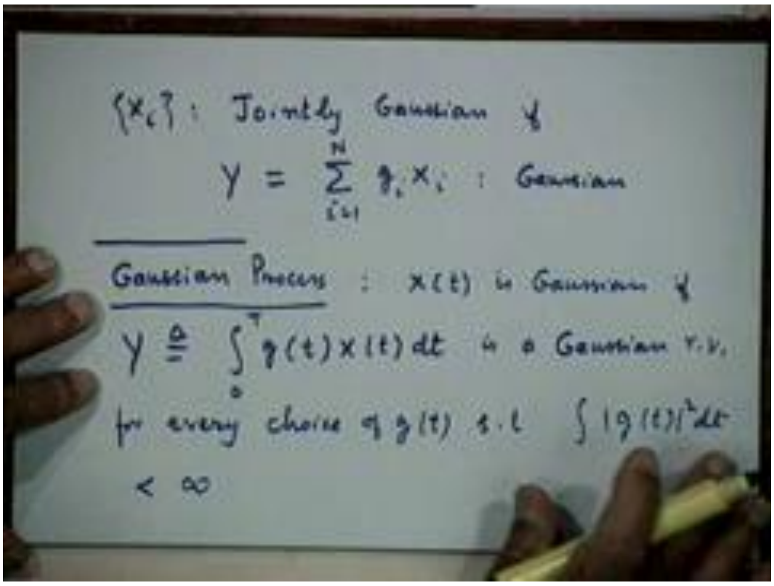
Communication Engineering
Prof. Surendra Prasad
Department of Electrical Engineering
Indian Institute of Technology, Delhi

Lecture - 32
Gaussian Random Processes

To continue with our discussion on Random Processes, we now discuss a set of very important processes namely Gaussian Random Processes. If you recollect, we have introduce the concept of Gaussian random variable earlier.

(Refer Slide Time: 01:40)



And yesterday I talked about, a set of random variables X i's to be jointly Gaussian, we say that, a set of random variables X 1 to X n are jointly Gaussian, this is the definition. If they are such that, any linear combination of these, like this here, any linear combination of this happens to be a constant random variable. So, if they are set of n random variables X 1 to X of N, if I take any arbitrary linear combination of these, resulting in a random variable Y and Y tends out to be Gaussian by every such linear combination.

Then we say that, X i is a jointly Gaussian, this is a definition of jointly Gaussian random processes. So, now this will look, is that, Y i is Gaussian this is Gaussian, we can now extend this definition to, define a Gaussian process, incidentally before I do that let me also mention, that in this particular case. The join distribution of X 1, X 2, X 3, there will be a joint distribution of these, would have a form, you would have a specific form which is called jointly Gaussian density function.

So, when they are jointly Gaussian there is also an associative density function, which is of course very easy to generate, if they was statistically independent. But, in general they may not even, this exercise need not be statistically independent for this definition to be valid. So, what the density function is, is something that I will discuss a few minutes later, but there is an associative joint Gaussian density function, of these n random variables, you see the form of the joint density function a few minutes.

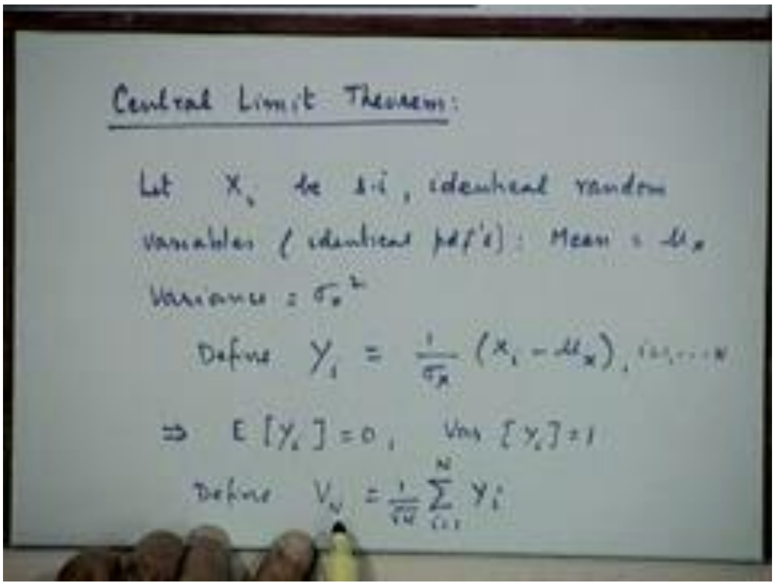
So, we say that a random process X t is Gaussian or a noise process X of t is Gaussian, by essentially generalizing this definition. If you have a random process X t rather discrete values X i, so these are you can think of i as a discrete index, going from 1 to n and t as a continuous index. So, how will you generalize this, essentially the integration and with g sub i replace with, some function g of t, so I generalize this and that gives the definition of a Gaussian random process.

So, we say X of t is Gaussian, if every Y that you construct like this or 0 to T or minus infinity to plus infinity does not really matter, that depends on the domain of t. If Y define like this, which is essentially once again, you can think of X of t denoting different random variables, except that it is a infinite set of random variables going from the going of the domain of t, a g of t is a corresponding multiplication of relative coefficient.

And you are summing up all these related random variables, exactly in the same that you are doing in this case. So, if this is a Gaussian random variable, so once again you are generated a Gaussian once again you generated a variable here, it is not a function of time, time will display in this. So, you are taking a linear combination of this random variables and generating a random variable here, if this happens to be a Gaussian random variable, we said that the process X t is Gaussian.

For, if this is so, for every choice of the compiling function g of t, these are the weighting function of the compiling function, with the constraint that this function should satisfy this condition. If this is true for every such g t which is a finite value of the function, if Y happens to be a constant random variable, then you say that process X t is Gaussian. Now, Gaussian random variables, as I mentioned some time ago and Gaussian processes, they are very commonly use models for physical phenomena. If you recollect and mention this earlier and the reason and justification for this, use is a result from statistics and probability theory, which is the central limit theorem.

(Refer Slide Time: 06:54)



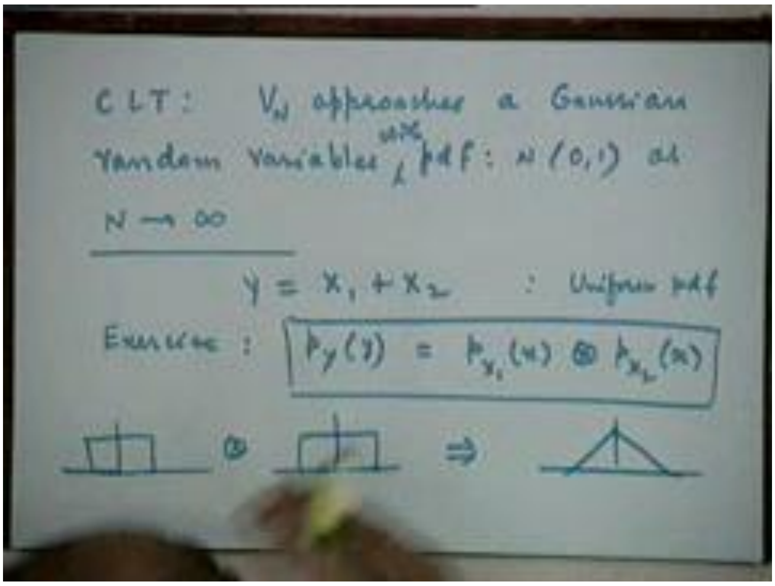
I just briefly mention this, for complete for the sake of completeness, a central limit theorem provides the mathematical justification for using, a Gaussian process model for many physical phenomena. So, let me state what this theorem is, let X sub i be statistically independent, identical random variables. By identical random variables I need, identical periods, they have identical, that is they have identical density functions.

That way, it does not mean that they are the same random variables yes, they are different random variables, identical periods and let us say the ith random variable has, the each of them has a mean equal to mu sub X and variance equal to sigma X square. That we define, a random variable Y sub i, which is 1 by sigma X, this your normalize random variable essentially, X i minus mu X like that, i going from 1 to N. So, from the ith X i, I generated a Yi.

How is this Y i different from this X i, it is normalize to have zero mean and unit variance. So, basically that is the purpose of defining this normalized random variables, so this normalization would imply, that the average value of this is 0 and variance of Y i is equal to 1. Let me further define, random variable V sub N, which is obtained by taking this sample average or taking the average of these N random variables, not exactly average, because I am not going to divide it by N, because I am going to divide it by root of N, it is not really average, you just adding of these and dividing it by root of N.

This division by root of N is again a kind of normalization, more than anything else, then the central limit theorem states, this is the most basic question of the central limit theorem, they are more advanced versions of this, which will generalize these results. The most basic question of the central limit theorem states that, as N approaches infinity, the random variable V sub N, will tend to be become a normal random variable, again with zero mean N, unit variants. So, that is the statement of the central limit theorem.

(Refer Slide Time: 10:16)



Central limit theorem states that, V N approaches a Gaussian random variable with PDF becoming, normal with zero mean infinity variance, as N tends to infinity. So, please note that, ((Refer Time: 10:54)) initially I have not put any kind of, I have not set what distribution X i’s could have, you could have any distribution, they need not be normal at all, they need not be Gaussian at all.

But, if I add a large number of such random variables, which may not even be Gaussian themselves, the result is a Gaussian random variable, that is the central limit theorem, it is not difficult to physically or intuitively see why this should be so. So, why because you already have some idea of these things, but in case you do not have, that we just give a physical picture for this, why this should happen. Suppose, I have two random variables X 1 and X 2 ((Refer Time: 11:45)) this is just to, I am not doing a very strictly doing here, just give a physical feed for why this results should ((Refer Time: 11:52))..

Suppose, I have two random variables X 1 and X 2 and I generate a random variable Y, which is the sum of these two. Let us say X 1 and X 2 about uniformly distributed random variables, both of them has uniform PDF, nowhere in the Gaussian PDF , totally different from it. Now, you know what will be the density function of Y, suppose X 1 and X 2 are statistically independent and I ask you, what is the density function of Y, could you able to say something, maybe not.

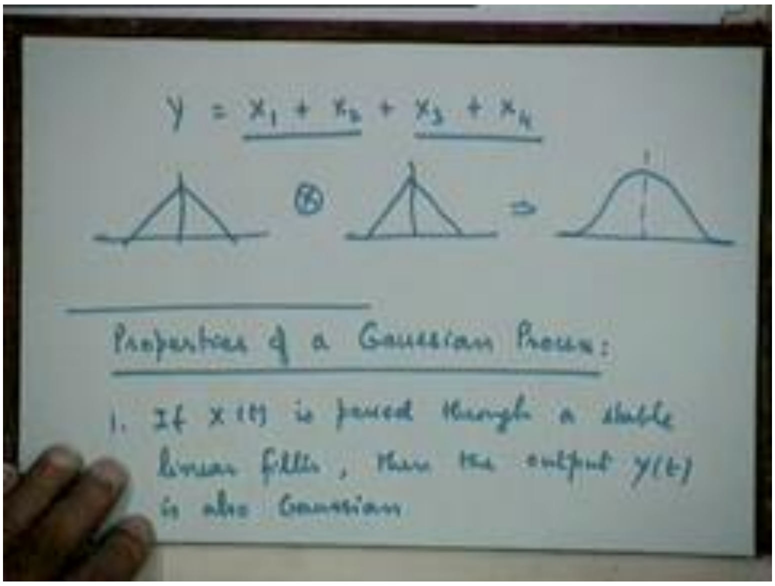
Will I will leave that an exercise for you to prove, this is consider this as an additional problem in the problems which there I given to you. Exercise is, to prove that P Y y, the density function of Y if X 1 and X 2 are statistically independent would be P X 1 x convolve with itself or P X 2 x. There is a reason, I mean that something very simple to prove by just very elementary consideration of the definition of density function and things like that.

So, let us start with this, assume this result is, which is very easy to prove and case from this we can argue, why the centre limits to hold. So, I just said, assume that both X 1 and X 2 have uniform period, then what may the distribution function of Y, here uniform is they are rectangular functions, it convolve to rectangular function, what you get?

Student: ((Refer Time: 13:33))

You will get a rectangular functions, so you start with things like this, this convolve with this, who lead to this. Now, let us say I add two more such random variables, to obtain Y.

(Refer Slide Time: 13:57)



So, let us say, Y is now define as X 1 plus X 2 plus X 3 plus X 4, all uniformly distributed. The sum of these two, as a triangular distribution, the sum of these two again, as a triangular distribution and you have convolving these two, what will you get now, we will get a parabolic function. So already you see, the resulting function is starting to resemble in Gaussian shape, is it not. So and as you add more and more such random variables this tendency to become normal will become bigger and bigger.

So, that intuitively gives a picture, as to whereas the result comes from, it basically say, result of this ((Refer Time: 14:51)) most basic probability theorem result, which is very easy to prove. I am giving that as an exercise for you to prove, ((Refer Time: 14:48)) given in most books or you can also prove it yourself. So, that is the reason, why Gaussian processes are used as models, because it tends out that, many physical phenomena in these can be express as some total of, a large number of micro phenomena.

And that is what you really observe, you observe the macro phenomena and not the micro phenomena. The micro phenomena may not be Gaussian in nature, but the macro phenomena that we are observing, will turn out to be Gaussian in nature. Now, since Gaussian process is a very important, let us study a few important characterizations or properties of Gaussian process, so let me talk about some properties of a Gaussian process.

Now, let me ask this question, suppose I pass a Gaussian random process, through a linear filter, what kind of process will I get at the output, can you say something, what will be the nature of the process at the output in the filter, I am not asking you to find the auto correlation function and then anything else, can you make any general statement about the nature of the output process or can we have ((Refer Time: 16:35)) make the question more focus, will it remain Gaussian?

Student: ((Refer Time: 16:38))

How does it follow?

Student: ((Refer Time: 16:40))

It follows from the definition of the Gaussian process, more or less, how do I define a Gaussian process. Just think about it, I define a Gaussian process X t to be so, such that every linear function which is integral of g t, X t, e t that tends out to be a Gaussian random variable and that convolution relation that holds, you essentially that kind of a relationship for every time instant. So, at every time instant, you are generating Gaussian random variable.

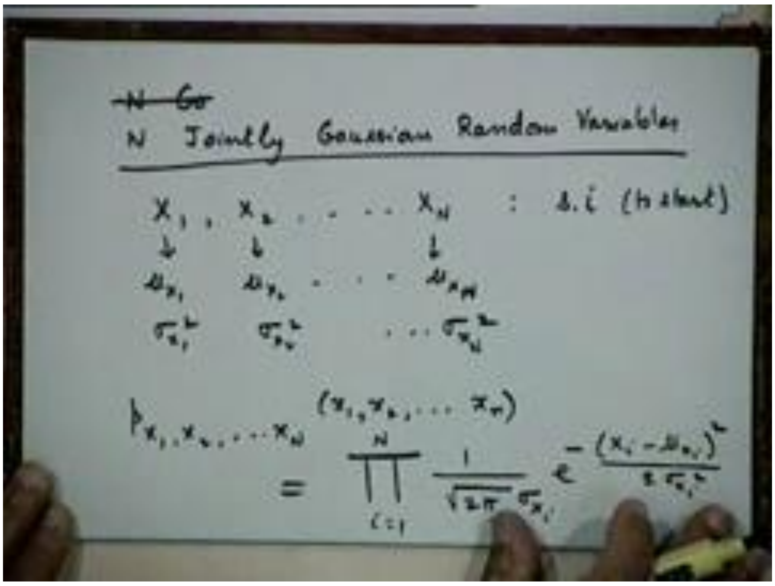
If you are generating Gaussian random variable at every time instant, this will have a same properties. So, it is of course, this is not a detailed prove, it just gives you idea of how to argue this out, but a detail proof is available in a book, I will write for yourself, so if X t is pass through a stable linear filter, then the output process Y t is also Gaussian. It is this a very interesting and useful property of a Gaussian process.

So, Gaussian process is not only, something that we can expect to occur in naturally to be found nature extensively, but as if they are very they have some very convenient properties in terms of the mathematical properties, which are therefore, in a useful when you are working with the mathematical models. The mathematical model should be easy to work with, in terms of this has some very nice properties, which makes it very easy to with.

So, the proof, leave it as an exercise, this stage that we come to that question which arise sometime ago, about how do you characterize the density function of a process, of a Gaussian process. Well, that equivalent to asking, how do you characterize the density function of N variables which are jointly Gaussian, is it not and that is the issue that very, because ultimately how do you characterize the density function of a process, you sample the process, generate and random variables and try to give the joint density function of this N random variables.

That is the most general characterization of any random process, if you can do that for every value of N, for every choice of the sampling instance, then you have a complete description. So, for a Gaussian process, so instead of talking about a Gaussian process, it is really more important talk about N Gaussian random variables.

(Refer Slide Time: 19:26)



N, jointly Gaussian random variables, how do you characterize their density function, let me start with a very simple physical picture, that say X 1 to X N or N Gaussian random variables, which are also statistically independent. In general they may not be, as I mentioned earlier, assumes they are statistically independent, to start with, only to start with, we will remove this assumption later, assumption of statistical independence will be removed later.

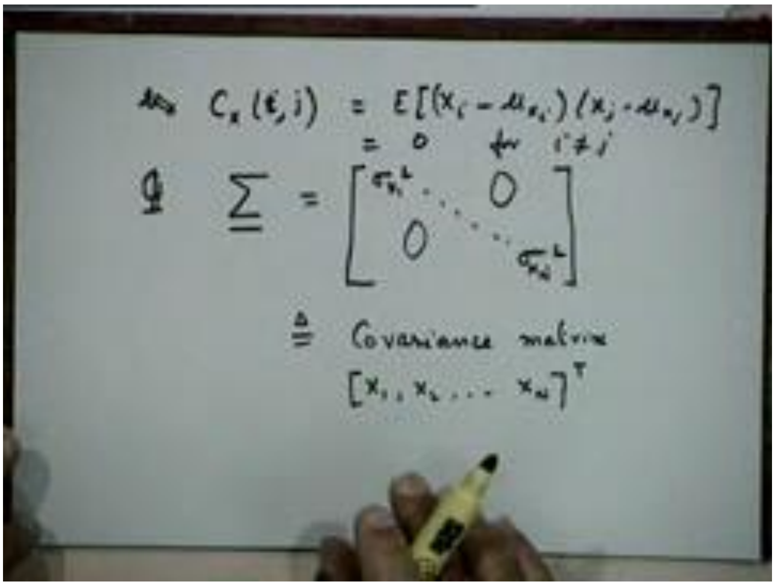
Now, let me say, that this is Gaussian or normal, with mean mu X 1, this as mean mu X 2 etcetera. Let us say the variance of this is sigma X 1 square, this is sigma X 2 square, this is sigma, this should be X sub capital N, X N square, now If I were to ask you, to write on the join density function of these, N random variables, is it easy to do or not, what is it, it is simply the, individually they are normal. So, what is the joint density function of N statistical independent normal random variables?

Student: ((Refer Time: 21:11)

Will be the product, is it not, so if I have to write, the joint density function of this N random variables, that is going to be equal to, the product of to going from i to N, i is equal to 1 to N of, 1 by root 2 pi sigma X i e to the power minus X i minus mu X i whole square upon 2 sigma X i square. This is easy, so if I have N jointly Gaussian, but statistically independent Gaussian random variables, writing a joint density function is trivial, you just multiply that.

Because, of the independence property, now what I want to do is, you move this assumption, but before I do that, let me try to rewrite this expression, in a manner which will help we to write the general expression, where I do not have to make the results, so basically that is the attempt that ((Refer Time: 22:37)). To do that, let me define a few things, that we define, I have to use what is called vector notation or matrix notation for writing this density function, let me define.

(Refer Slide Time: 23:01)



This is of course the definition already, that we define a variance between, the ith and jth random variables, as X i minus mu X i, expected value of X i minus mu X i into X j minus mu X j, that is the definition of the covariance. variance X i minus mu X i square, that is sigma X i square and I define C X or C, maybe sigma, this is the more commonly used definition. As a matrix, whose i j th element is C X i j, so if this N random variables are statistically independent, what is the nature of this matrix.

The i, j th element of sigma, this is the matrix, this is just a notation for a matrix which I am calling sigma, what will be, the i j th element is C x i j, what is the value of C x i j when these are independent?

Student: ((Refer Time: 24:20))

No, it is 0, this is a central limit, remember that?

Student: ((Refer Time: 24:26))

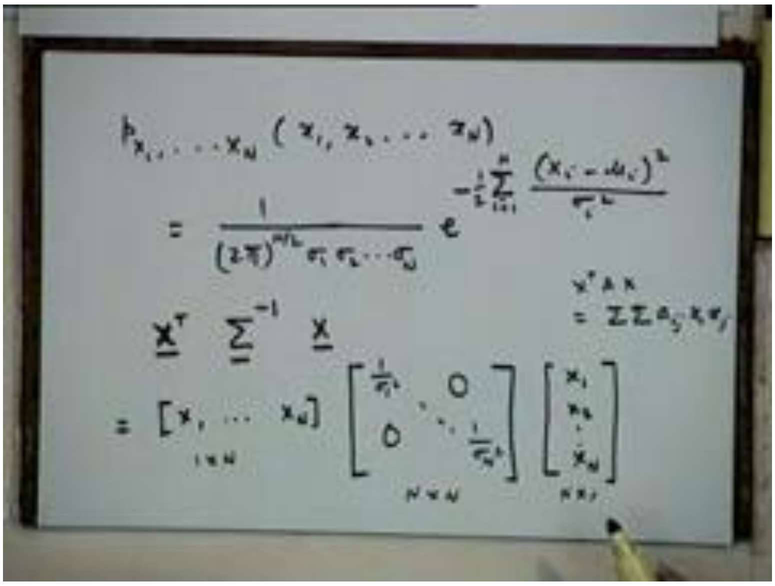
Yes, it is product, but each expectation itself is 0, is it not, the first movement the first central movement is always 0, so yes this is equal to, expected value of this into this, but each of the expected value is 0 so this is 0 for, i not equal to j. So, if I have to think of sigma as a matrix, whose i, j th element is given by this, what kind of a matrix is this, it is a diagonal matrix, is it clear. what are the diagonal elements, the variance is sigma X 1 square to sigma X N square.

If you drop this X 1 ((Refer Time: 25:09))this X 1 square it is not suffering any process, sigma 1 square to sigma N square and all the other elements are 0. This matrix is defined to be the covariance matrix associated with the N random variables X 1, X 2 to X N. In fact, let us think of this collection of N random variables, as a vector, you can represent a collection of N quantities as a vector, is it not, that we show this as a column vector, so I have column vector whose N components or the N random variables I am talking about.

And the covariance matrix therefore represents, the covariance between every pair of the components of this vector, the ijth pair, will have a covariance which is this and that is the i j th element of this. For statistically independent random variables, the covariance matrix would be diagonal, for random variables which are for vectors, whose components are not statistically independent, this will not be a diagrammatic, it will be a full ((Refer Time: 26:32)) matrix with non zero entries for all elements.

But, let us at the movement, come back to this, ((Refer Time: 26:43)) we had this, we want to write this expression now, in terms of these two entities, can you do that, more precise ((Refer Time: 26:55)) to if you ask you look at this exponent. Before we do that, I think we need to do one more step, I come back to this expression, this product I can take as a summation over this exponents, is it not.

(Refer Slide Time: 27:14)



I can write therefore, the joint density function as 1 by, if I multiply a root of 2 pi, N times with itself, I get 2 pi to the power N by 2, similarly sigma 1 to sigma N, so this will be a product sigma 1, I am drooping the X here for simplicity of writing, this will be sigma 1 to sigma N, multiplying sigma 1 to sigma N. So, that is, as for as the coefficient is concern, this gets modify to this form, what about the exponent part, this becomes e to the power minus summation over, i is equal to 1 to N of all these terms, is it.

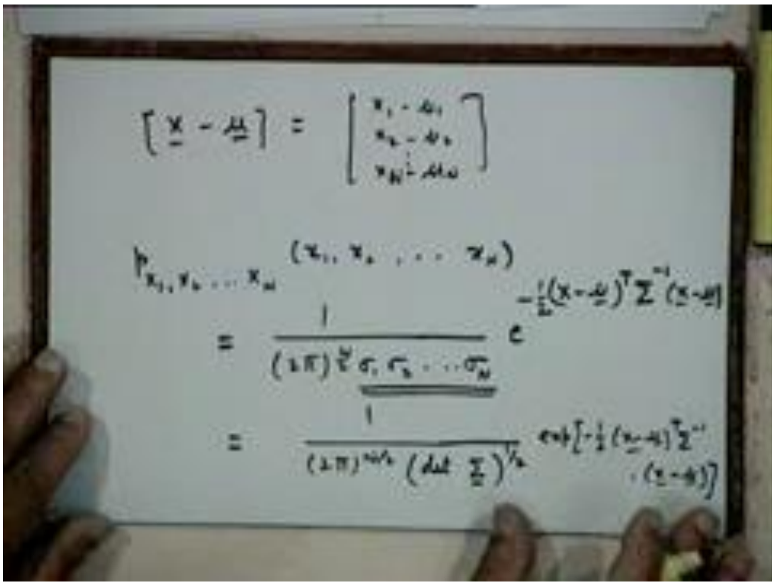
Maybe I can take the half outside, is it fine, I just drop the subscript X at everywhere for convenience, you will got this, so basically, this product here becomes the summation here. Now, try to imagine, writing this summation in terms of a vector matrix product of some kind ((Refer Time: 28:37)), can you see, can you express this summation in terms of these quantities, this matrix sigma and this vector X. Let me give a hint, consider this product, let me denote this as a vector X.

So, vector X is a column vector, whose elements are X 1 to X N, so what will be X transpose, what kind of vector is this, zero vector whose elements are X 1 to X N. Let me multiply that with sigma inverse, consider this quantity, X transpose sigma inverse X, this is equal to X 1 to X N, inverse of a diagonal matrix is very easy, what is it, all the diagonal elements will become you will take the reciprocal of those, will be 1 by sigma 1 square to 1 by sigma N square, 0 is here and what you get here, X 1, X 2 to X N.

Now, can you tell me what this product will be, this is actually what is called quadratic form of a matrix A, you know things like X transverse A X, then X is a vector is call the quadratic form of A and this actually can be written as a double summation a i j x i into x j, is very easy to prove this, we just we have to multiply this matrices. In fact, it is obvious, what this will be, you would not you have a double summation because, the diagonal element only diagonal elements are non zero.

If you have to expand in multiply this row vector with this matrix and this resulting matrix for this column vector, actually you will get a scalar, is it not. This is 1 into N, this is N into N, this is N to 1, what is the overall product like, it is a scalar, 1 into 1 and the value of the scalar is 1 by sigma 1 square into X 1 into X 1, plus 1 by sigma 2 square. So, actually this term whereas precisely use, So that I am not multiplying with X 1, it should become X 1 minus mu 1, this should become X N minus mu N.

(Refer Slide Time: 31:26)



So, if I want to define the vector, X minus mu as this vector X 1 minus mu 1 X 2 minus mu 2 etcetera. Now, you can see, how to write their expression, a joint Gaussian density function, ((Refer Time: 32:01)) especially becomes 1 by 2 pi t power N by 2 sigma 1, sigma 2, sigma N and you have e to the power minus X minus mu transpose, sigma inverse into X minus mu, this sigma is not summation, this is a notation for the covariance matrix and half here.

Student: ((Refer Time: 32:35))

No, that is taken care of by sigma i sigma t, is it not, ((Refer Time: 32:42)) let precisely what sigma inverse is, is that, everyone. Please see, you just what we are saying is at this quantity is equal to this, except that I am not put the mu i is here, there the same thing. So, I am defining X minus mu, instead of X 1 instead of the vector X, I take the vector X minus mu here, the vector X minus mu here and you got this.

Let me, I think simplify this further, can you discuss this product in terms of this matrix sigma, remember what is sigma, sigma is this matrix whose diagonal elements are sigma 1 square to sigma N square. So, can you identify sigma 1 into sigma 2 into sigma N something relating to this matrix, some property of this matrix?

Student: ((Refer Time: 33:33))

It is varying that the determinant square root of the determinant of this matrix, so I can write this as further as 1 by 2 pi, in a more complex form to the power N.

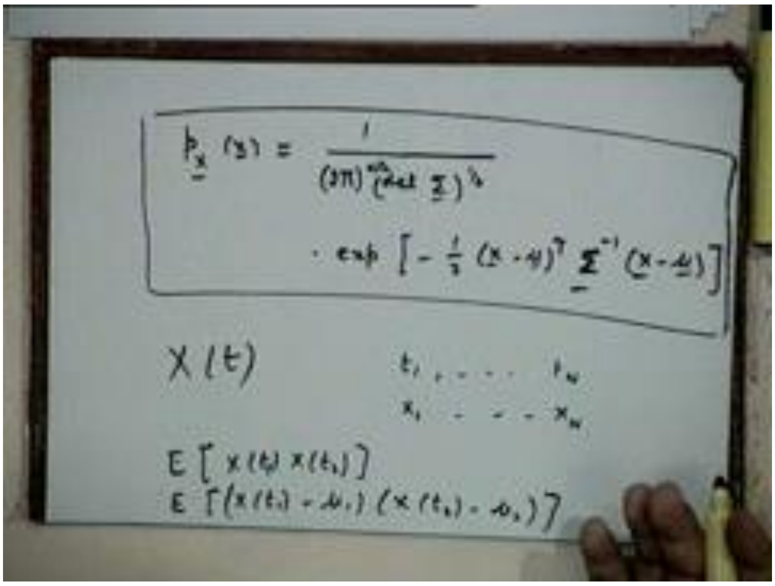
Student: ((Refer Time: 33:48))

Yes, yes this should N by 2 and this we can write as determinant of sigma, square root of that into exponential minus half etcetera. So, here is the general expression, I hope you are with the, if I have N jointly Gaussian random variables, which as statistically independent, I can write the simple looking expression by a more complicated looking expression like this and why should I do that, why should I go into these vector matrix notation.

Because, now I can just generalize this one, any set of random variables, whether they are independent or not, because the covariance matrix will capture the information about the statistical dependencies and this becomes the special case and sigma becomes diagonal, this becomes a special case. So, that is what you need to know, so an even this joint density function, I do not have to write like this, I use the vector notation and get a compact notation for that, P X X.

This simply write that, it means this, I have a vector X, I am writing the density function this vector X which essentially means, the joint density function of a it is components. So, when I talk about the density function of a vector, what it really means is, we talking about the joint density function of it is components, is it clear, this is a notation, nothing else.

(Refer Slide Time: 35:46)



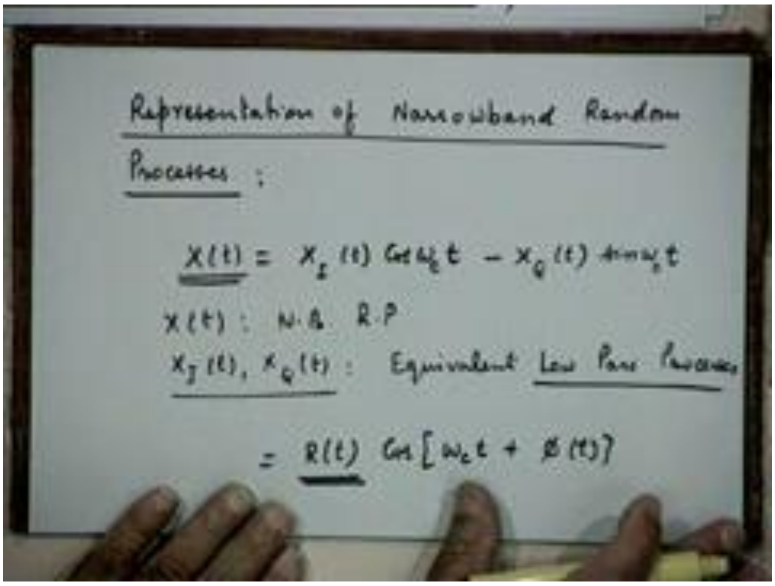
So, that is the most general form, for the expression less movement expression for the jointly Gaussian random variables, 2 pi to the power N by 2, determinant of sigma to the power half into exponential of minus half X minus mu transpose sigma inverse X minus mu. So, if I now give you a Gaussian process X t, can you characterize it, yes I can, because I know that, if I choose some sampling instance t 1 to t N any ((Refer Time: 36:41)) that is N is 1 or 2 or 3.

They will be an associated set of random variables X 1 to X N and what I need to know the corresponding density function is, their mean values and their covariances. If I have all that information, then I can write this density function and the covariance information, can we obtain from the auto correlation function, is in it, because what is auto correlation function, it is X t into X t plus tau or X t Y into X t 2. The covariance corresponding covariance information will be, expected value of X t 1 minus mu 1 into X t 2 minus mu 2.

If I know this and if I know the mean values ((Refer Time: 37:36)), so if I know the auto correlation function, I can write the density function for any set of N random variables that I may encounter in a Gaussian process, or that I can generate from a Gaussian process. The information in the auto correlation function is sufficient to write the complete density function, which is a manifestation of a fact which we earlier know, that a Gaussian random variable is completely characterize in terms of, it is first two movements.

Similarly, Gaussian process is completely characterize in terms of the mean value function and the auto correlation function, because from these two informations, I can generate all the information that is necessary, writing any density function of this kind. Are you all with me, I think that is sufficient information for Gaussian process to start with, let me wind up this discussion on Random processes.

(Refer Slide Time: 38:44)



By just one last result about representation of narrowband Gaussian processes, narrowband processes random processes, which is the straight for extension of our understanding of representation of narrowband signals, narrowband deterministics. You remember, what the result there was, how do you represent the narrowband signal?

Student: ((Refer Time: 39:23))

We had a, you are develop a quadrature representation, every narrowband signal could be express in terms of two quadrature components, which are low pass signals and the corresponding center frequency. So, this result can be, and this is an important issue to work, this is an issue important issue for us, because you know, I mentioned earlier that most of the time the random signals that we will work with, which is essentially, typically noise, will be wide dense signals.

We talk about white noise, which is very white bandwidth, but it is not always any work with wide dense signals. Suppose, the same white noise that is coming along with the signal, is pass through a filter, that is a narrowband filter and you communication receives a full of band filters, is in it, filters to into some frequency f c. Now, what will be the nature of the process noise of the output of this, it will not be anymore wide band, it was wide band at the input, but at the output of the filter, it will be a narrowband filter narrowband noise.

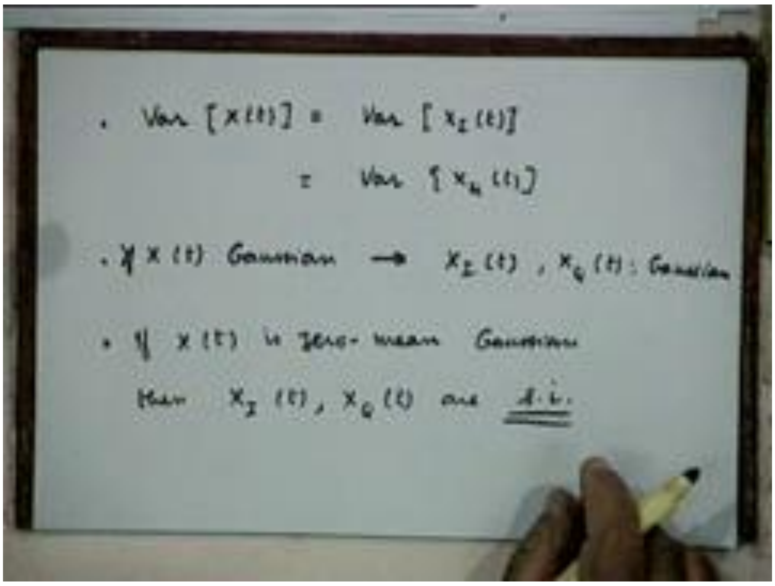
So, you need to invoke this representation, so just like you had for a deterministic signal, a quadrature representation of signal for very first narrowband, you can have a similar quadrature representation for the narrowband noise. That is, we can express X of t, in terms of a in phase component X I t, multiply with cosine omega c t plus or minus the quadrature component X sub Q t into sin omega c t, where x of t is a narrowband noise or narrowband random process and X sub I and X sub Q would be equivalent low pass processes.

That is, the information about X of t is really containing in these two quadrature components, is it, of suitable bandwidth. It will be maybe I should give you a couple of exercises in this connection, I like you to, show for example, that you see, first of all appreciate that, this is a rapidly fluctuating process and these are slowly fluctuating process, you can simply also write this in terms of envelop and phase. So, there is an envelope process, you know R of t into cosine omega c t plus pi of t.

You can also think of this as this, your pi of t would be, tan inverse of X Q upon X I and R of t would be square root of X I square plus X Q square. So, you can think of this as an envelope random process, a random process we describes the envelope of the narrowband signal and this is, the phase process which describes the, instantaneous phase of the narrowband noise and these are important representations, because when we discuss either amplitude modulation or phase modulation and it is the impact of noise from that, then it is obvious that, this as well as this representation very useful.

Now, a few things are you likely find out is, show that the average power of each of these three processes, namely the original narrowband signal and the quadrature representations, the two quadrature representations, they have the same every power that is the same, they have to same variance.

(Refer Slide Time: 43:51)



So, please show that, the variance of X t, is equal to the variance of X I t, is equal to the variance of X Q t also, this is one property. The second property is, I think maybe for this property they ((Refer Time: 44:21)) would scaling factor is involved, or it is considerably modify this definition by maybe a 1 by root 2 have to be involved. So, please check of, let me know whether this is fine or whether you need to multiply this with root 2 or something.

Basically, this is some there is a normalize representation. So, that this result was, if this any scaling, please work it out and let me know, the second result is, if X t is a narrowband Gaussian process, so I am not saying just narrowband, I am saying narrowband Gaussian process. If X t is Gaussian, then both X sub I and X sub Q will also be Gaussian, please work out in argument suitable argument for this and as a further special situation if X t is zero mean Gaussian, that is the mean value function is 0.

Then, the quadrature components are statistically independent, so these are some of the properties, please learn the proof of these properties as well as other additional properties which are done here, which are available in the book. So, because of every time, it is difficult for me to cover random processes exhaustively, but I think, I have covered sufficiently exhaustively for us to, now will comfortable with it, provide you do enough problems from the book and we are ready to use them now. For as models that we need to use for modeling noise and communications system, as studying the effect of noise and communication systems. So, I stop here and please review this quickly, so that we can get on their job.

Thank you very much.