Communication Engineering
Prof. Surendra Prasad
Department of Electrical Engineering
Indian Institute of Technology, Delhi

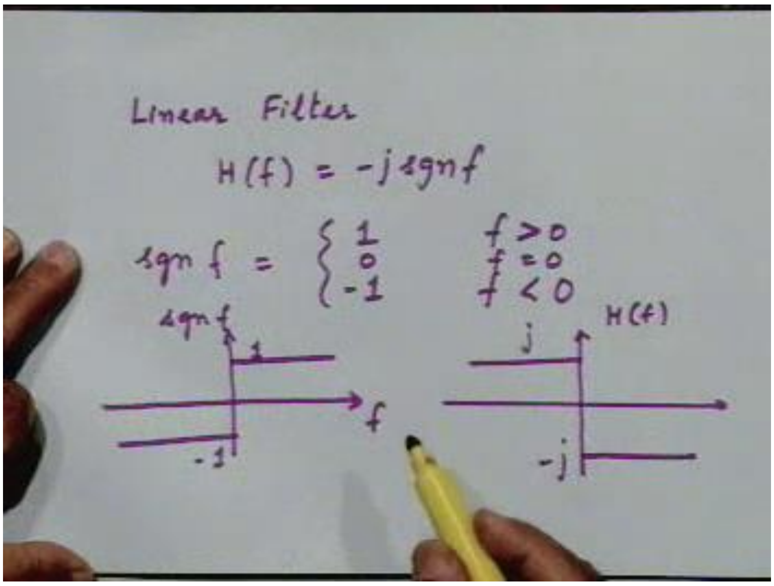
Lecture - 4
The Hilbert Transform

Today, we will discuss the Hilbert transform. Now, unlike the transforms that we have learnt earlier, namely the Fourier transform and the Laplace transform, which take you from let us say the time domain to the frequency domain. This transform is of a slightly different kind; it takes a time domain signal to another time domain signal. So you are not changing the domains.

The reason, why I am discussing this before, I have even started with the regular content of the communication course; that is what we are going to follow in this class, because this transform is a very useful transform, for representation of certain kind of signals which, we shall be most commonly dealing with, namely the representation of band pass signals. Since, we are going to deal with band pass signals quite a lot, it is useful therefore, to have this analytical tool available to us in this representation task.

Later on, we will find that the, it has a very particular, very important application in particular, for the case of certain for representing certain kinds of modulation schemes, for example, the single side band EM modulation scheme.So, that will be one of the major applications of this representation, so since this application will only come later. Today, we will only discuss what the Hilbert transform is all about. What it is, how it can be used for representing band pass signals in general; that is our mainly for today. To start with, let us consider what Hilbert transform is all about.

(Refer Slide Time: 02:53)



Let me consider a linear filter, or a linear time invariant system; this transfer function is something like this. Are you familiar with this function signum of f; it is a very simple function. A signum of f is equal to 1 for f greater than 0 is minus 1, for f less than 0. So what does it look like of frequency axis or f axis, this is a function, plot of the signum function, signum of f, plus 1. Here, we will define the value at 0 to be 0 itself; the mean value between the two extreme values.

Now, with this definition of the signal function, what is the ratio of the transfer function that we are talking about here? Suppose I was to plot H of f, as depicted here, what will it look like; just the opposite of this.Let this value be plus j, this value be minus j. Now that is the mathematical picture of this filter. Can you tell me what will this filter actually do to an input signal? How will it affect the input signal? Can you say something about the effect that such a filter will have on any input that is applied to ((Refer Time: 05:03)).

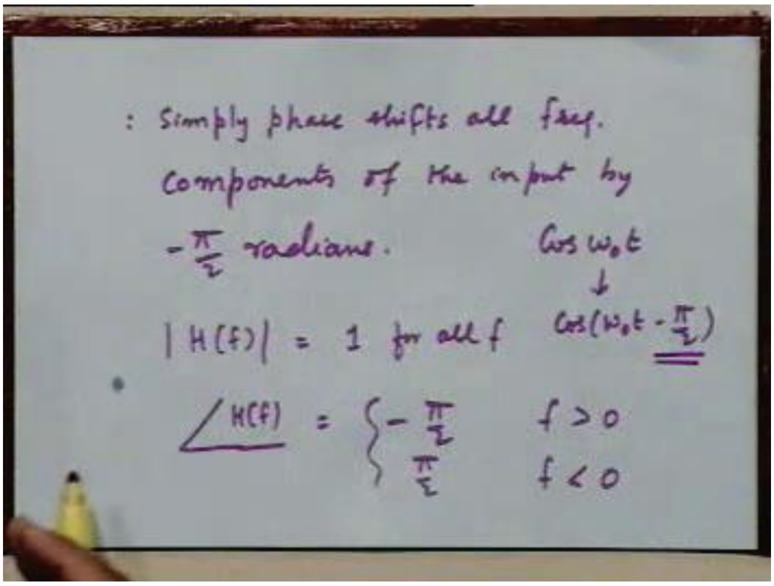
Student: ((Refer Time: 05:04))

Please speak a little loud.

Student: ((Refer Time: 05:07))

There is going to be a phase shift of minus pi by 2 for every frequency in the input signal; is it clear? So, this filter, we phase shift every frequency component present in a signal by a value of minus pi by 2, so that is what the Hilbert transform does.

(Refer Slide Time: 05:37)



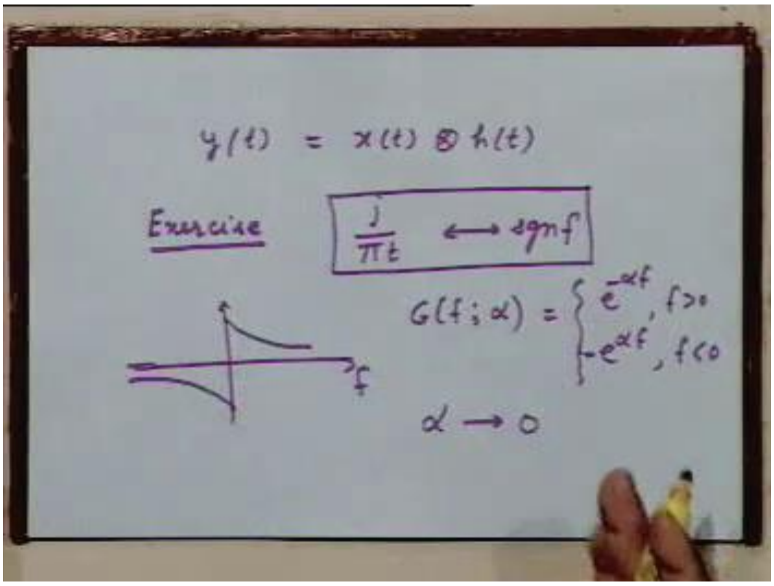
It phase shifts simply phase shifts all frequency components of the input by this amount. So, if you were to look at magnitude characteristics of this filter, what will it be? It is equal to 1 for all f. But this is going to be equal to minus pi by 2 of thefor fand since, it has to be an odd symmetric function, it will be pi by 2 for f less than 0. So, that is what Hilbert transform is. It is a filter which carries out a constant phase shift of minus pi by 2 radiance for every frequency component that is present in input signal.

Student: ((Refer Time: 07:02))

You see, real signals may have both positive frequency as well as negative frequencies, when it gets phase shifted; phase shifted by minus pi by 2, the corresponding negative frequency component. Every real signal will have a positive frequency component and a negative frequency component. The positive frequency component will get shifted to the minus pi by 2 and the negative by pi by 2, and the real signal for example, is a cosine wave, if it is cosine omega naught t.

What will happen to this? It will become cosine omega naught t, minus pi by 2. Now look at the spectrum of this in the positive frequencies. This will become minus j pi by 2 and negative sign will become e to the power plus j pi by 2. So, the real phase shift is minus pi by 2, but that is only a mathematical effect of that one. Imagine negative frequency components are always present in a real valued signal. So, that is something that is purely a mathematical property.

(Refer Slide Time: 08:29)



Now, let us look at the same operation in different way. We know that the output of any filter; see output of some signal, we want to understand what is the nature of the output signal. We have said something about it, but we like to say something more about it. What we have said is every frequency component gets phase shifted by minus pi by 2 radiance. Can I express the waveform y of t in this case in terms of x of t? That is the next question.

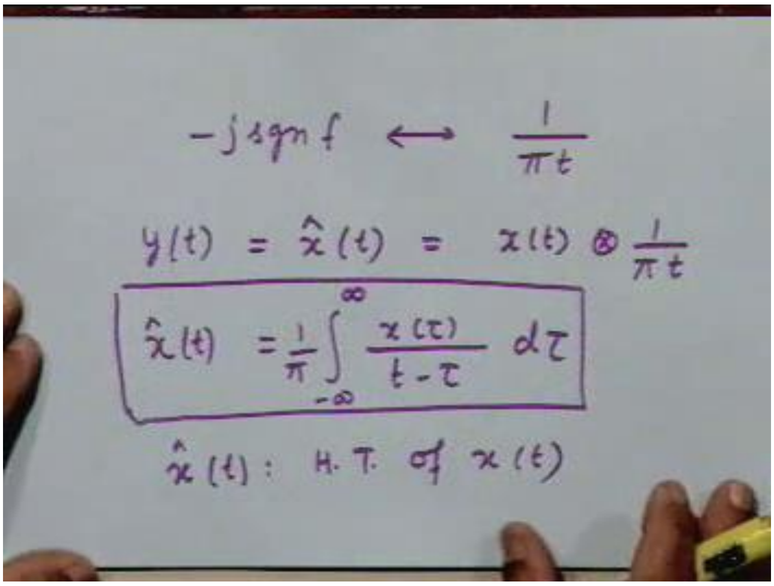
What is the actual waveform like, for a given input waveform x of t and I know how to find that in our review of signals and systems. We learnt that the output is a convolution of an input signal and the impulse response. So, y of t is x t convolved with h t. So the question arises; what is h of t for this filter so that, if I know h of t, I can affect this convolution and find out what is the nature of y of t. So, now I will give you a small exercise to do, because this is a simple exercise in Fourier transformation.

I will also give you a hint as how to do it, but I like you to do it yourself. Exercise is to prove that these two functions are Fourier transform pairs. Prove that j by pi t, and signum of f are Fourier transform pairs. To prove this, I will just give you a hint. Do not start with the signum function directly, because you will have a problem. It does not satisfy the condition directly. So, a problem, it is a constant amplitude signal; it is not a finite energy signal in the frequency domain. Iit does not satisfy the condition.

So, you will have a problem doing it directly. So do it slightly indirectly. You consider this function, which is basically, let us call this. Let me define this function G f with a parameter alpha, and is defined as e to the power minus alpha f, for f greater than 0, and e to the power alpha f, or minus e to the power alpha f, for f less than zero. Now, it is ok. You can see that ((Refer Time: 11:17)), it is a finite energy in the frequency domain; this is of course f.

So, first find the Fourier transform of this function or the inverse Fourier transform of this function and take the limit as alpha tends to 0. So, if you carry out this exercise, you will be able to prove that these two functions are Fourier transform pairs. So, I have more or less done the job for you. You just have to sit down and write down the equations to confirm that this is the result. Can I assume that you can do this? It will also be a good check for yourself, that you can handle the Fourier transform, nicely by yourself.

(Refer Slide Time: 12:11)



So, suppose we have this result; what does it mean? So, what is the Fourier transform of minus inverse Fourier transform of minus j signum f, because the impulse response is looking for the inverse Fourier transform of the system function or the transfer function. They transfer function is minus j signum f, Signum f inverse Fourier transform is jpi t. So this will simply become 1 by pi t, because minus j square is equal to 1. So, the inverse Fourier transform of minus j signum f is 1 by pi t.

And therefore, we can now write; what y of t will be like? Incidentally, instead of calling it a general signal y of t, since it is a very special operation, I denote this y of t, by a special notation. I call it x hat t. x cap t. This symbol is usually called a cap symbol. We call it x cap t and it is equal to the convolution of the actual input signal x t with the impulse response 1 by pi t which, you can write and that is the definition of the Hilbert transform.

X hat t, which is the output of this constant pi by 2 phase shift filter is related to x t, through this convolution relation, and x hat t here, is a Hilbert transform of x t. ((Refer Time: 13:57)) this convolution is this integral; x tau into h t minus tau d tau, h t minus tau here would be, 1 by pi t minus tau and that is how this integral comes. I hope it is so ok. You can do the Hilbert transform in two ways; one purely as this integral relating to the input signal x of t to the Hilbert transform x hat t, or the other as an operation in which, every frequency component gets phase shifted by minus pi by 2 radiance.

Any questions?

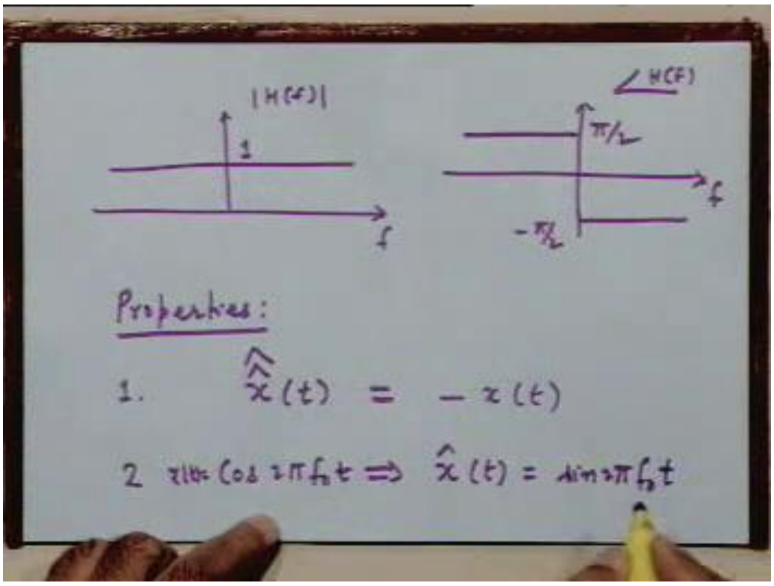
Student:((Refer Time: 14:52))

Why is it called a linear filter, because it satisfies the principle of superposition, and it is only linear filter, that we can express in terms of the transfer functions or the impulse response. So all filters that we normally deal with are linear filters. The superposition integral is nothing, but the convolution integral. The convolution integral is basically, coming out of the linearity of the filter. So, any filter which is specified by transfer function or by it is impulse response. We are implicitly assuming it to be a linear filter. Yes please.

Student: ((Refer Time: 15:35))

In this particular case, h f is not complex; it is really imaginary only. You see, in general h of f, if any filter would be a complex valued function, and you will to represent a complex valued function typically, you will have to make 2 plots. A magnitude plot plotting magnitude of h of f, and a phase plot. We can do that also here in this case. It happens to be purely imaginary, so one plot is good enough. If it is either purely real or purely imaginary, one plot is easy to we will describe the whole thing.

(Refer Slide Time: 16:23)



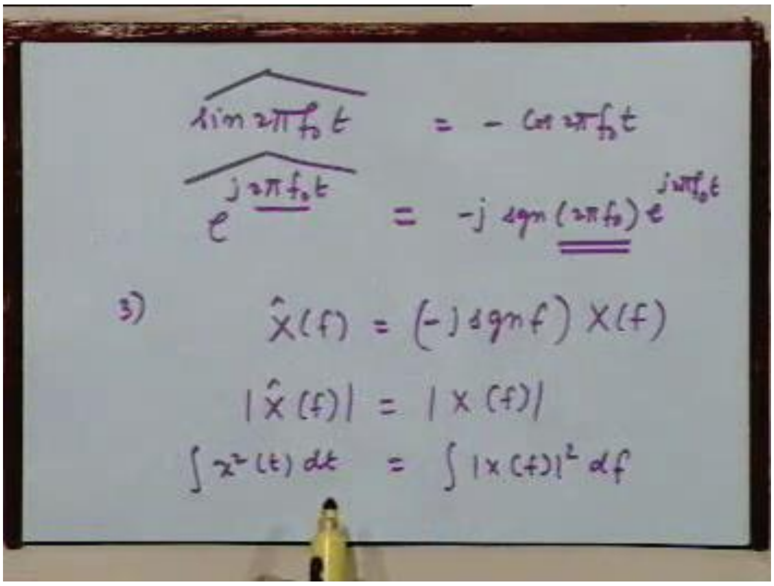
But if you wish, you can still go back to the 2 plot system and what will be the 2 plots. In this case, the amplitude plot transfer function will be a constant equal to 1, and the phase plot angle of H f; what will this be? Minus pi by 2 for this, for positive frequencies and plus pi by 2 for negative frequencies. So, this also is a description of the Hilbert transform. This is the magnitude plot. This is the phase plot. So, the question was how do we show the magnitude and phase plot in the Hilbert transform. This is how they look like all.

Let us look at some interesting properties of the Hilbert transform in this slide. We appreciate the properties of the Fourier transform and the Laplace transform; that is, because some simple basic interesting properties of the Hilbert transform. Let me ask you; suppose I take the Hilbert transform of x hat t; what will I get back without doing any mathematics, can you tell me?

If I take the Hilbert transform once again, of the Hilbert transform of x t, minus x t, because the total phase shift will become minus pi for every frequency component. So, every frequency component goes through a phase reversal. So, the result that the whole signal, this simply becomes negative of the original signal. So, this is very easy to prove. I will leave the proof. It is obvious, I hope it is obvious.

The second one is, so Hilbert transform in the first one is the Hilbert transform or the Hilbert transform of x t is minus x t. So, carry out the Hilbert transformation twice; you get the same signal back except for change in sign. Now of course, let us look at some special cases. Suppose I have a very simple signal x of t equal to cosine 2 pi f naught t. What is your Hilbert transform for this? If x of t is cosine 2 pi f naught t, what is it is Hilbert transform, sin 2 pi f naught t?

(Refer Slide Time: 19:04)



What is the Hilbert transform of sin 2 pi f naught t? So, the Hilbert transform of this signal, I depict it like this is minus cosine 2 pi f naught t. So, these are very simple signals. So, you can find obviously, without going through that convolution operation. You have to judge for yourself whether, it is more beneficial to use a frequency domain relationship to find the answer or to use a time domain relationship. In this case, it is much better to look at the frequency domain, because in a frequency domain, we have a single frequency component.

So, I know what happens to the signal. So, the Hilbert transform of sin 2 pi f naught t is this. In general, suppose I now consider the complex sinusoid and take the Hilbert transform of that; what will that be equal to?

Student: ((Refer Time: 20:04))

Minus j times, you have to be little careful. It will depend on the sign of f naught also. If you we are assuming of course, that f naught here is positive. But suppose f naught is negative, then it will be plus j, because we have so minus j into signum of whatever, is the this parameter here, into e power j 2 pi f naught t. Please convince yourself that this is correct.You have to take into account the sign of this exponent; it is exponent is positive. See, when you have a real signal, there are both positive frequency components as well as negative frequency components.

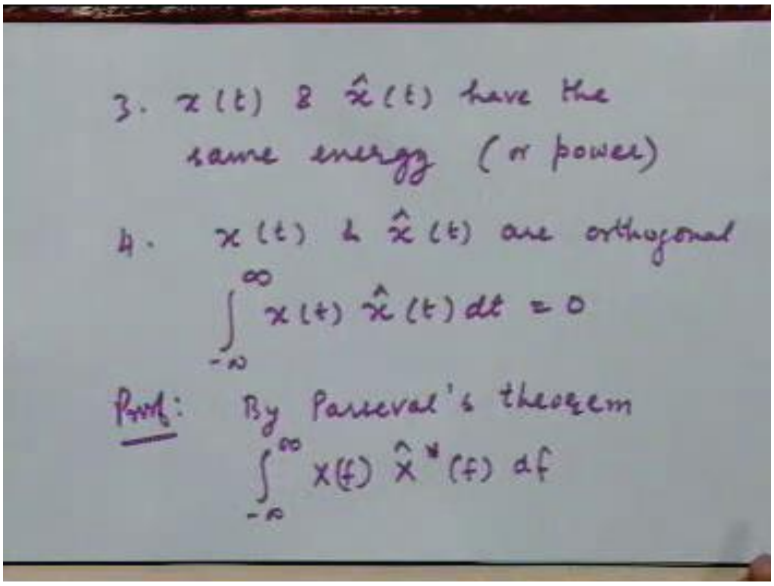
So, it makes sense to say that the positive frequency gets shifted by minus pi by 2 and the corresponding negative frequency component gets shifted by plus pi by 2 or multiply, just multiply with plus j and minus j. But when you have complex exponential, we have only a single frequency component, either that is positive or it is negative. So when it gets multiplied by minus j or plus j, will depend on whether this is positive or whether, it is negative; are you with me on this everyone? Good, some other properties.

If you have any questions please raise it. I will try to make sure that all you understand what I am saying. So I can assume that it is ok. What can you say about the energies of x t and x hat t? What is the relationship between the suppose x of t is an energy signal.

Student: ((Refer Time: 22:18))

They will be same; why because mod of h of f is equal to 1, so the, if you look at in the frequency domain x hat f in the Fourier transform of the Hilbert transform of x t, I denote by x hat f, but this is equal to minus j signum f into x of f. So, look at the magnitude of this; it is equal to and from Parseval's theorem. We know that, the energy function can be written in terms of this equability holds. So, this is the same for both of them. That means, energy of x t and x hat t are the same. Similarly, if x t was a power signal, rather than an energy signal, it could have been checked that the powers will remain the same all.

(Refer Slide Time: 23:59)



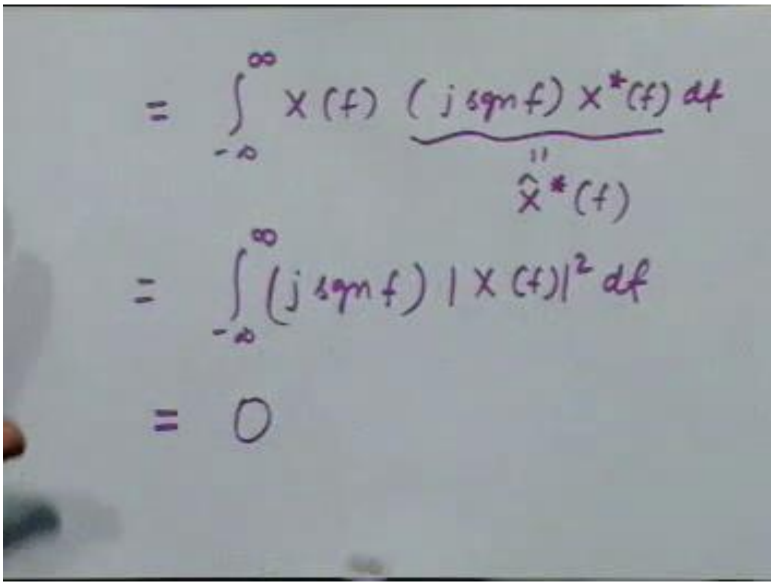
So, the property that we have just discussed is that x t and x hat t have the same energy or power as the case may be all. Next property and this one is also quite interesting. We can almost deduce it from the case of the simple case that we discussed with the case of sin wave. What is the Hilbert transform of cosine 2 pi f naught t, sin 2 pi f naught t? What is the relationship between these two? They are orthogonal to each other. In general, one can show very easily that x of n Hilbert transform would be 2 orthogonal signals. So x t and x hat t are mutually orthogonal.

Mathematically, this is written like this. Of course, this statement is for energy signals for the power signals. There will be a slight modification of this statement mathematically. How do you prove this; proof, any ideas? One way is to substitute for x hat t in terms of x t the convolution integral and see what happens to this integral; that will be rather complicated, I can assure you. Is there a simpler way? If we can somehow, go from the time domain integral to the frequency domain integral, and how do you do that; a slight generalization of the Parseval's theorem.

There is a generalized Parseval's theorem available, which talks about two functions, integral of the product of two functions. I am sure you know about it, but in case you do not, I will just tell you that. So, by Parseval's theorem, which is actually, the most general form of the Parseval's theorem, this time domain integral again, I will leave the proof for you to work out yourself or look it up in some book. This will be equal to x f into the conjugate of the Fourier transform of the second function.

So, this will become x conjugate f df. You can see that the normal Parseval's theorem is a special case of this. You can think of mod of x t square as for a real signal x square t as x t into x t. So, when you apply the generalized Parseval's theorem on that, you will get x f into x conjugate f. In this case, you will get x f and conjugate of the Hilbert transform, the Fourier transform of x hat t, which is x hat f. So, as far as the proof of this is concerned, as you know I am leaving it, as an exercise for you. Please see if you can do it.

(Refer Slide Time: 27:12)

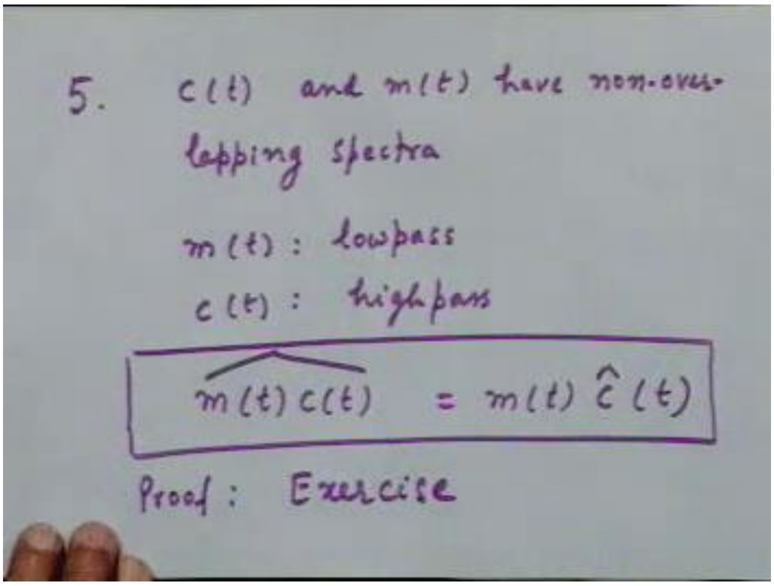


If you have this, what is the value; what is the value of this integral? So, what you are saying is integral of x t x hat t b t is equal to integral of x hat f x hat conjugate f, sorry, x f into x hat conjugate f b f. Let us substitute for x hat f. It is equal to x hat conjugate f is equal to j signum f. We have taken it as conjugate into x conjugate f minus j signum f becomes plus j signum f, because of the convolution operation. So, it becomes x conjugate f, because of the conjugate operation.

This is equal to x hat conjugate f. I am substituting for this. This is equal to j signum f into mod of x f square d f and this is equal to what is the value of this integral; it is obviously, 0. What kind of a function do you have inside the integral? It is an odd function of f,x f mod square for a real signal, is an even function, magnitude square function or magnitude function. Magnitude spectrum is an even symmetric function. Signum f is an odd symmetric function. The product would be an odd symmetric function and therefore, the value of this integral is 0, which proves our result.

So, you must judge in the case of problems in Hilbert transform ((Refer Time: 29:08)) to do the required operation in frequency domain or time domain. It is so easy to show this in the frequency domain, but if you were to proceed in the time domain, it will be much more involved to prove the same result.

(Refer Slide Time: 29:33)

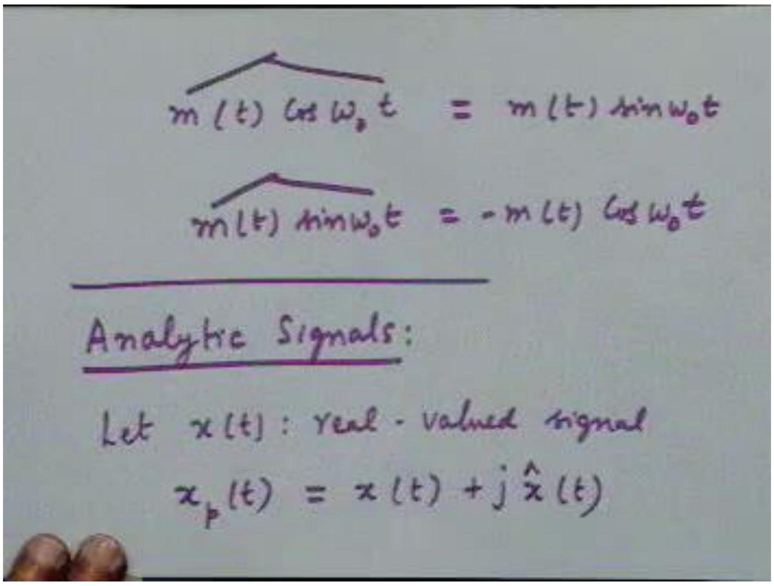


The last, but very important property is the following. This is something slightly more involved. I will just take the result here, and again ask you, to see if you can prove it. If not, we will discuss it in one of the problems, ((Refer Time: 29:55)), but try. Suppose you have two signals; I will call them c t and m t and they have what we say non overlapping spectra. That is in a frequency domain they do not overlap ((Refer Time: 30:25))signals, but in a frequency domain, they do not overlap.

Since obviously, if they are overlapping, one of them will contain lower frequency components. The other will contain higher frequency components. So, without loss of generality, let m t be the low pass function. So, we say that m t is low pass means, it is spectrum easily occupies the lower frequency components and c t is high pass in that sense. It could be the other way round, but we have just decided to do this. Then if you consider the Hilbert transform of the product of these two functions, these two signals.

If you look at the Hilbert transform of m t into c t, it will turn out to be nothing, but m t. The same low pass signal the m t, which is being used here, multiplied by the Hilbert transform of c t. So, proof of this at the moment; I am leaving that as an exercise. If necessary we will discuss it in the tutorials, but it is something that you have to sit down and prove. It is not obvious so; therefore, do not attempt to visualize it away. You will have to sit down and work it out.

(Refer Slide Time: 32:11)



But as a consequence of this the two special cases, which are particularly relevant to us, because we deal with such situations quite commonly, is when c of t, the so called high pass signal happens to be a, what we call carrier signal of some high frequency omega naught. And m t is a signal with much lower frequency band; that is the highest frequency component m t is smaller than omega naught. Then, that condition will be satisfied and what will happen to the Hilbert transform of this thing if you apply that result?

It will become m t into the Hilbert transform of cosine omega naught t. It will be m t sin omega naught t. So, this is a very important special case of that. Similarly, m t sin omega naught t will be minus m t cosine omega naught t; are you with me? So, this more or less is a reasonably good introduction to the Hilbert transform definition and its properties. We next look at the application of the Hilbert transform in representation of signals particularly, representation of band pass signals, which is what I mentioned in the beginning that it has major applications there.

To do that, if you have any questions, if you want a discussion on some concept that we have discussed so far, you can take a couple of minutes or I proceed further now. I like to kept this as interactive as possible. So, if you have any questions, please come up with them; no questions? So, it is fair for me to assume, that the concept of simple enough for you to have totally absorbed at the moment; is it ok? Let us look at the next concept. The next concept that I am going to introduce based on Hilbert transform introduction that we just had is a concept of analytic signals.

Let us say, x t is a real valued signal and then let me first, give you a little bit of motivation.We have discussed earlier, and have discussed at length. I am sure, when you had course on signals and systems, that it is much more convenient for analysis, to work with the complex representation of a sinusoid, then the sinusoid, then the trigonometric representation of the sinusoid. Example; instead of working with cosine omega naught t, it is much more simple to work with e to the power j omega naught t.

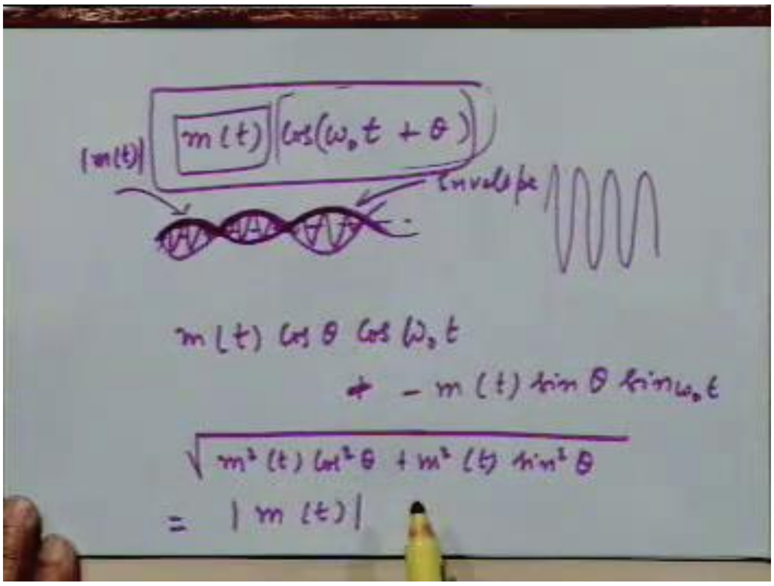
So, in this case, in that sense, e to the power j omega naught t is an analytic representation of cosine omega naught t. You want to generalize this concept of arbitrary signals, any signal; not necessarily only sinusoidal signals. That is our motivation behind what I am presenting here. And obviously, from our analogy with the real signal, from our analogy with the sinusoidal signal, such analytical signal must necessarily be a complex signal.

So, we need to define or introduce the concept of an analytical, which is obviously, going to be very, very complex and it would have the similar simplification, which is offered by the normal complex exponential representation of a sin wave. So, we introduce for real valued signal x of t, it complex analytical representation as x sub p t. We will soon come to the point as to why we use the subscript p in a few minutes.

Its real part is x of t, x p t is a complex valued signal, its real part is the same real values signal with which we started, but it is imaginary part; it has an imaginary part is the Hilbert transform of x t. So, that is the concept of an analytical signal. An analytic signal has this composition. It will now see that your complex exponential is a special case of this representation, is not it, this is cosine omega naught t, this becomes j sin omega naught t, which is e power j omega naught t. So, that becomes a special case of this.

And, this is more general form of that analytic representation, we will see. We will try to get used to this, and try to develop a feel for this representation as to what it can do for us. Now, one of the things that you can see is before I do that, let me again discuss conceptually as to some what is significance of this representation.

(Refer Slide Time: 38:16)



Suppose I work with a signal like this. Let us consider the signal m t cosine omega naught t. Suppose I ask you a question as to or may be let us make it, slightly more general, omega naught t plus theta. Suppose I ask you to visualize the signal as a waveform that I mean, it is a product of two signals. We are going to discuss this kind of product signals in detail when we discuss modulations. Let this be the, as be the time as, any other to discuss, what is the nature of the signal; what does it look like?

Assume that m t is a ((Refer Time: 39:00)) signal of which has a ((Refer Time: 39:02)) low frequencies. It is a low pass signal and when I say a low pass signal, essentially, it contains lower frequencies. Much lower than so called what we typically, gave the name as carrier frequency omega naught. So, this is the highest frequency component of m t is much smaller than omega. So, let me make it more ((Refer Time: 39:27)), let us say, this is the message signal m t, this is what it looks like the waveform m t.

And, the carrier looks something like this or the cosine wave looks something like this much higher frequency. So, it will be ((Refer Time: 39:42)) look like. Can you say something, is it easy for you to imagine that this is what the product will look like? Let us if not, let me do some manipulations. Basically to visualize this, you have to answer only one question. It is obvious that the product signal is a fluctuating signal, fluctuating at this rate. It is the same cosine wave signal, cosine omega naught t signal with some instantaneous amplitude, which is governed by the value from t.

Suppose I ask you the question; what is the instantaneous amplitude at time t? How will you answer that question? It is m t or something else. Let me do a slight expansion of the trigonometric identity. This is m t cosine theta, cosine omega naught t, minus m t sin theta sin omega naught t. So, what is this instantaneous amplitude?

Student: ((Refer Time: 41:12))

This no, it is if you remember, if I want to go from here to here, this always has to be positive quantity and what will this be equal to? It will be m square t cosine square theta, plus m square t sin square theta. This is simple trigonometry for you; simple trigonometry; is it ok, which if you have to simplify, a will be square root of m square t, which is how much? Well ((Refer Time: 41:51)) mod of m t, the sin information will be lost and it has to be a positive function.

So, really speaking, if I write something like this, what I am saying is that instantaneous amplitude is governed by the modulus of m t, the positive value of m t; m t itself will be positive and negative. For example, the original signal was like this. It was going positive and negative. But if I look at this signal, as if I want to look upon this as a waveform, I can ask the question, it is a sinusoidal. I know this is sinusoid of high frequency to visualize it. The important question is what is the instantaneous amplitude of that sinusoid?

Student: ((Refer Time: 42:34))

I am talking of amplitude as against this function m t. So, I am not saying this is equal to mod of m t cosine omega naught t plus theta. But to visualize it of asking, what is the amplitude of the signal at time t? The answer is the amplitude is whatever, is the magnitude of m t.

Student: ((Refer Time: 43:00))

No, amplitude has only one. See the question; you have appreciate amplitude has only one significance. If I say a cosine theta or minus a cosine theta; what is the amplitude a ? That is what I am referring to. You are appreciating what I am saying. Now the amplitude does not, minus the minus sign has to be incorporated to the phase information. Is it clear?

So, I am talking about instantaneous amplitude. So if you look, so that is one way of visualizing this signal. The important signal, why I brought this up here is, at such a signal, you can see, in terms of some fine variations, whose amplitude variations are dependent on the magnitude of m t.You can think of this. For example, this waveform has a very important attribute of this product wave form and this waveform, which basically, is a trace of the amplitude of the signal of this sin wave. As a function of time, you can think of this trace of the amplitude of this waveform as a function of time; that we call the envelop function of such a signal.

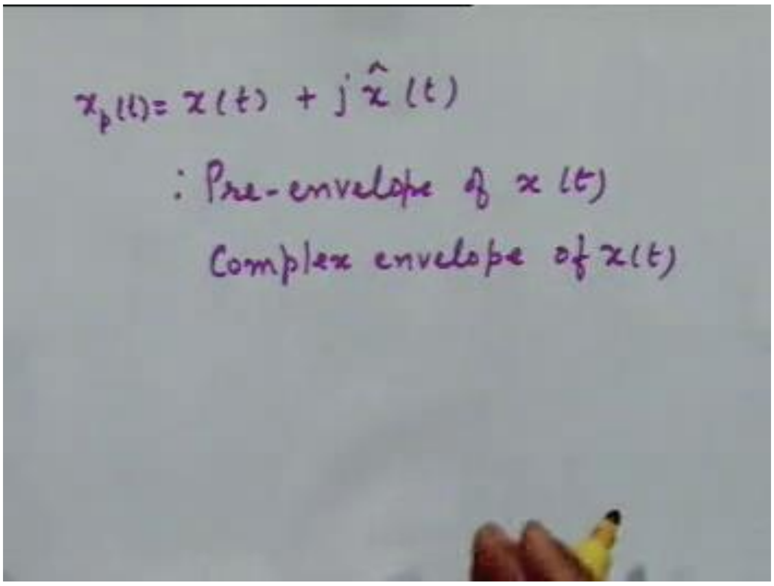
So, you know, actually band pass signal here, and those ((Refer Time: 44:29)), which you can define here, which you can call the envelope of the signal. The envelope of the signal is a trace of the peak of the sin wave that we are working with; the sin wave that you are working with this. If you trace out the peaks of this that forms the envelope of the signal, and it is easy to visualize this kind of an envelope when you have product of a low pass signal and high pass signal.

When you have a sinusoid like this multiplied by a low frequency function like m of t, this slope fluctuation, these envelope fluctuations are slope fluctuations. But the sign of the fluctuations are depicted or dictated by the value of omega naught, which is much larger. So, the point that I am basically making is that for band pass signals of this kind, this is a band pass signal hopefully obvious, because the Fourier transform of this signal will be translated version of the Fourier transform of m t; frequency translated version, and therefore, it becomes a signal which is a band pass signal.

So, in general a band pass signal of this kind, you can identify an important attribute of that waveform, which you can call the envelope of the waveform, and that contains some useful information about m t itself. In this case, these are exactly equal to empty. It is a modulus of empty as you can see, empty goes like this; the envelope goes like this; the envelope is this which is mod of m t. The actual signal is this, but the envelope is a trace of only the peaks of these fluctuations, is mod of m t.

We will discuss this waveform in much more detail later, when we discuss the amplitude modulation. For the moment, my only simple point that I want you to ((Refer Time: 46:36)) was, that such a signal is associated with the concept of an envelope. And I wanted to introduce this, because actually in general, for any kind of signal, whether or not it is band pass nature, mathematically one can define the concept of an envelope in exactly the same way, that I have defined here.

(Refer Slide Time: 46:59)



And the next concept therefore, that I will take up in the next class is the concept that x p t is sometimes also called the pre envelope of x t, or simply, the complex envelope, but before introducing this concept, I wanted you to understand what envelope means and that is why I ((Refer Time: 47:28)) a little bit, so complex envelope of x of t. So, this analytic representation x sub p t, which is defined as x t plus j times x hat t, we also call the pre envelope of x t or the complex envelope of x t. Precisely, what is the relationship of this x sub p t to the envelope; that we had just discussed that is something we will take up in the next class.

Thank you very much.