the individual or the marginal distributions that is

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$$

Therefore, the Joint distribution of the random variables is equal to the product of the marginal densities product of the marginal densities and therefore, we have

$$F_{X,Y}(x,y) = \frac{1}{\sqrt{\pi}} e^{-x^2} \frac{1}{\sqrt{\pi}} e^{-y^2}$$

$$= \frac{1}{\pi} e^{-(x^2+y^2)}$$

This is the Joint distribution of the x and y. So, what we have done is we have characterized the distribution that probability density function of the Fading Channel Coefficient in terms of the real and imaginary parts of the Fading Channel Coefficient. So, we have characterize it is distribution this is one way to characterize the distribution of the Fading Channel Coefficient.

However a more interesting and a more useful way helpful way to characterize the distribution of the Fading Channel Coefficient h is to characterize it rather in terms of the real and imaginary components is to characterize it in terms of the magnitude and phase of the fading channel caution that uses an idea of the power of the received signal.

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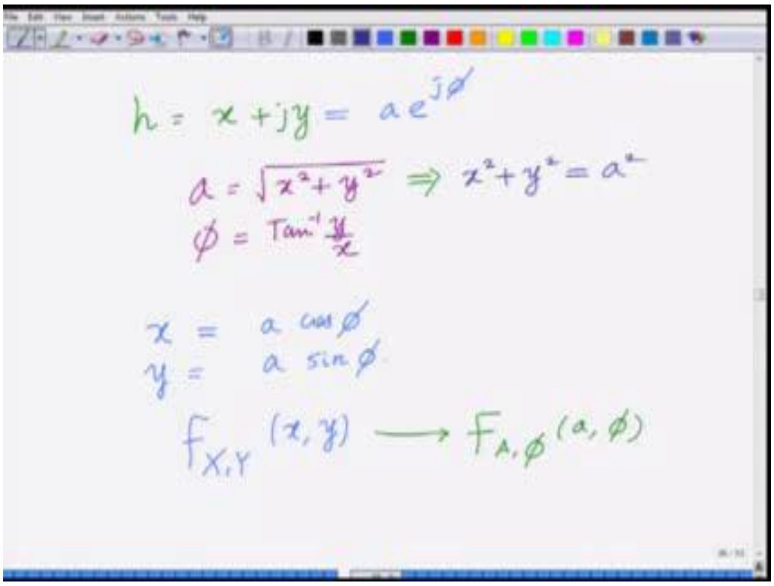

And the phase of the received signal therefore, what we are going to do is we are going to now convert h which we have presented as

$h = x + j\,y = a\,e^{j\phi}$

where a, is the amplitude of the Fading Channel Coefficient and it is equal to

$a = \sqrt{(x^2 + y^2}$ and $\phi = \tan^{-1}(\frac{y}{x})$

Also we can write this the other way around as

$x = a\cos\phi$ and $y = a\sin\phi$

Now, what we want to do we are given the Joint distribution in terms of x and y. So, we have the Joint distribution in terms of the real and imaginary components x, y; what we would like to do is we would like to derive the Joint distribution in terms of the amplitude and phase factors A, ϕ and this can be done as follows.

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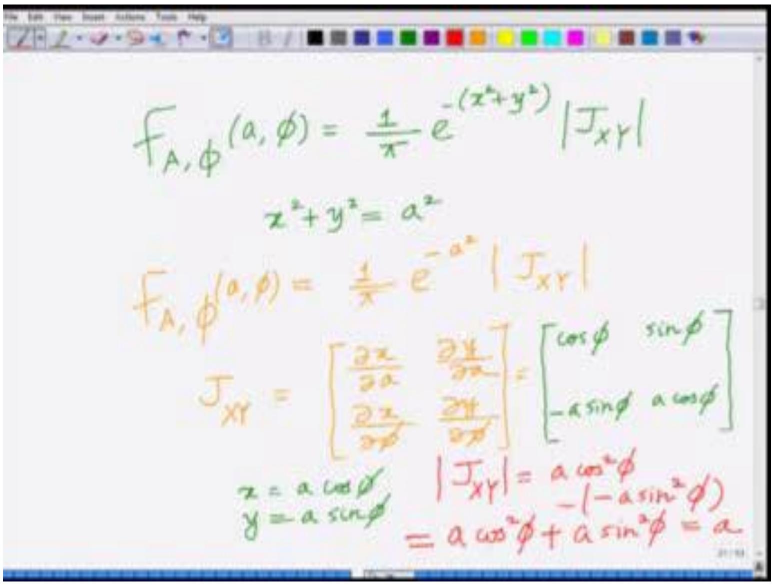

What we have is the following thing; we have

$$F_{A,ø}(a,ø) = \frac{1}{\pi} e^{-(x^2+y^2)} \, | J_{XY} |$$

But now you can see that $x^2 + y^2 = a^2$; therefore, I can write this Joint distribution as

$$F_{A,ø}(a,ø) = \frac{1}{\pi} e^{-a^2} \, | J_{XY} |$$

This Jacobian of X Y can be determined as follows :

$$J_{XY} = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial ø} & \frac{\partial y}{\partial ø} \end{bmatrix}$$

as we have $x = a \cos ø$ and $y = a \sin ø$

and therefore, $J_{XY} = \begin{bmatrix} \cos ø & \sin ø \\ -a \sin ø & a \cos ø \end{bmatrix}$ So, this is the Jacobian matrix the determinant of this equals

$$| J_{XY} | = a(\cos ø)^2 + a(\sin ø)^2 = a$$

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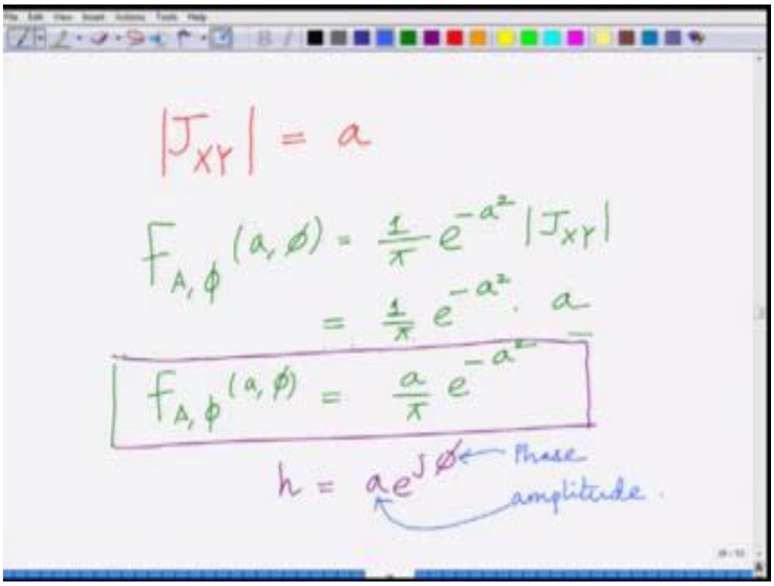

Therefore, we have the determinant of this Jacobian matrix is simply equal to a; and therefore, the Joint distribution with respect to the magnitude and phase components equals

$$F_{A,ø}(a,ø) = \frac{1}{\pi} e^{-a^2} \, |J_{XY}| = \frac{1}{\pi} e^{-a^2} . a$$

$$F_{A,ø}(a,ø) = \frac{a}{\pi} e^{-a^2}$$

This is the Joint distribution of the channel coefficient in terms of the magnitude and phase component. This is the distribution of h or the Joint distribution of A and ø which are the amplitude and phase of the Fading Channel Coefficient h because remember we have

$$h = a\, e^{jø}$$

which means this is the phase and A is the amplitude of this Fading Channel Coefficient and therefore, we have the Joint distribution in terms of the magnitude and phase as

$$F_{A,ø}(a,ø) = \frac{a}{\pi} e^{-a^2}$$

Now what we would like to do at this point is let us try to derive the marginal distributions from the Joint distribution with respect to this magnitude and phase; let us derive the marginal distributions that is let us drive the individual distribution of the amplitude A and the phase that is ø.

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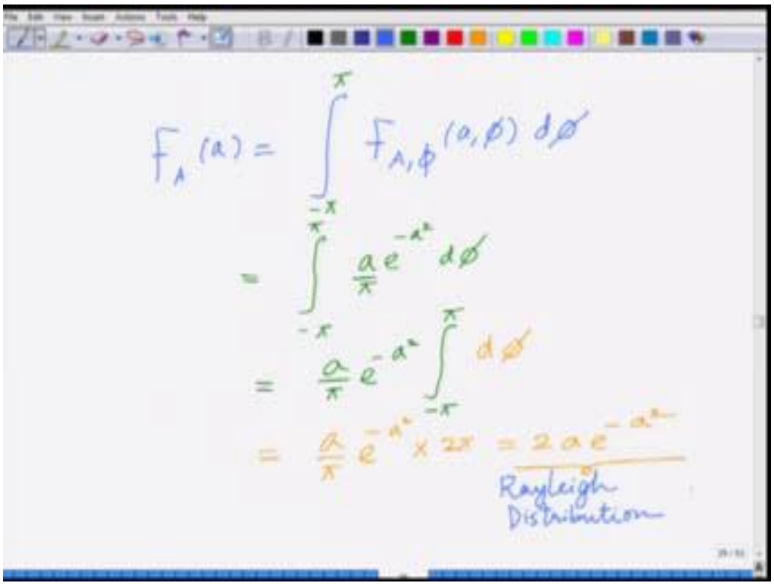

The distribution of the amplitude that is $F_A(a)$; is I have to take the Joint distribution $F_{A,ø}(a,ø)$ and integrate it with respect to the phase that is between and the phase the limit of the phase is between $-\pi$ and π.

$$F_A(a) = \int_{-\pi}^{\pi} F_{A,ø}(a,ø)\, dø$$

$$= \int_{-\pi}^{\pi} \frac{a}{\pi} e^{-a^2}\, dø$$

$$= \frac{a}{\pi} e^{-a^2} \int_{-\pi}^{\pi} dø$$

$$= \frac{a}{\pi} e^{-a^2}\, 2\pi$$

$$= 2a\, e^{-a^2}$$

this is known as Rayleigh fading and this know as the Rayleigh Distribution.

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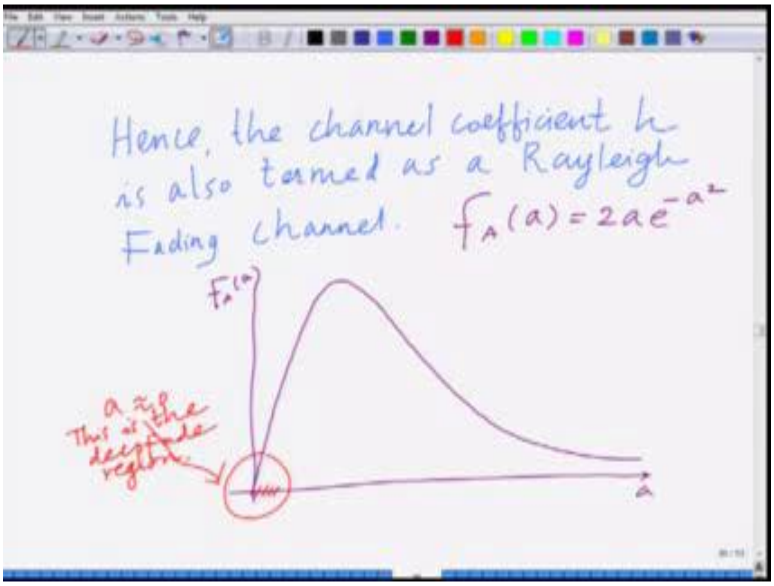

Hence, this channel coefficient h is also known as a Rayleigh Fading Channel. So, it is distribution is given as

$$F_A(a) = 2a\, e^{-a^2}$$

If you can look at this region this region which is close to 0 which corresponds to a; approximately equal to 0 that is where the channel coefficient the gain of the channel coefficient is very small it is close to 0 this is termed as the deep fade.

So, this region where the Fading Channel Coefficient is very small that is the probability this region and the corresponding probability. So, this region corresponds to the deep fade regions where the Fading Channel Coefficient the magnitude of the Fading Channel Coefficient is very small and the corresponding probability is termed as the probability of deep fade. So, the probability that a; is close to 0 which means the probability density integrated over this region is basically termed as the probability of the deep fade.

Remember we had earlier talked about the deep fade as an event where the channel takes a dips to take a very small that is where the different multi path components cancel out each, almost cancel out each other resulting in a very small gain or resulting in a very high attenuation of the received signal that corresponds to the deep fade event and what we are saying here is that basically this region where the channel coefficient. The magnitude of the channel coefficient is very close to 0 correspond to the deep fade event and the corresponding probability is the probability of the deep fade.

So, we will stop this module here and we will continue with the next module.

Thank you very much.