Advanced Digital Signal Processing
Wavelets and Multirate
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Lecture No # 18
Time Bandwidth Product Uncertainty Bound

A very warm welcome to the eighteenth lecture, on the subject of wavelets and multirate digital signal processing. We build further in this lecture the idea of uncertainty and what is called the time band width product. The product of sigma t squared and sigma omega squared has been defined them in the last time. Now, we shall just recall a few ideas from the previous lecture to put our discussion in perspective. So, what we intend to talk about today is what is called the time bandwidth product, essentially as I said the product of the time variance and the frequency variance. And, we wish to build a lower bound on that product based on some very fundamental principles that could be explained when we look at functions as vectors. So, with that background, let us recall some of the definitions that we had brought out in the previous lecture.

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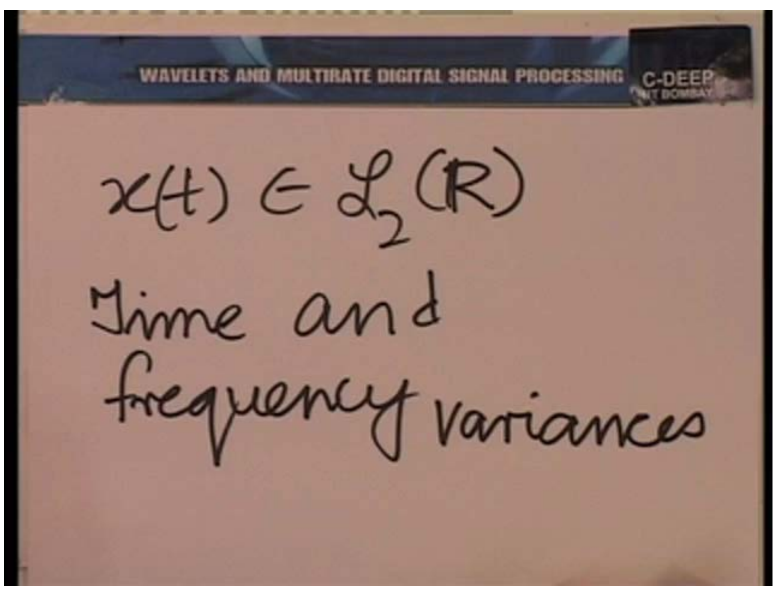

Last time, we had agreed to confine ourselves to a function x t belonging to L 2 R. And, we said we could define it as time and frequency variances. In fact, the time variance was first defined by choosing what is called the time centre.

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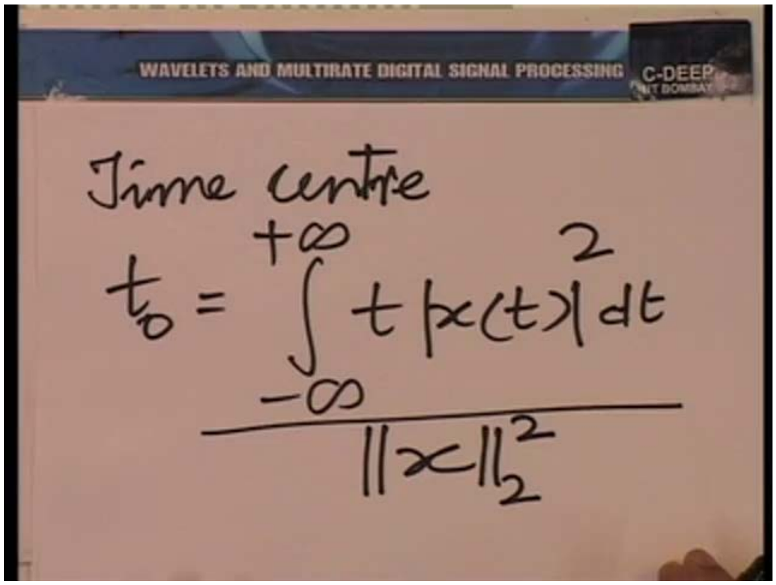

The time centre t 0 is defined according to this. Essentially, treating x t or mod x t squared as a mass in terms of t and then looking for the centre of mass. That is the interpretation of this. And, we assumed the centre was a finite number. I mean, that is the reasonable assumption in almost all cases. That we will talk about.

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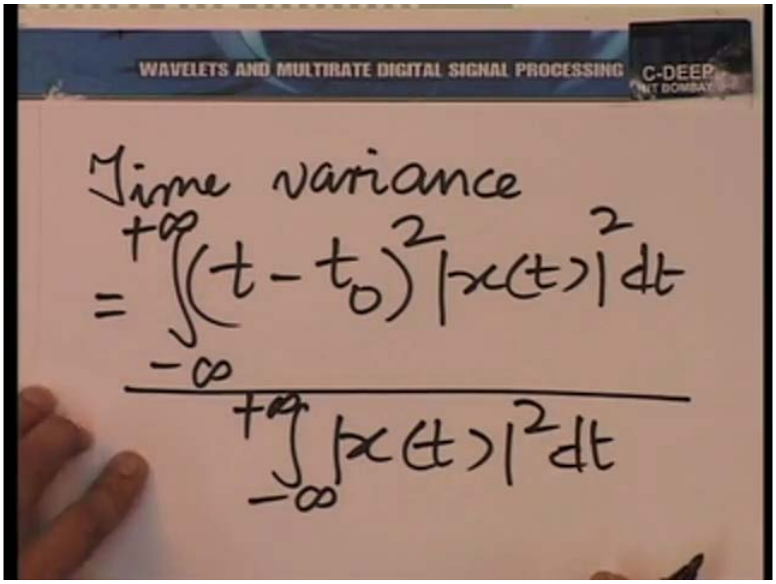

From the time centre, we define the time variance. So, time variance is given by t minus t 0 the whole square mod x t square d t integrated over all time divided by the integral of mod x t squared. Essentially, this time variance was indeed the variance of a density, which we constructed out of that mass; mod x t squared divided by the norm of x the whole squared. Now, similarly we could talk about the frequency centre and the frequency variance.

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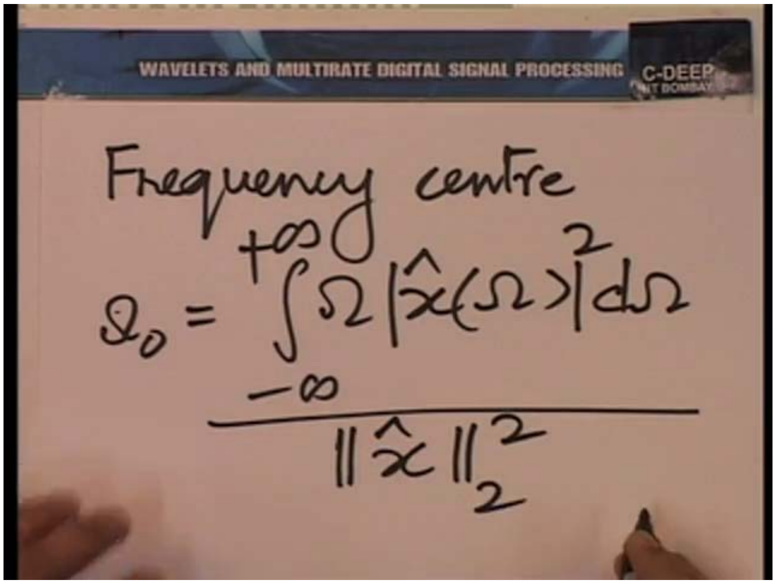

So, we defined the frequency centre to be capital omega 0 given by minus to plus infinity integral omega mod x cap omega the whole squared divided by the norm x cap squared. Again, we assume the frequency centre as finite, which is of course almost always the case. There is no problem and we also noted that for real x t, this frequency centre is 0.

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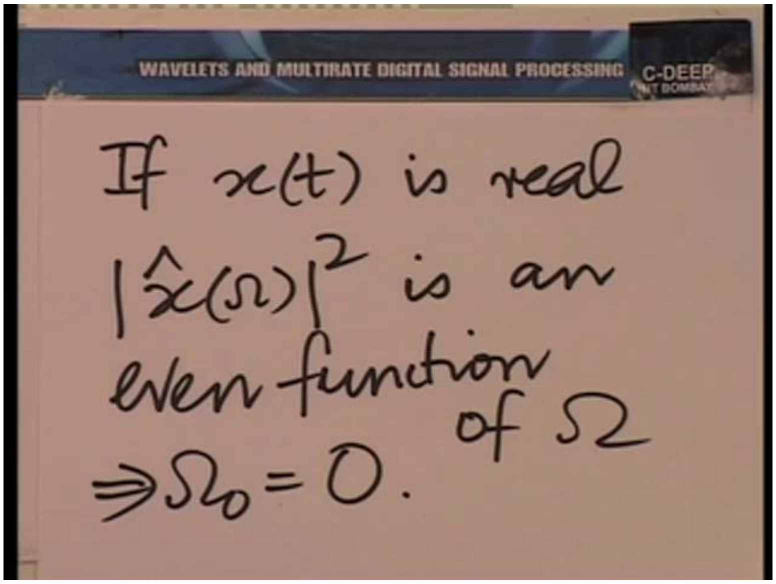

So, we said if x t is real, then mod x cap omega squared is an even function of omega. And, that implies that omega naught is 0 because of symmetry. Now, we could similarly define the frequency variance.

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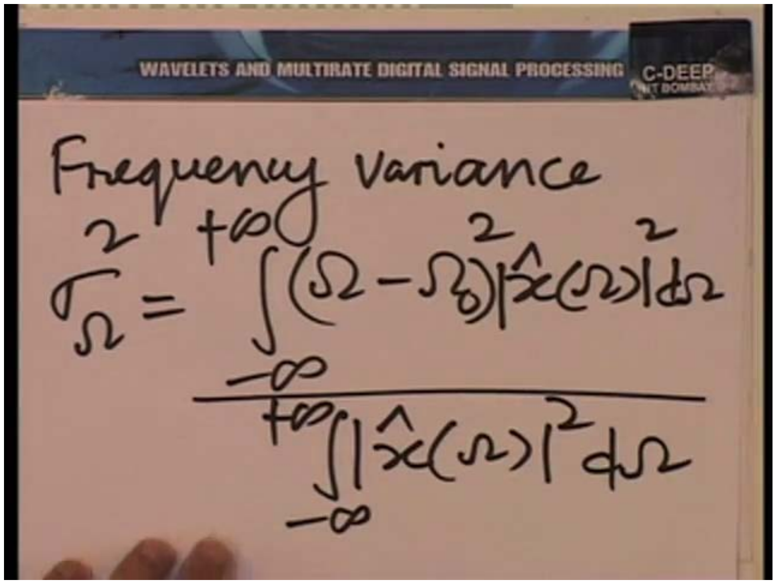

So, frequency variance sigma omega square; so, to speak, essentially integral omega minus omega naught squared modulus squared of x cap omega divided by a similar integral of mod x cap squared. So, once again the frequency variance is like a variance on the density constructed out of the modulus squared of the Fourier's transform.

Now, let us see what happens when we shift a functioning time. As we expect, when we shift a function in time there is no change in the magnitude of the Fourier transform. In fact, all that happens is at the centre shifts.

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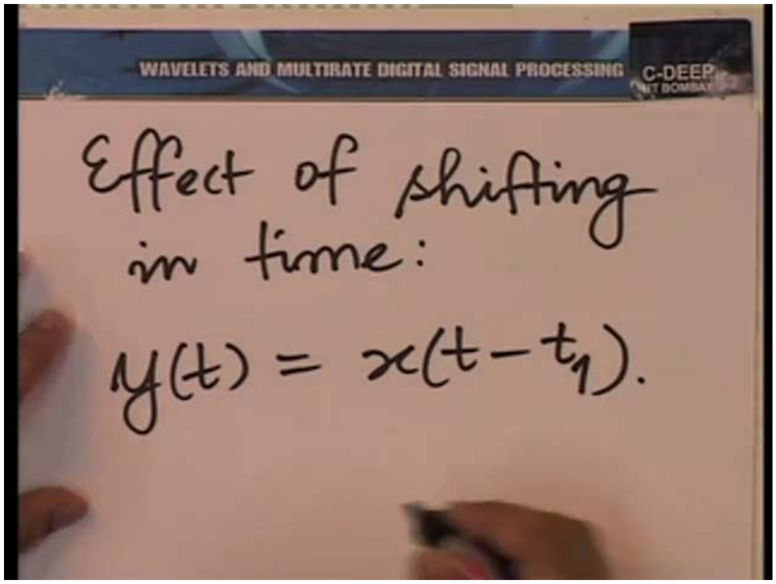

So, effect of shifting in time. So, let us consider y of t to be x of t minus t 1. Now, we can easily write down the integrals corresponding to y t. And, they are not very difficult to evaluate. I shall not go through all the details of working, but I shall straight away put down some of the important results here.

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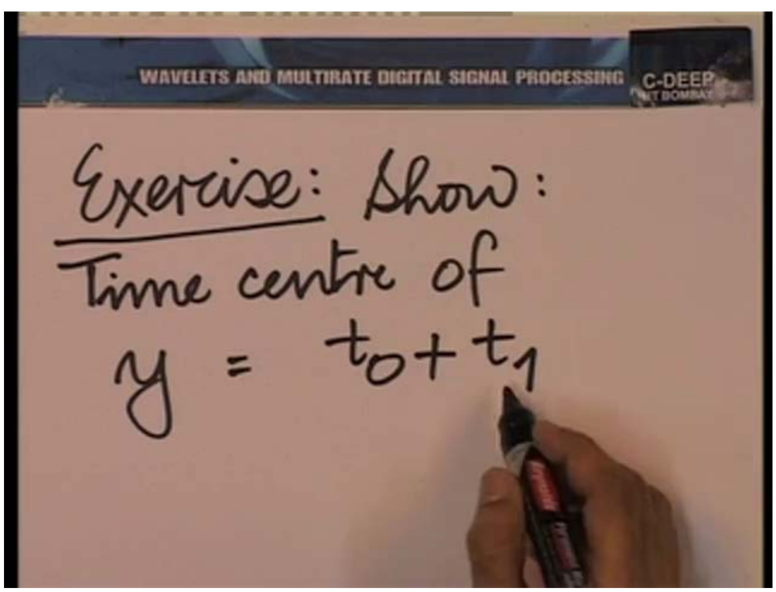

In fact, I put this as an exercise. We can show that the time centre of y would essentially be… you see I take an example. Suppose, for example, you have a function centred at the point t equal to five. So, t 0 is equal to five. Now, if you shift the function backward by five units in time, you would get a function centred at zero. That is easy to visualise. And, if one just writes the integrals down carefully that is easy to prove.

So, essentially what it means is that, when you shift backwards by the centre, you are bringing it to be centred around zero. And, with that, we can generalise saying that the time centre of y; essentially, t 0 plus t 1. So, for example, if t 1 is equal to minus t 0, then the centre of y would be zero. Anyway, so one can show the time centre of y 0.

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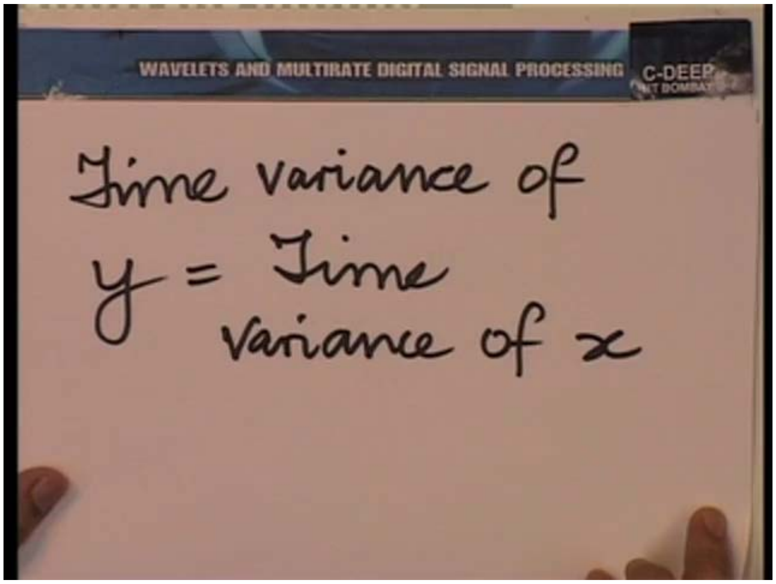

The time variance of y is equal to the time variance of x. Again, this is a very easy result to show just amongst writing the integrals down carefully and making a transformation of variable. So, I would not go in to the full proof here. Similarly, we can now analyse what happens in frequency. What happens in the Fourier transform, let us look at that first.

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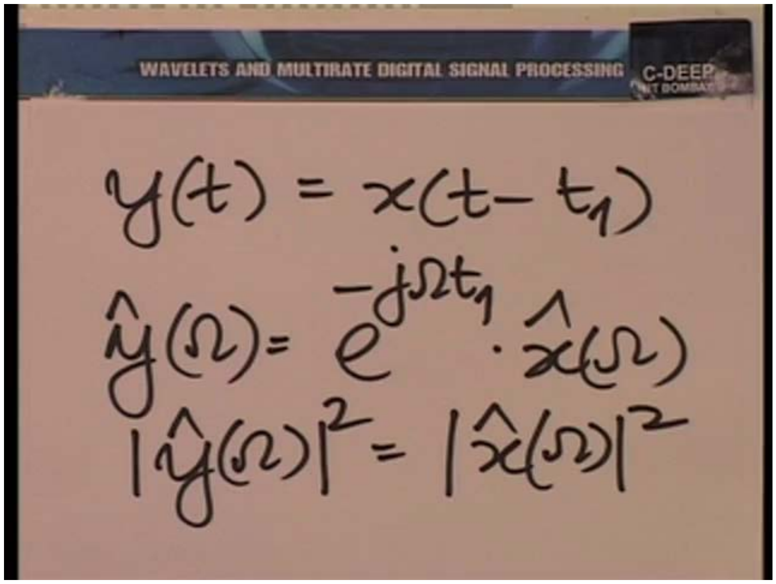

See, if y of t is equal to x of t minus t 1, the Fourier transform y cap omega is e raised to the power of minus j omega t 1 times the Fourier transform of x. And, very clearly mod y cap omega the whole squared is equal to mod x cap omega the whole square. That is very easy to see here. So, you see under a shift in time, the magnitudes square, magnitude of the Fourier transform is unchanged, the mass as seen on the frequency axis remains unchanged in the Fourier domain.

So, you can shift as much as you desire and it has no effect on the Fourier transform magnitude squared. As you can see it effects the phase, but not the magnitude. And, the phase is not of consequence to us, as far as the centre and the variance are concerned. So, anyway with that, you understood what happens when we shift. Now, let us take the dual operation of shifting, namely modulation.

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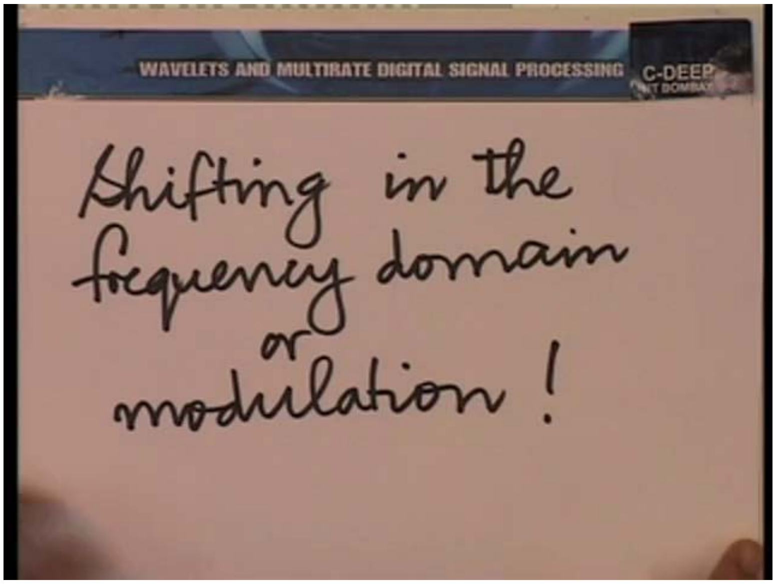

So, in fact let us look at it, as I shift in the frequency domain. So, let us consider y t to be a modulated version of x t.

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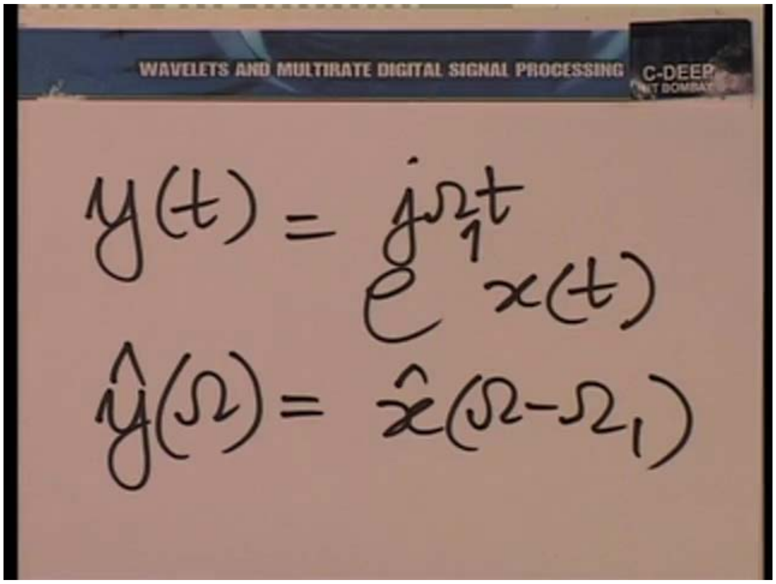

So, let us consider y t equal e raise to the power j omega 1 t times x t. Now, of course it is very easy to see the Fourier transform, y cap omega is x cap omega minus omega 1. You will recall that this idea is used in amplitude modulation or for that matter in frequency modulation in Communication Engineering.

When we shift a signal on the frequency axis, we are modulating in time. Now of course here we are talking about modulation by a complex exponential. If we modulate by a sine wave, we are essentially modulating by two complex exponentials and adding up these two modulated signals. Just, to put the idea of modulation in perspective. Anyway, coming back to this; so, when we modulate with e raise the power j omega 1 t, we essentially shift on the omega axis as we have done here. So, y cap omega is x cap omega minus omega 1.

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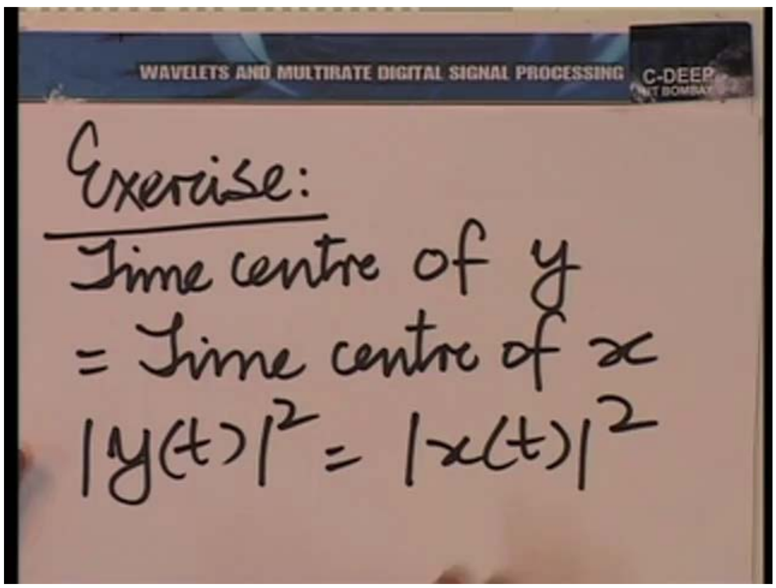

And, of course it is again easy to work out the following exercise. The time centre of y is equal to the time centre of x. In fact, that is very easy to see mod y t squared is equal to mod x t squared. And, first as far as that time domain is concerned, as far as the variable t is concerned, there is no change in the mass that is placed on t. So, it is not surprising that the time centre is unchanged and also the time variance, which we will now show.

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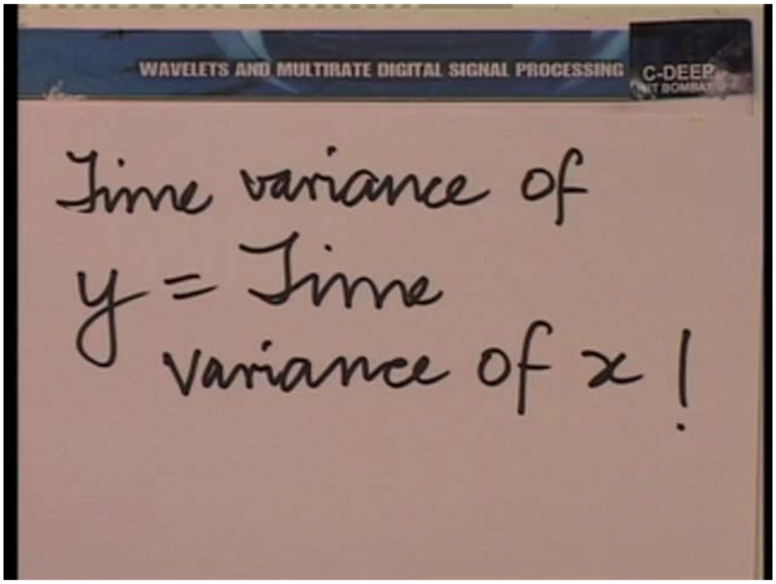

The time variance of y is equal to the time variance of x. What about the frequency variance and the frequency centre? That goes exactly dual to what happened, when we shifted in time.

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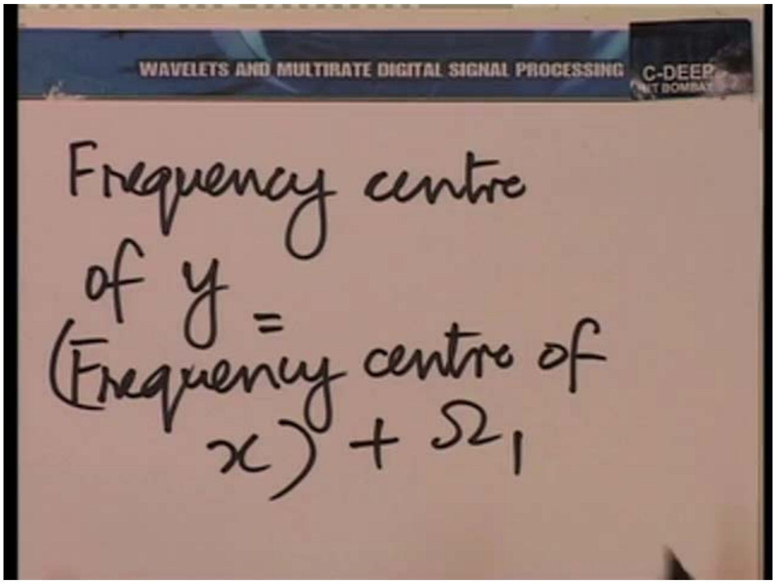

So, frequency centre, the frequency centre of y is now going to be the frequency centre of x plus omega 1. And, put a bracket there. And, what happens to the frequency variance? While shifting on the frequency axis, has no effect on the variance. Again, I leave it to you to show this by writing down the integral carefully. It is a simple exercise. I shall put down the final result.

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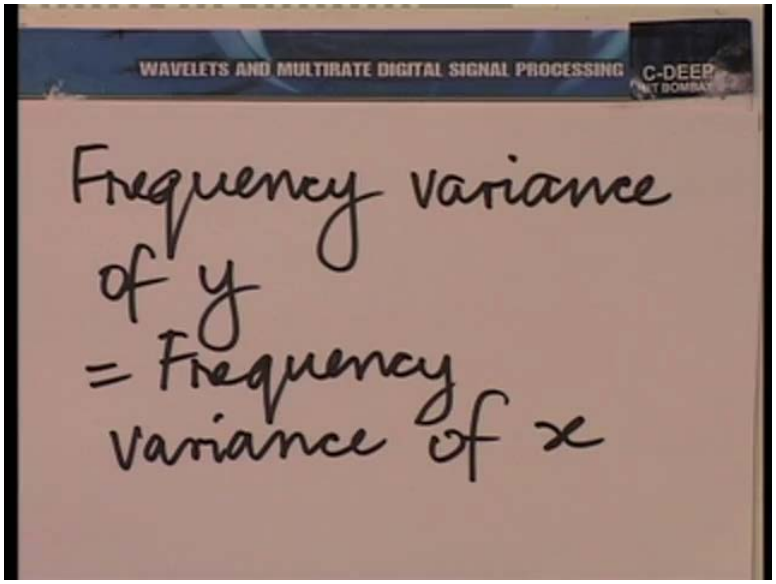

So, the frequency variance of y is equal to the frequency variance of x. So far, so good. So, we have taken two dual operations; the time shift and the frequency shift. And, both of them we have had no change in the variances. And, obviously when there is no change in the variances, there is going to be no change in the product of the variances. So, the sigma t squared sigma omega squared product, which we shall now give a name; we call it as the time bandwidth product.

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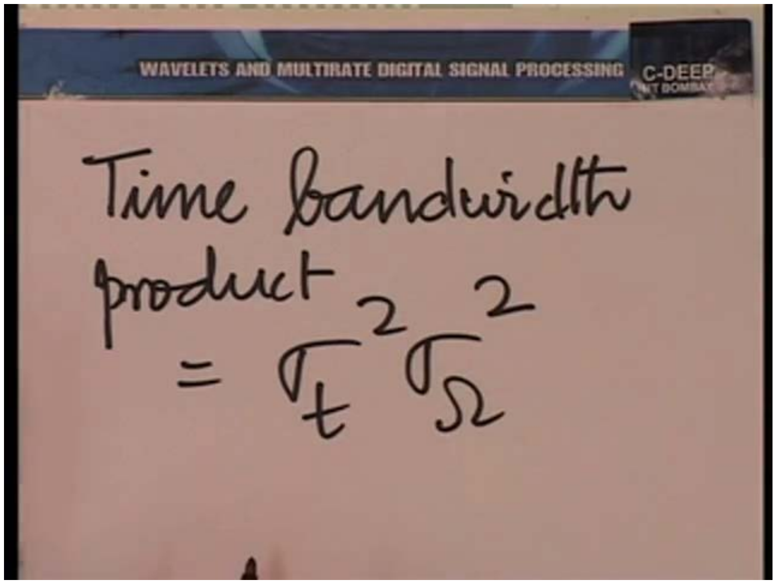

So, we introduce the term for time bandwidth product, given by the product sigma t squared times sigma omega squared.

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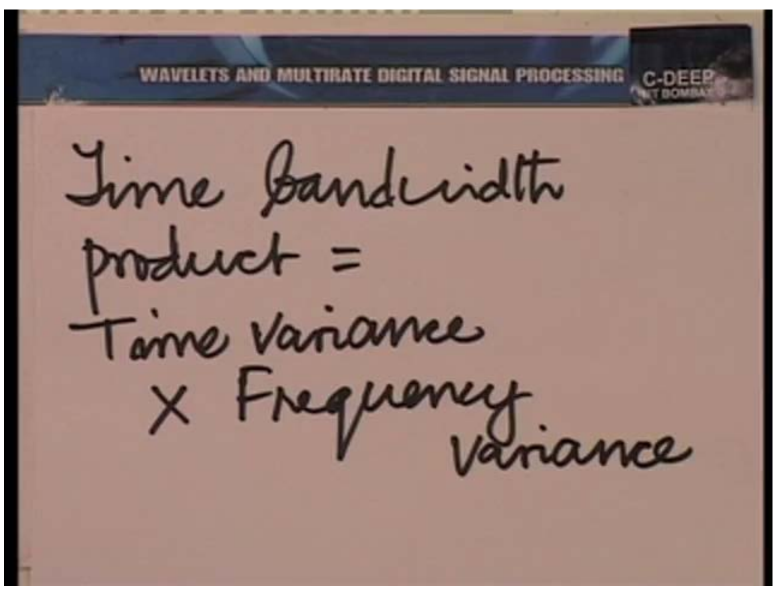

Essentially, the time bandwidth product is the product of the time variance and the frequency variance. And, this product is very important from a number of perspectives. As you can see, there is a strong invariance that this product exhibits. When you shift a function in time or when you shift it in frequency; in other words, modulated in time, this product is unchanged.

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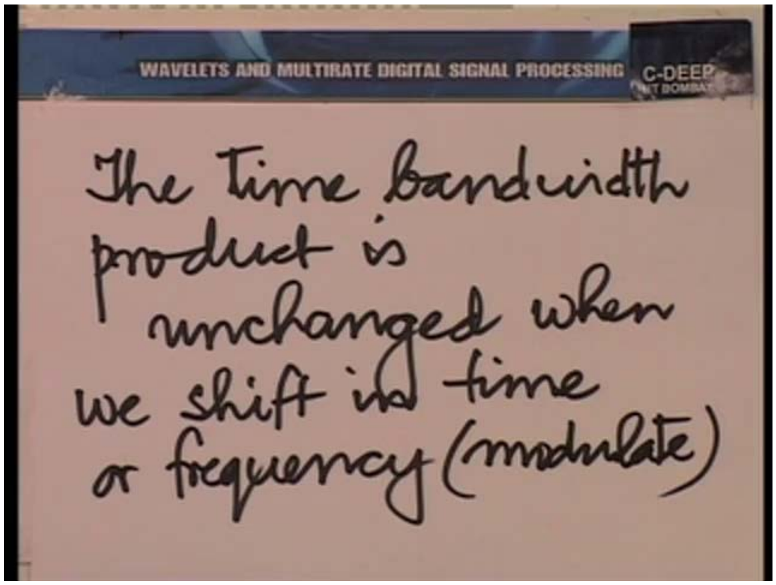

Let us make a note of that. The time bandwidth product is unchanged, when we shift in time or frequency. And remember, shifting in frequency is modulation in time. Now, we would like to look it this product in a little greater depth. What happens to this product when we stretch or compress. You see in the wavelet transform or for that matter in all he discussion that we had in some of the previous lectures, we have talked about moving along the frequency axis by a process of dilation.

We saw that, when we stretch the compressed function, the band pass function, it is equivalent to moving that function on the frequency axis; moving in a constant quality factor fashion keeping the ratio of the centred frequency to the bandwidth a constant. Now, in this process of compression or expansion together called dilation, what is happening to this fundamental quantity the time bandwidth product; is the question that we would now like to ask and answer.

So, the first thing that we observe is that you know, if you look at these let us… let us look at that process. So, suppose you have a function, you multiplied by a construct; that is a trivial thing. We will settle that matter first. So, we had, see here, we are talking about scaling the independent variable. When we stretch or compress, we are talking about scaling the independent variable. What happens when you scale the dependent variable? So, if you multiply the function by a constant. You must answer that question first.

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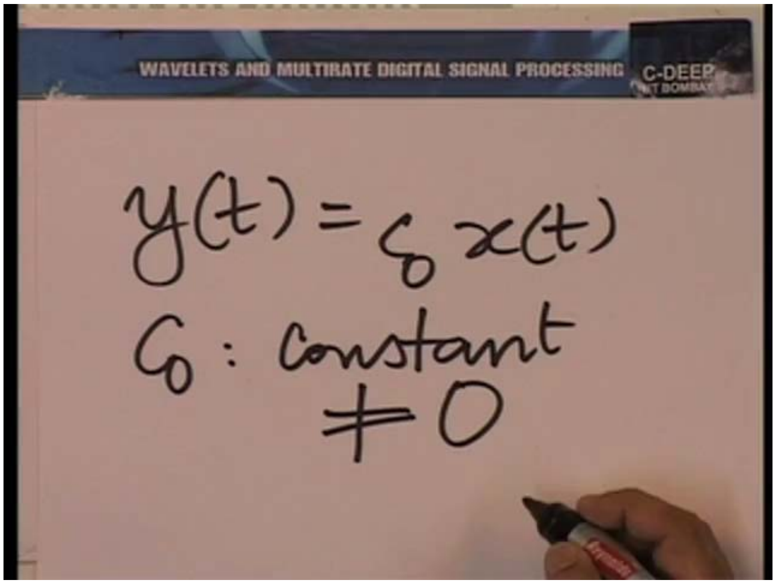

So, again it is very easy to show that if y t is equal to some constant, let us say C 0 times x t, C 0 is the constant. Of course, unequal to zero; that is straight forward. So, C 0 is a constant.

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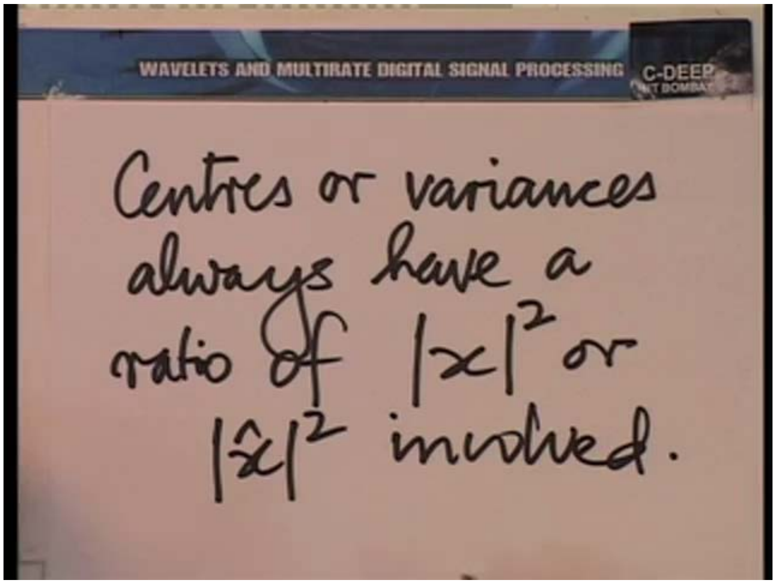

It is very easy to see that in the process of defining the centre or the variance, the centres or variance always have a ratio of mod x squared or mod x cap squared, involved.

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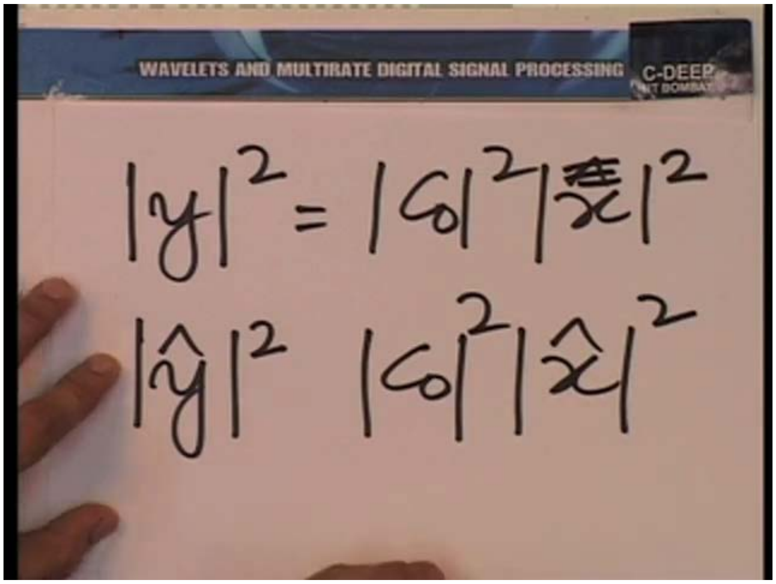

Now, you know, if y t is C 0 times x t mod y squared or for that matter, mod y cap squared is going to be mod C 0 squared times mod x squared. I am sorry, here it is just x.

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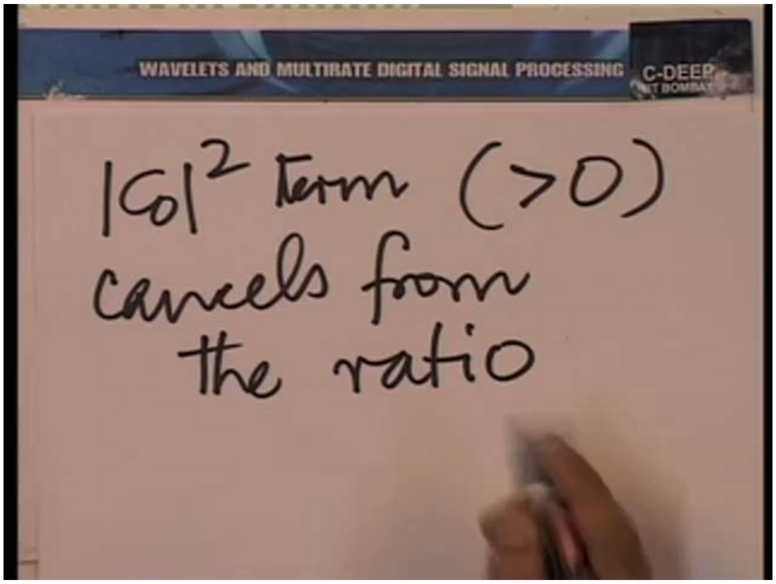

So, here to its mod C 0 squared mod x cap squared. Now, when we take a ratio, this mod C 0 squared being non-zero is going to cancel from the numerator and denominator. And therefore, when we scale the dependent variable there is no effect on the time centre or the frequency centre. Let us make a note of that. The mod C 0 square term, of course greater than 0 strictly, cancels from the ratio.

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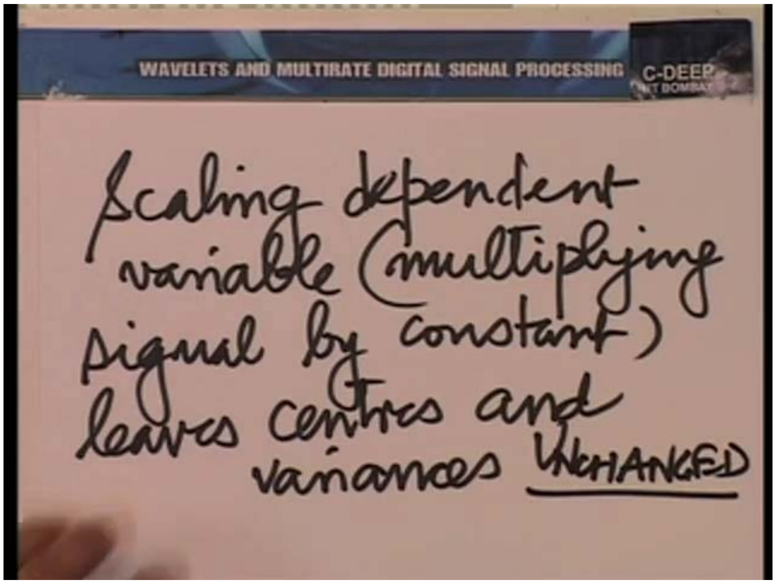

And therefore, scaling the dependent variable or essentially multiplication by a constant, multiplying the signal by a constant leaves the centres and variances unchanged. This is important. Scaling the dependent variable leaves the centres and the variances unchanged. And, now the most difficult of them, all in yet elegant and beautiful. What happens when we scale the independent variable?

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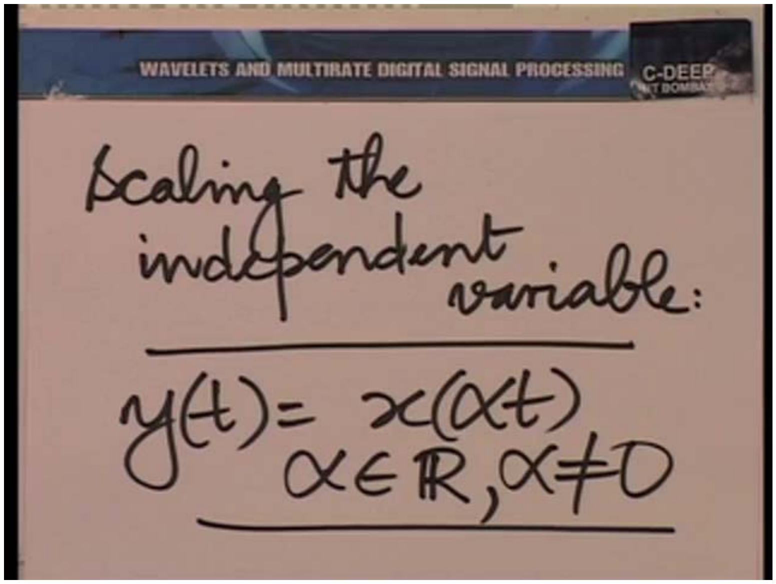

In other words, let us consider y t equal to x of alpha t. and, alpha is the real number. Alpha is not equal to 0. Now, we know what happens to the Fourier transform in this.

So, you will recall that when we scale the t variable by alpha, the Fourier variable capital omega is scaled by 1 by alpha, but that is also one by mod alpha factor that emerges outside.

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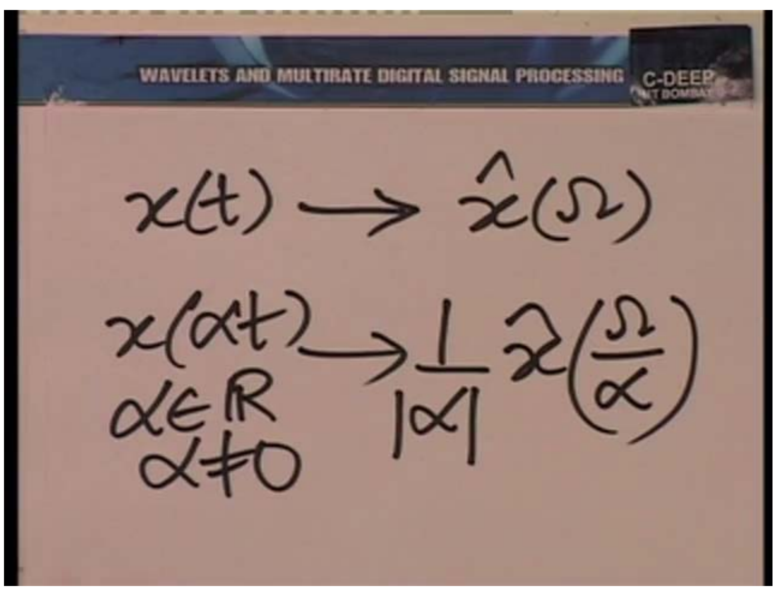

So, what we saying is, if x t has the Fourier transform x cap omega and x of alpha t with alpha real number, alpha not equal to 0 has the Fourier transform1 by mod alpha x cap omega by alpha. Here, it is alpha; here, it is mod alpha. So, in fact it is adequate for us to see what happens to one of them, the time or the frequency domain. And, the other one can be interpreted. Let us take the time domain for simplicity.

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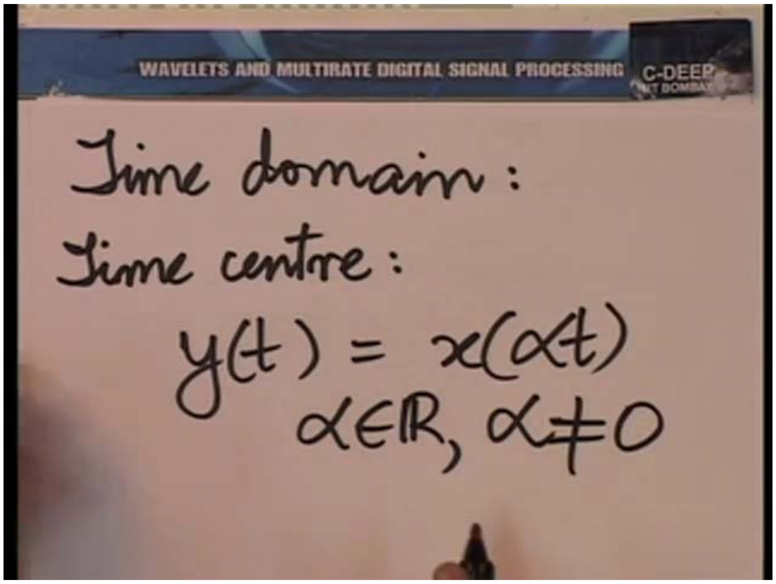

So, consider the time domain. Let us see what happens to the time centre. Let y t be equal to x of alpha t; as usual alpha real, alpha not equal to 0. And, let us ask what is the norm of y t, norm squared of y t in the L 2 R sense against the norm of x.

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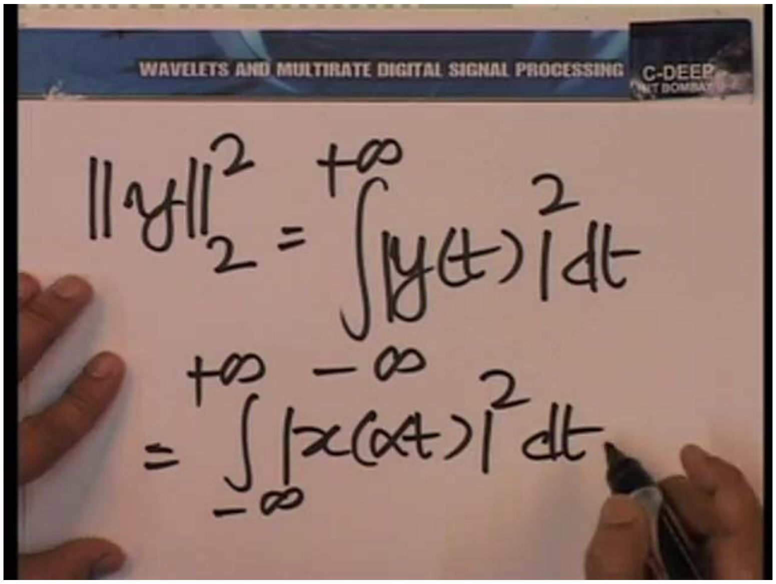

So, the L 2 norm of y is this. L 2 norm squared I mean and of course, this is easily seen to be mod x alpha t squared d t. Now, it is an easy integral to evaluate. All that we need to do is to make a transformation of variable.

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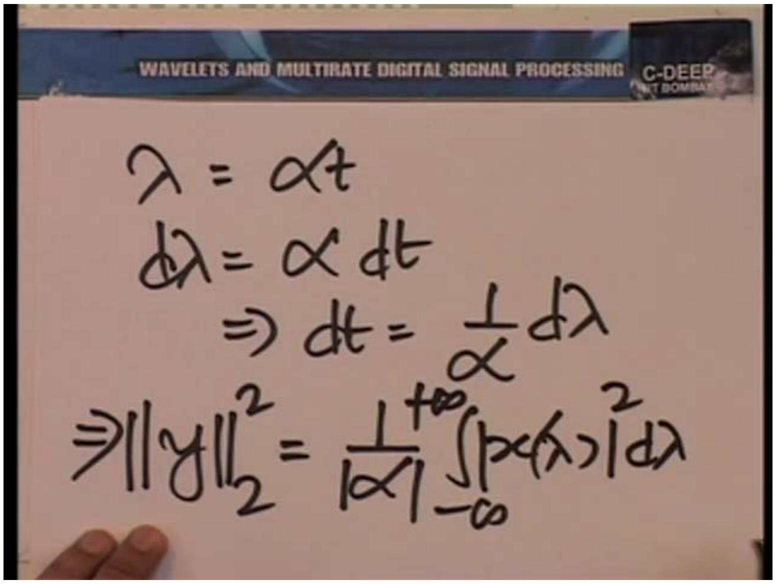

So, we have lambda is alpha t, where upon d lambda is alpha d t. And therefore, d t is 1 by alpha d lambda. Now, here in the integral you see, let me just bring the integral back here, the relevant integral.

So, we are going to replace d t by d lambda by alpha. And, depending upon whether alpha is positive or negative. If alpha is positive, the limits still remain from minus to plus infinity. If alpha is negative, the limits go from plus infinity to minus infinity, but then there is also l by alpha negative there. So, 1 by alpha is negative. And therefore, if you take the negative and the reverse integral together, all in all we always have the following. The norm of y in the L 2 R sense is always 1 by mod alpha times integral from minus to plus infinity x lambda mod squared d lambda.

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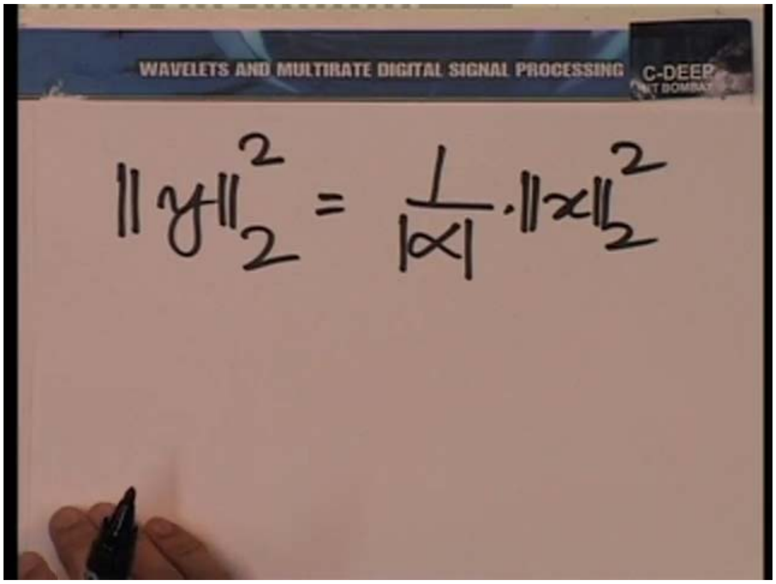

Where upon, what we have concluded is norm y in the L 2 R square sense squared is 1 by mod alpha times the norm of x in the L 2 R sense squared. Now, of course we can use the same reasoning. I have illustrated the thought behind the reasoning. And, if we write down the time centre and the frequency centre expressions, let us write down the time centre. . As we said we are going to focus on time.

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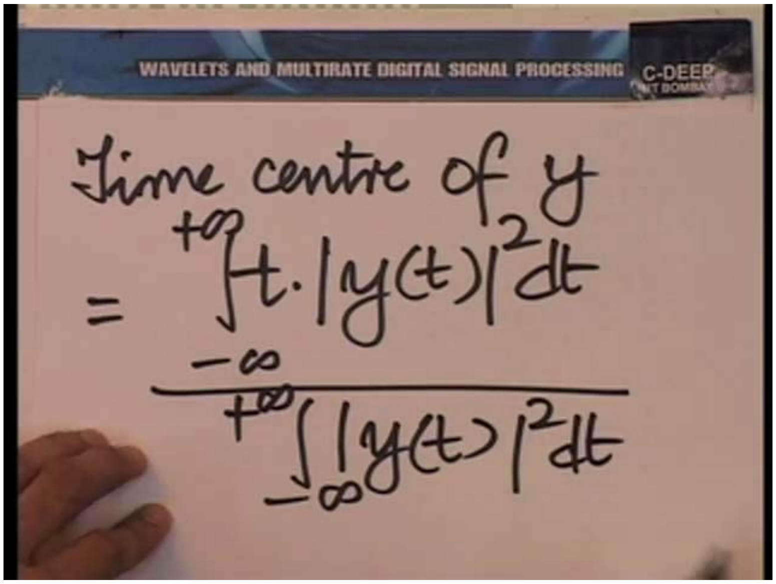

So, let us write down the time centre expressions. The time centre of y is going to be given by t times mod y t squared d t integrated, divided by integral mod y t square d t. Now, again I shall not repeat all the working that I have done to relate the norms of y and x. Essentially, the central idea there is a transformation of variable. Put lambda equal to alpha t and do this all throughout the integral. And, one can straight away write down the final result.

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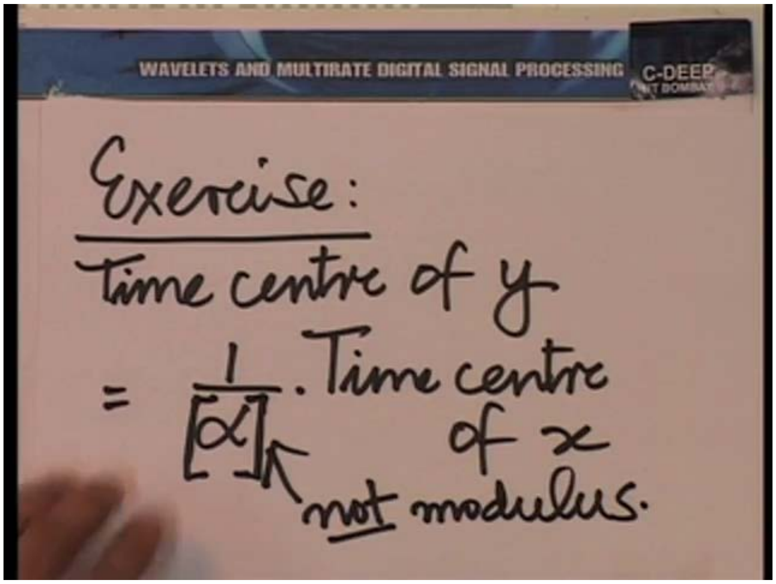

It is very easy to show by making that substitution that the time centre of y is equal to 1 by mod alpha times the time centre of x. Actually not mod alpha, I am sorry, it should be just one by alpha because we must take in to account the sign as well, not modulus. You must know where it is modulus and where it is not. Here, it is not modulus.

So, for example, I will take the example of alpha equal to minus 1. If alpha is minus 1, then the time centre also reflects it. So, the time centre of x was t 0; the time centre of y becomes minus t 0. So, it is not mod alpha, it is just alpha. Sometimes, simple intuitive reasoning can also clarify certain points for us. Anyway, here it is not modulus of alpha.

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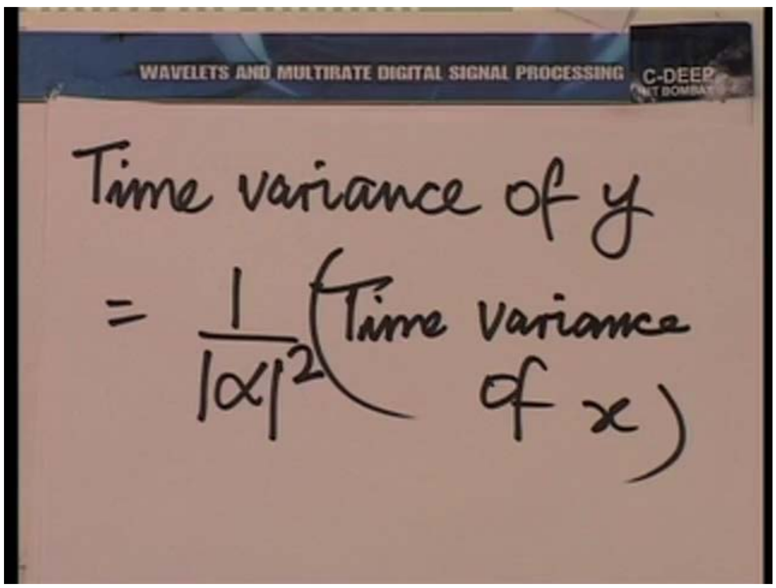

Similarly, one could use the same transformation and show the time variance of y. Now, here it is modulus. So, it will be one by modulus squared of the time variance of x. And in fact, if we now note that what we are doing in time is reversed in frequency. So, in time, we have t being replaced by alpha t; in frequency, capital omega gets replaced by capital omega divided by alpha with the scaling of one by mod alpha. But, please remember the mod alpha scaling is not going to affect either the time centre or the time variance or the frequency centre or the frequency variance. So, scaling the independent variable, scaling the dependent variable has no effect. We saw that few minutes ago. Scaling the independent variable has an effect. Let us make a note of this.

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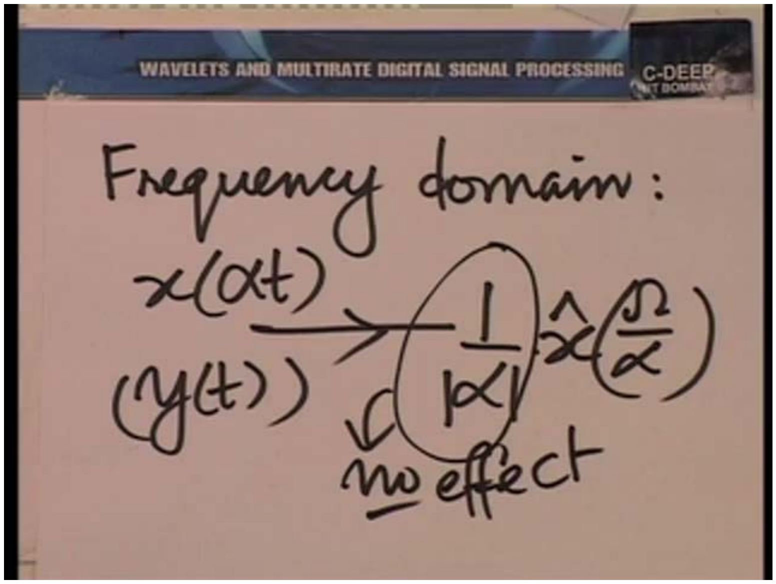

So, we have, in the frequency domain x of alpha t, which is essentially y t corresponds to l by mod alpha x cap omega by alpha. And, this has no effect. No effect on the variances or the centre as we saw a minute ago. It is essentially, this that we need to look at carefully; this one, the one by alpha scaling. And, one can use set of steps very similar to what we did in the time domain and arrive at the following conclusions.

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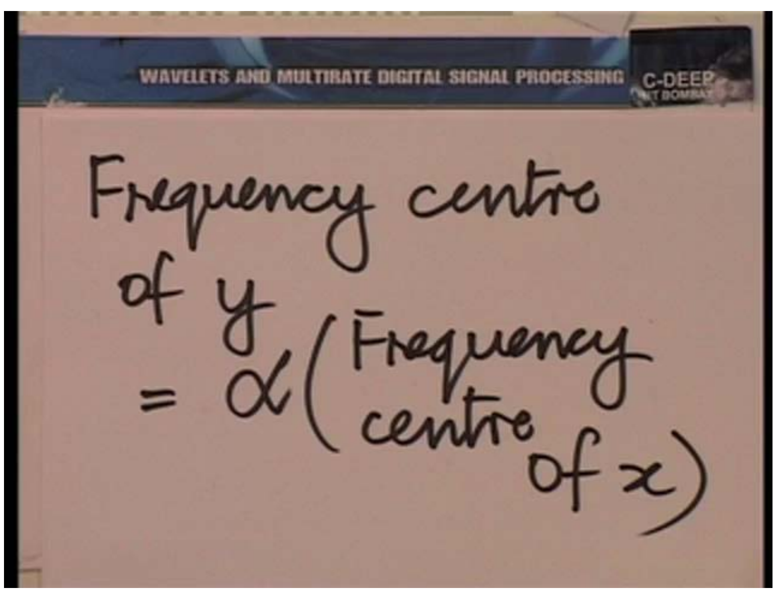

So, the conclusions are the frequency centre of y is equal to alpha times the frequency centre of x.

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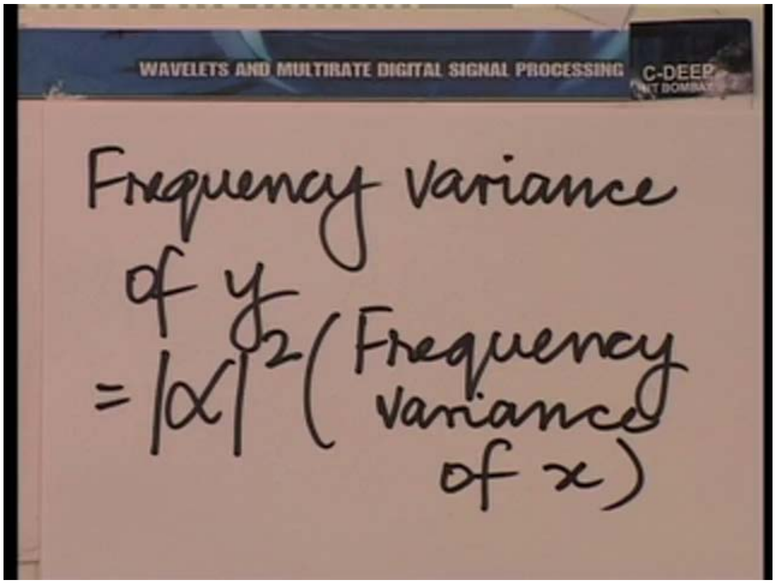

And, as far as the frequency variance course, the frequency variance of y is more alpha times the frequency variance of x. I leave it to you as an exercise to proof these. It is easy to do by making the same kind of substitution of variable. Anyway, now let us look at the time bandwidth product.

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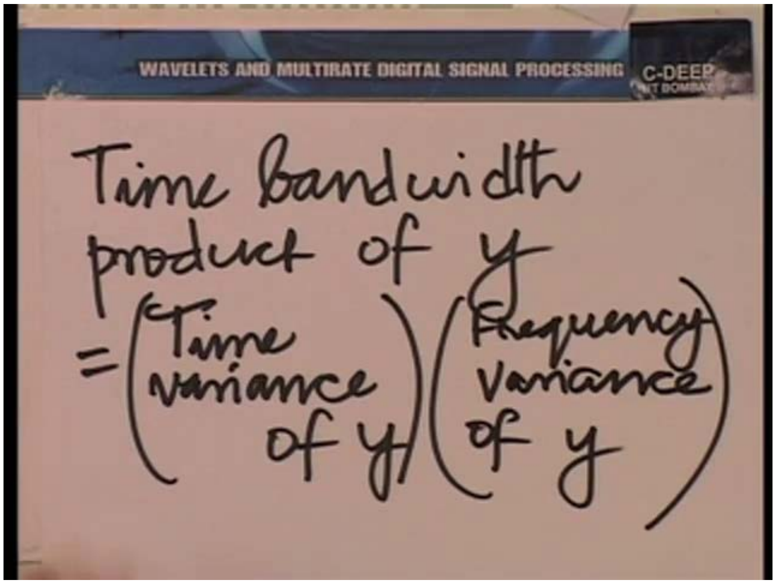

The time bandwidth product of y is essentially the time variance of y multiplied by the frequency variance of y.

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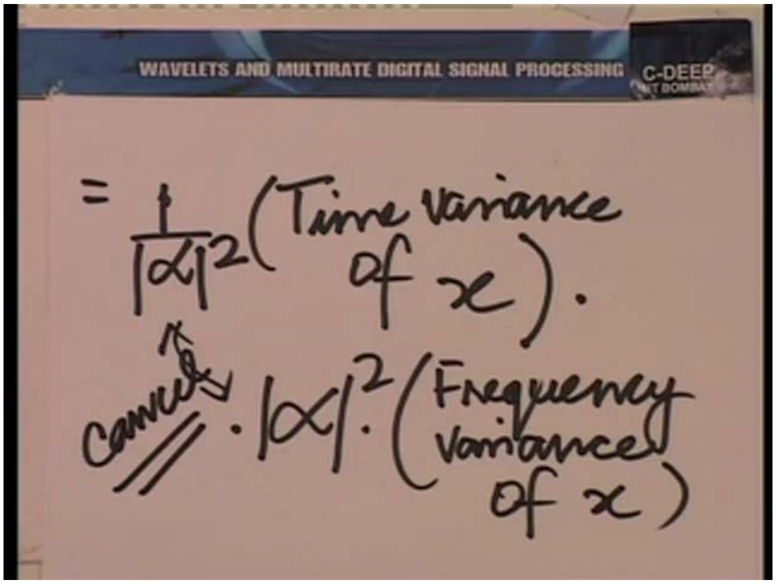

But, the time variance of y, by this argument is clearly seen to be 1 by mod alpha squared times the time variance of x multiplied by… So, as I am continuing this as a product here, multiplied by the frequency variance of y is mod alpha squared times the frequency variance of x. That is very interesting. The time variance is multiplied by 1 by mod alpha squared and the frequency variance is multiplied by mod alpha squared.

So, when you take a product, these two cancel. And, because they cancel, you get just the time variance of x times the frequency variance of x, which is essentially the time bandwidth product of x.

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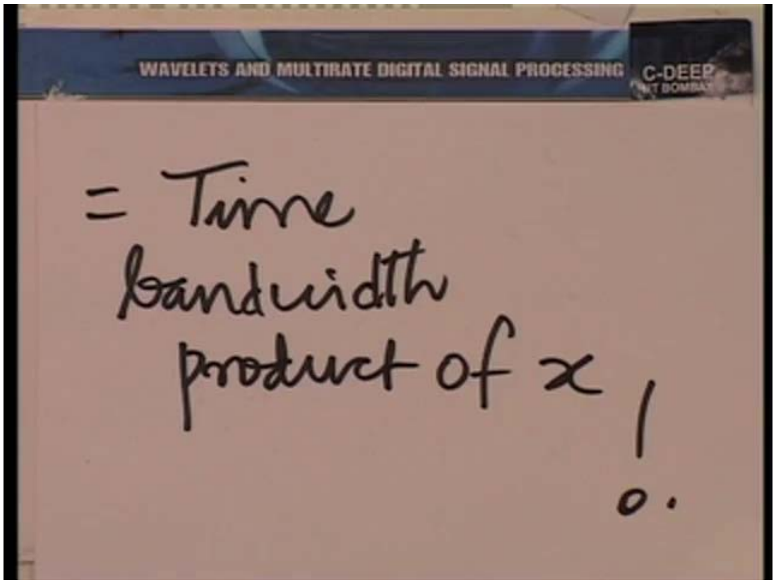

This is a very significant statement that we have made. What we have just shown is that the time bandwidth product is unaffected by scaling of the independent variable as well. It is very strongly invariant. Thus, we can see it is a very serious statement. Let us write it down.

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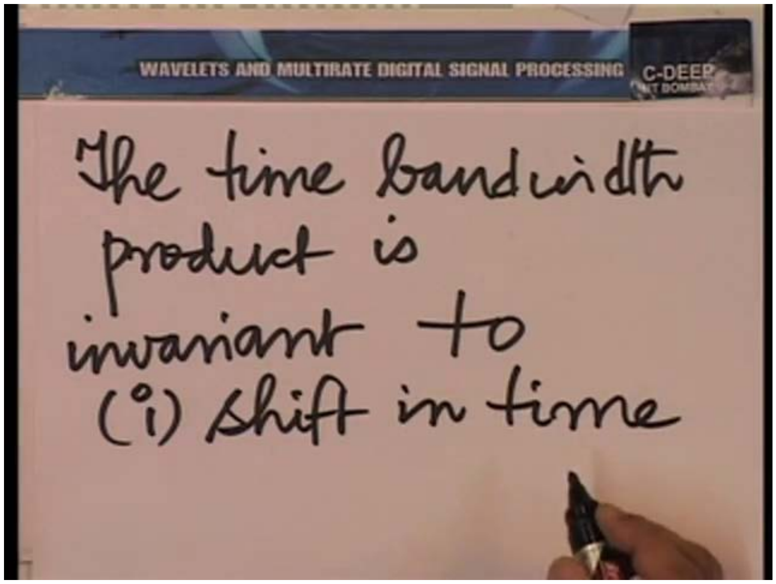

The time bandwidth product is invariant to, number 1: shift in time.

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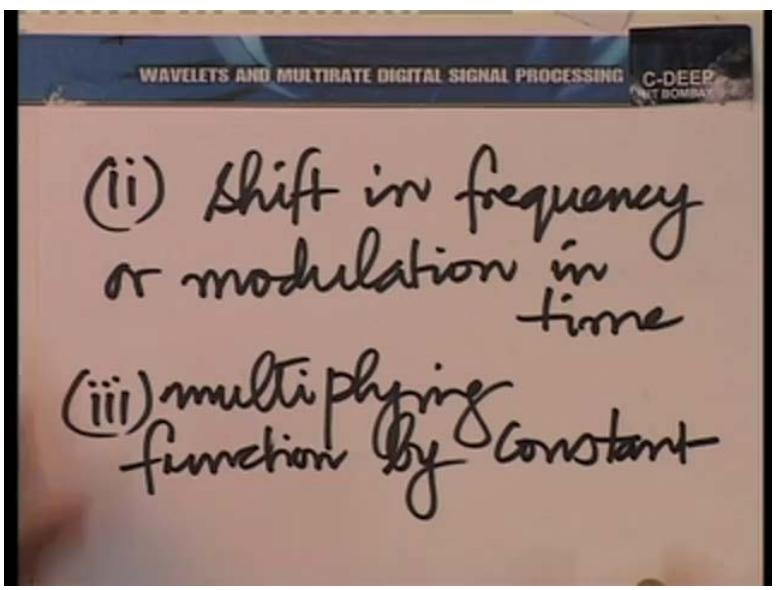

Number 2: shift in frequency or modulation in time. Number three: multiplication by a constant, multiplying the function by a constant.

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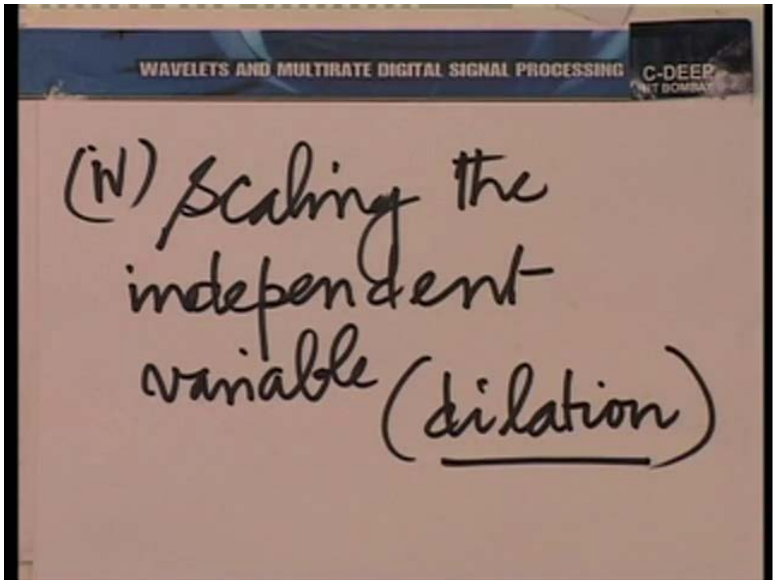

And finally, number four: the most interesting of them all. Scaling the independent variable, essentially what we have been calling dilation, all these while dilation by alpha. So, it is invariant to dilation. So, the time bandwidth product is something very fundamental about a function. What is it then variant two, what is it change with. A change is essentially with the shape. So, different shapes have certain time bandwidth product associated with them. That is interesting. Now, I put before you a question for thought. I like to put certain questions to challenge your imagination. And, this is one of them.

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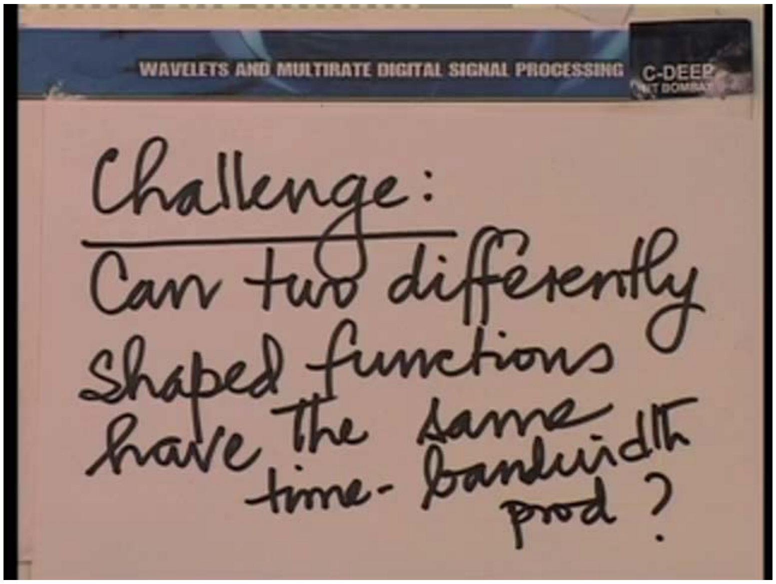

Can two different shaped functions have the same time-bandwidth product? And, so I will not leave you entirely in the dark, I will give you a hint. What could happen, if you took the Fourier transform of a function? Can one employ the property of duality somewhere and with that slight hint, which I am sure should take you rather far, I shall leave the rest of the reasoning to you to answer this question. Anyway, that could also bring forth one more kind of invariance.

So, it is in some sense, giving you two gifts all at once. And, answer to this question and one more kind of invariance at this time-bandwidth product exhibits. Anyway, so, now what we have seen is that this time-bandwidth product is a very important quantity in the context of functions. It, in some sense, is characteristic of the shape. Though, I must remark not unique to the shape, but characteristic of the shape. Now, we saw what was the time-bandwidth product of this pulse in the previous lecture.

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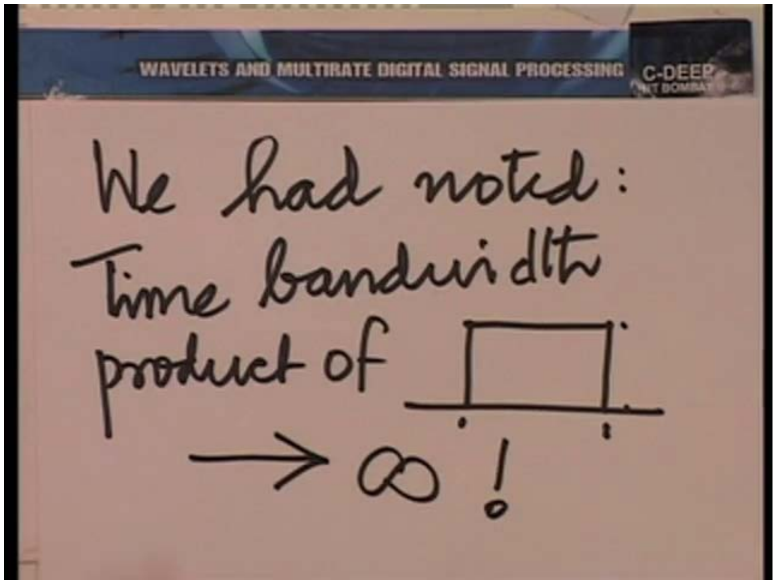

Last time we had noted that the time bandwidth product of this pulse, now you will agree with me that I do not really need to write the limits here, nor do I need to give you the extent of this nor even, I need to give you the height of this. Just the shape would do because whatever be this extent, whatever be this height, the time bandwidth product is all the same. But, as you noted, the time bandwidth of this product tends to infinity. And, we had asked why this is terrible… told us why we are not contended with the Haar.

We are dealing with Haar in terms of the scaling function or the wavelet function. You are essentially working with functions of an infinite time bandwidth product. Having the large time bandwidth product is bad. It means that, I cannot localise nicely in time and frequency. So, the whole objective, the whole game is to see how small you can make this time bandwidth product.

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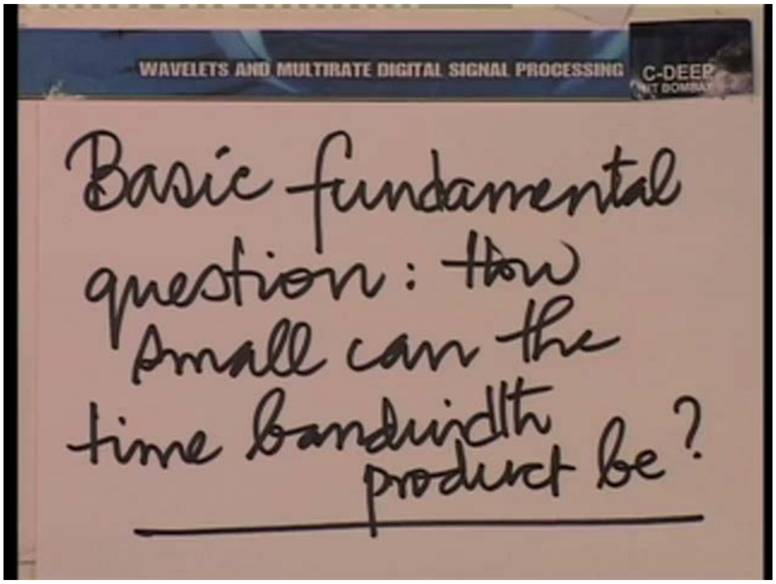

So, that is the question that the answer that the principle is going to ask. And, we shall formulate that question today; a basic fundamental question. How small can the time bandwidth product be? The Haar terrible infinity. Well, we shall have good news very soon.

We shall be able to come up with functions, whose time bandwidth product is not bad at all. But then, we will again see what nature often does to us, poor scientists and engineers. You can reach within a certain level of this bound. That, we are going to come too soon. But, reaching the bound itself is impossible. Anyway, that is just a prelude. Now, what we need to do is to establish at bound. And, to establish at bound, let us first simplify our work as we should always do. Try and take away unnecessary trappings from the problem, let us identify the essence of the core of the problem and then try and solve the problem.

So, what we are trying to establish is the fundamental, know a bound on the time bandwidth product. We already noted the invariance of the time bandwidth product to a number of different operations namely shifting in time, shifting in frequency, multiplication by a constant and scaling the independent variable. Now, what we shall do therefore? Is to do away with the requirement of time and frequency centre.

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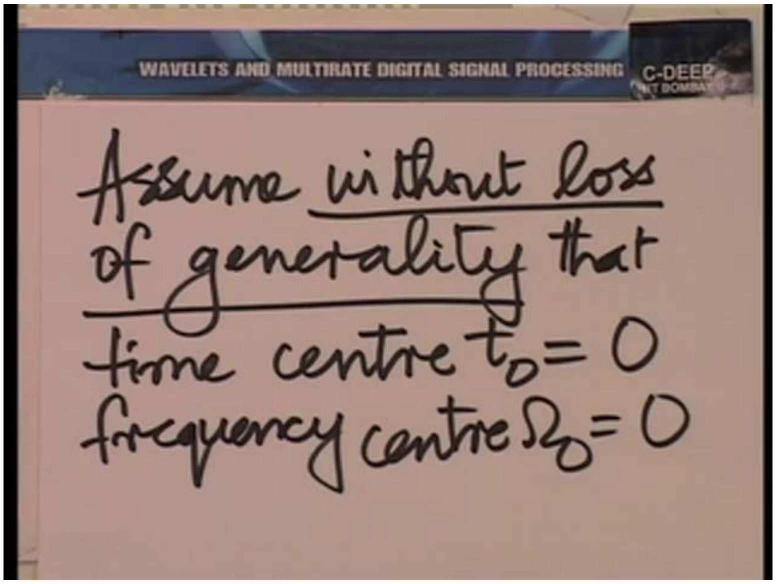

So, let us first, to answer this question, assume without loss of generality. And, this without loss of generality is because of all that I just said that the time centre and the frequency centres are both 0. After all, if the time centre is not 0, we could always shift and make the time centre 0 in time, without affecting a time bandwidth product. If the frequency centre is not 0, we can always shift in frequency; which means modulate in time and bring this frequency centre to 0. Of course if the function if real, we do not even need to do that. So, this is without loss of generality.

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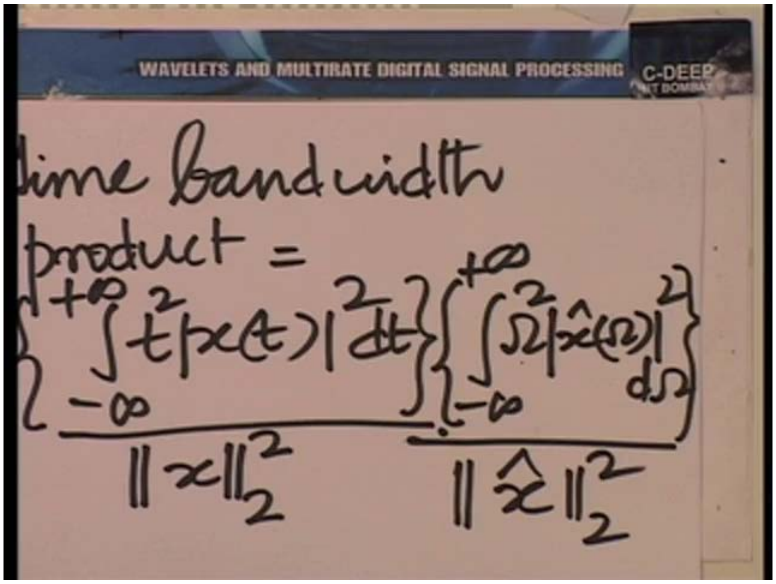

Let us make the time centre and the frequency centre zero; where upon, we then have the time bandwidth product to be essentially the following. It is integral minus to plus infinity the whole squared of x multiplied by t squared d t divided by the norm of x squared times something similar in frequency. So, we will keep this here; minus to plus infinity omega squared mod x cap omega squared d omega. And, here again divided by the norm of x cap the whole squared. Now, you have already made an observation about this. And, we formally make that observation again.

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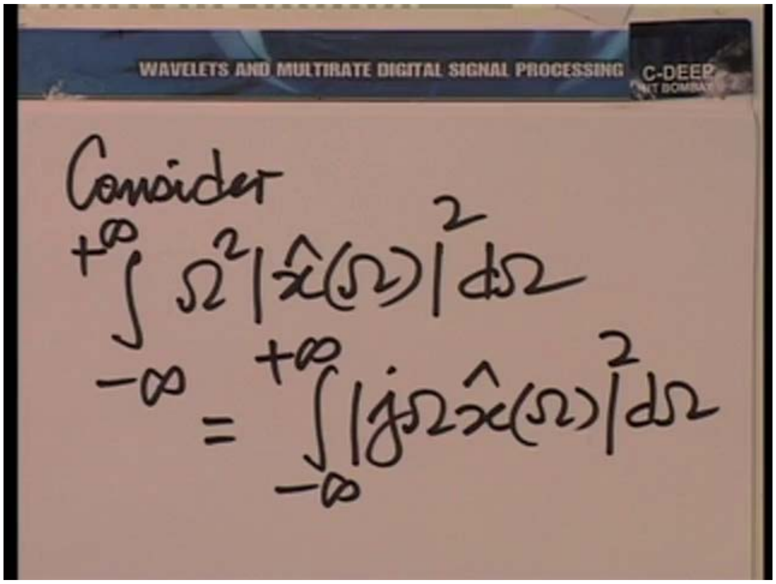

We already seen that, this essentially is minus to plus infinity j omega x cap omega mod whole squared d omega.

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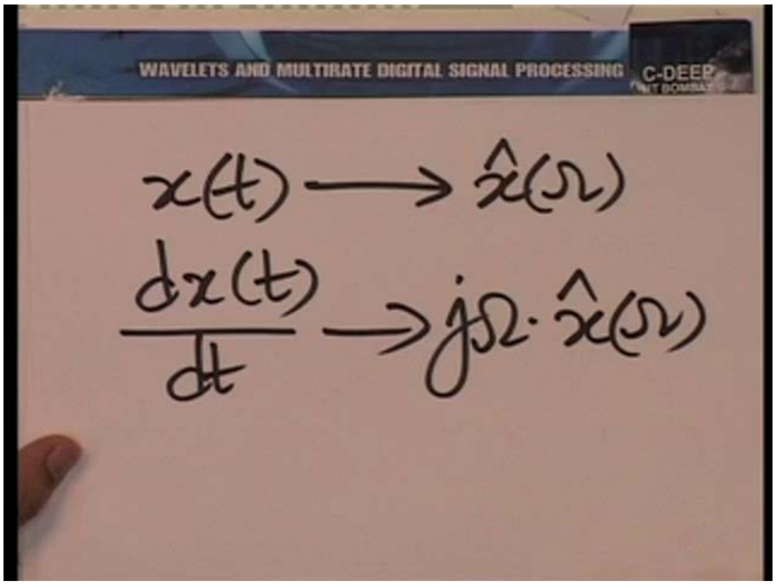

And that, we have noted is essentially, you see, noting that if x t has the Fourier transform x cap omega, then d x t d t has the Fourier transform j omega times x cap omega.

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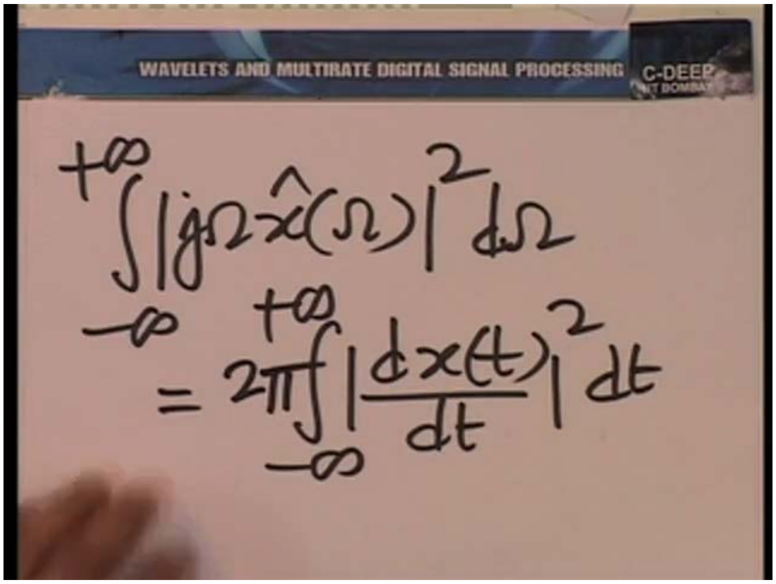

Where upon, integral minus to plus infinity j omega x cap omega mod squared d omega is essentially integral from minus to plus infinity d x t mod d x t d t mod squared d t, but thus a factor of 2 pi. So, it is 2 pi times this.

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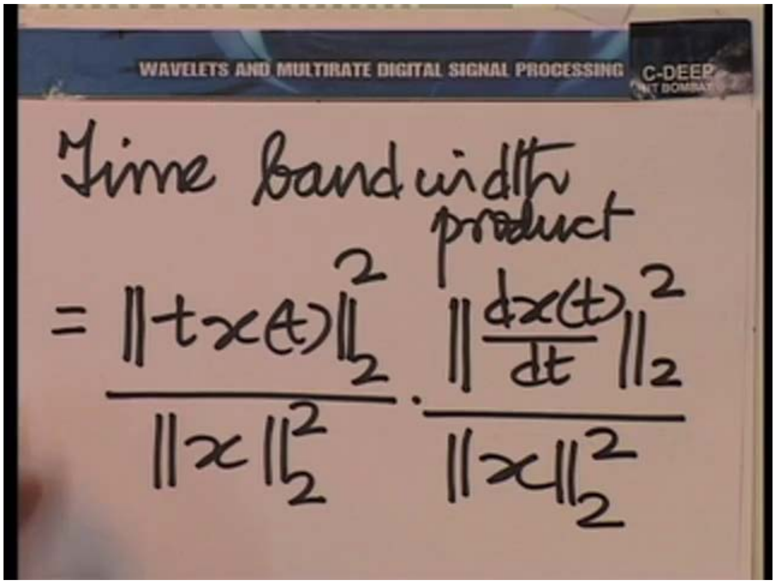

And, of course we also see that norm of x cap squared is 2 pi times the norm of x squared. And therefore, if we now put back all together here; when I put all these together, I keep these as the

R in the time bandwidth product. And, I note that this is essentially the norm squared or the energy in the derivative divided by the energy in the function, the factor of 2 pi cancels from the numerator and denominator here.

So, we could write down all in all; thus the time bandwidth product can be written as essentially the energy, the L 2 norm of t x t whole squared divided by the L 2 norm of x multiplied by the L 2 norm of d x t d t whole squared divided by the L 2 norm of x the whole squared. A very interesting result. I will just remind you that when we put back the expression for the time bandwidth product, this is essentially the L 2 norm of t x t. and, that is how we come to this conclusion.

So, here of course, there is the bit of abuse of notation. We are talking about a function t x t and then taking its L 2 norm. So, this is ratio of two products of L 2 norms. And, now here, we have all set to minimise this product. In fact, that would be our next step in the beginning of the next lecture. What is the minimum value of this product, minimum overall x that can exist in L 2 R. And, that would give us a very fundamental bound in nature, which is called the uncertainty bound. We shall derive that uncertainty bound in the coming lecture. Thank you.

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