Relativistic Quantum Mechanics
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Lecture - 6
Electromagnetic interactions, Gyromagnetic ratio, Lorentz force, Larmor precession

Last time we discussed the free particle solutions of the Dirac equation, and in particular we could identify the various components of the conserved current, and also normalised the wave function accordingly, so that we have a interpretation for a density as well as some Lorentz invariant quantity like psi bar psi. Now, we are prepared to extend this a free particle solutions to the case of interactions.

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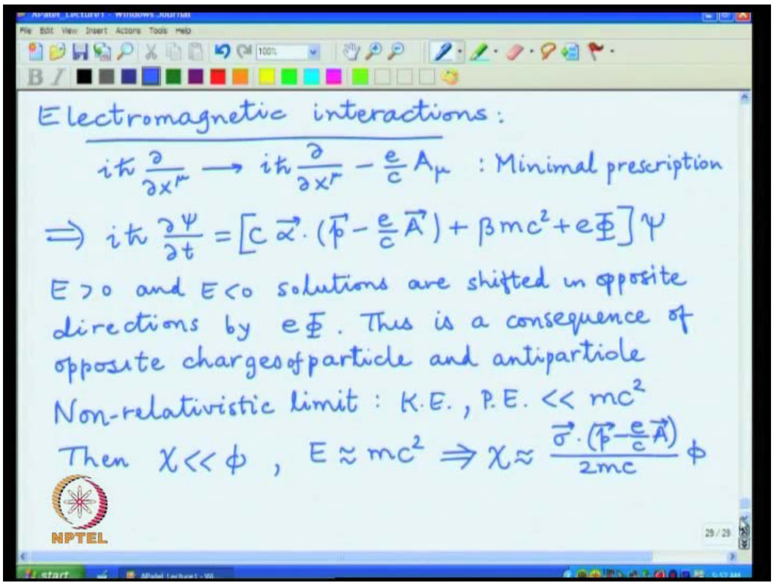

And, the simplest interaction to deal with is the electromagnetic, and that is achieved by the so called minimal coupling prescriptions. So, we have to replace the momentum or equivalently the gradient operator by the simple rule that del mu goes to del mu minus e A mu. This mini times is also called covariant derivative. We will investigate the consequences of the electromagnetic coupling into the Dirac equation, and see what kind of effects it produces.

It is very easy to just include this in the standard Dirac equation form, and I will write it using the momentum operators. It says a bit of notation where the electromagnetic vector potential is explicitly broken up into its time component. It produces the A 0, or the which is equal to phi in this particular notation, and the space component which are written as vector components of A.

If you directly do this substitution, the e phi is actually on the other side of the equation, but I have just transferred it in the convenient form. So, the time derivative defines the evaluation, and this whole term produces the effective energy which defines the rate of change in time. So, now, one can ask how does the electrostatic potential which is phi change the energy levels, how does the vector potential change the energy expression.

It is quite easy to see that the electromagnetic potential acts little differently than the rest mass from m c square. And in particular there is a matrix here beta with the rest mass, but no such matrix with the electrostatic potential. So, if you look at the whole equation in the conventional Dirac basis, there will be positive and negative energy solutions with beta being diagonal 1 and minus 1.

And, relative to that energy magnitude, the effect of the electrostatic potential is in opposite direction for positive energy as well as negative energy. And so E greater than 0, and E less than 0 solutions are shifted in opposite directions by this term e phi. And this is a consequence of what I have mentioned earlier, that the charges carried by particle and the antiparticle are always opposite in sign. And we will have an explicit example to solve for a solution to this problem which is the hydrogen atom problem, and phi is the coulomb potential. We will come to that later.

Now, let us look at how the vector potential shifts. Here the matrix which is involved is the alpha, and in Dirac basis it is of diagonal matrix. So, we need to do some simplification to understand the effect of the vector potential on the energy levels of the particle. And here it is convenient to take the non relativistic limit of this equation and see what emerges out of it. And non relativistic limit basically means that the kinetic energy as well as the potential energy terms in this equation are much smaller than m c square, which is the rest mass energy.

And so we expect in the non relativistic limit the alpha term to be small as well as the contribution of e phi also to be small. This is the dominant term. And in that case, we can now expand the equation into its upper components and the lower components which are defined by this matrix beta, and see what is the next correction in addition to the m c square.

We can follow the same methodology as I did last time. And that is, write this equation in this 2 component notation, phi and chi. Chi which is going to be much smaller than phi in the non relativistic limit. Hence, this off diagonal contribution which is coming from alpha dot p terms here, is going to be negligible compared to the rest mass terms. So, in this notation we have, chi is much less than phi. And if you now look at the solution, the total energy can be approximated by m c square that is a leading term. And these 2 conditions then give the explicit ratios from the matrix equation which I mentioned before; that chi now is approximately sigma dot p minus e by c A. This is the ratio for the equation. Actually there is a c times alpha, so the numerator has the factor of c and denominator had a factor of energy plus m c square; and energy is approximately m c square; and I cancelled a factor of c, so that is why this is the convenient form.

And now this approximation can be substituted back into the original equation, and one can write down the equation motion of phi itself. So, this term, we will write the equation for upper 2 components which is essentially phi. This is diagonal, this term is diagonal, and this is diagonal. This term produces the off diagonal corrections, but that is chi, and we already know which is the expression for chi, so we can just plug it back in. There is the same factor ratio which is here, the appearing over here, and the whole term basically gets squared.

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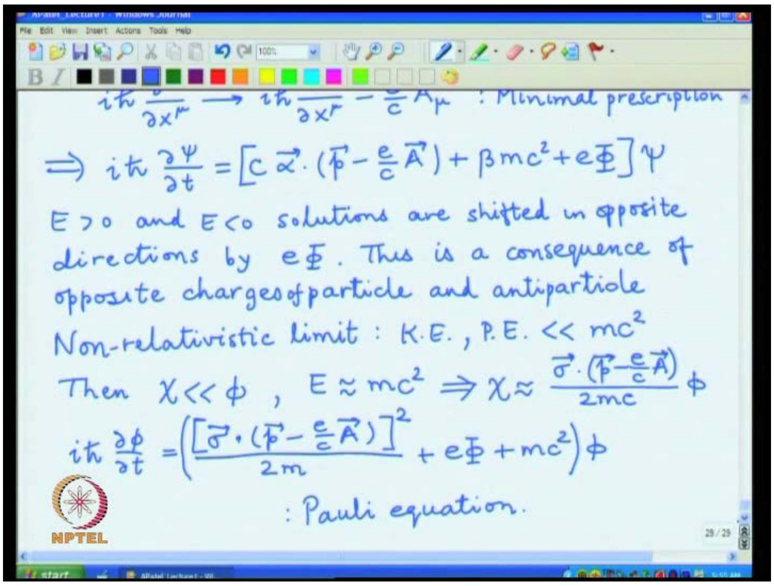

And, then we have the equation which can be written as; there is a, this whole thing getting basically squared and at the factor of c. So, it is sigma dot p minus e by c A. This is the whole operator, and what we have is a square of this operator, divided by 2 m plus there is a electrostatic part which is this, and then there is a m c square which is the rest mass term m, and the whole equation takes this particular form. So, now, this equation is satisfied by this 2 component object phi, and it involves the operators which are the Pauli matrices.

Many times this equation is also referred to as the Pauli equation which is a 2 component generalisation of the Schrodinger equation. Schrodinger equation just that no sigma; it just p square by 2 m; there was a electrostatic energy; and this m c square is the shift in the overall scale which occurred through the rest mass term in the energy; we can drop it, if you want to shift the 0 of the energy it does not mean very much. And the generalisation from Schrodinger equation to Pauli equation is just by introduction of this Pauli matrices.

So, instead of having p square, now we have essentially sigma dot p whole thing square. And this equation is the non relativistic limit. And actually it was written down earlier after Pauli incorporated the spin degree of freedom into the Schrodinger equation, and this was the natural generalisation.

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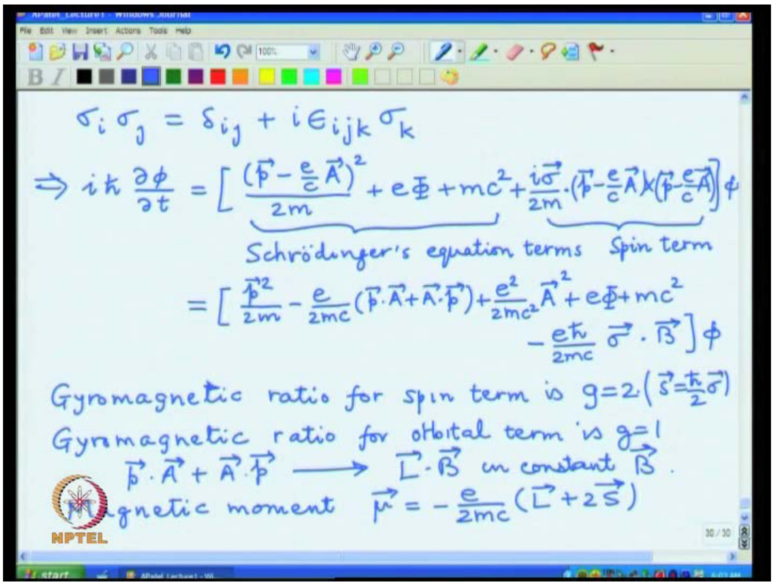

We can try to simplify it, and see what are the various terms appearing in it and that can be done by using the identities of the Pauli matrix which is essentially we need a general expression for multiplying 2 Pauli matrices which produces delta i j, it is a chronicle delta, and epsilon i j k times of the third Pauli matrix. Once you put both those things inside, one can have now the simplified version where some of these terms look more conventional. So, it is the delta i j part just produces square of the momentum term.

So, these 3 terms make up exactly the operator which appears in the Schrodinger equation. And now we have a mu term which comes from this epsilon symbol, and that produces a cross product of these 2 momentum operator. So, this is the Schrodinger, and this is the new interaction which is the contributions of the extra degree of freedom described by Pauli matrices which happens to be the spin.

So, we can now work out these things in a very specific potentials term by term, or some of these things can be written down even more generally. Now, this is a cross product of the 2 identical operators, now it will be 0 if the operators commutate, but they do not commute, and so the non-trivial effect actually arises from when the momentum acts on the vector potential. This is a gradient operator, the cross product with A easily produces curl A which is equal to the magnetic field, and the i gets eaten up by the fact that converting momentum to gradient we have to include a factor of i. So, that term is easy to interpret.

The first term can be written down as the usual Schrodinger's equation expansion. So, it is p square, p dot A plus A dot p, and then A square; then the other stuff is essentially the same. And this now converted into curl A, looks like e h cross by 2 m c sigma dotted with B. And so this is now the more explicit form; that in addition to having Schrodinger equation you have an extra sigma dot B term which produces the Zeeman effect for the spin angular momentum carried by the electron.

And, it is very easy to see that the gyromagnetic ratio for the spin term is g equal to 2. Remember that this definition of spin is h cross by 2 times sigma. So, the half which is here is actually part of the spin, half description of the problem. On the other hand, the orbital part has a gyromagnetic ratio is just equal to 1. And that comes from the fact that in external magnetic field which is generally easy to consider a weak field in some particular direction. And then this combination simplifies to L dot B when B is equal to constant.

This is just the standard result for Schrodinger equation, and you have L dot B plus S dot B. The only thing is their relative normalisation is different, and which it explains the experimental fact that the orbital term is normalised to this gyromagnetic ratio 1, in that same convention the gyromagnetic ratio for the spin term is 2. And the convention for the magnetic moment is then the orbital contributions accompanied by the appropriate gyromagnetic ratio. So, it is L plus 2 S.

And, this is a major success of Dirac equation in explaining the properties of the electron because electron indeed behaves in this way, that the spin angular momentum couples twice as strongly to the magnetic field compared to the orbital angular momentum. So, these equations are kind of straight forward to derive, and it was very helpful to get the spin automatically coming out from the Dirac formulation and also with the correct properties, and that is why Dirac called these equation a theory of electron.

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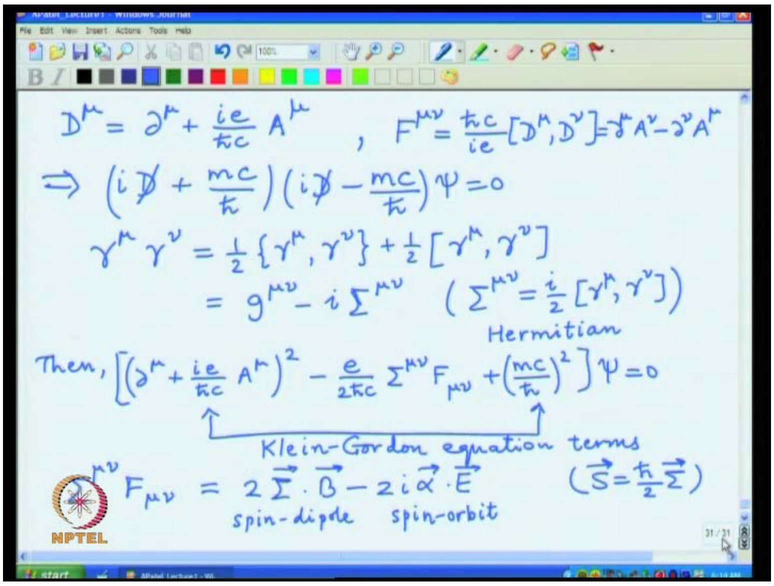

Now, one can look at the same structure in a covariant language also. Sometimes it is helpful to understand that as well, and so we will look at the equation again, but now using the 4 vector notation in terms of these, so called covariant derivative. So, we have the ordinary derivative plus the correction from the electromagnetic potential. With this substitution, now the Dirac equation can be written as; actually it is only this operator acting on psi is equal to 0 is the Dirac equation.

We are just multiplying by another overall factor to be able to take the limit of non relativistic approximation easily. And now it is convenient to just expand out these terms. There are, these factors of the 2 operators will just multiply those things together, and use the gamma matrix algebra where gamma mu, gamma nu, the anticommutator produces 2 different terms.

So, one can write gamma mu gamma nu as half the anticommutator, and so half the commutator. This result is the Minkowski metric. And we will define this particular quantity to be a new object it is related to spin, and we will call this minus i times sigma mu nu, or equivalently the definition of this object is i by 2 gamma mu gamma nu. The anticommutator is antiHermitian object, and that is the reason for sticking in this factor of i, explicitly. So, the sigma mu nu, these are Hermitian objects.

And now with this convention we can expand out the product of 2 d slashes. One of the term will just produce this g mu nu; that means, they will be trivially contracted, and then there will be an extra term coming out which coefficient sigma mu nu; so with that convention. And then the cross terms in the multiplication cancels because m c by h cross is just a number.

So, then the equation now takes the form that it is del mu plus i e by h cross c A mu, whole thing is trivially squared; that is the g mu nu part. Then there is this extra term involving sigma mu nu, and it gets multiplied by the field strength tensor constructed from this covariant derivative. And then there is the trivial term of the compton wavelength inverse, whole thing squared.

The explicit definition of this tensor is, follows from the definition of covariant derivative. So, it can be defined as f mu nu is h cross c divided by i e times the commutator of 2 covariant derivatives; and in terms of explicit evaluation d mu A nu minus d nu A mu. All these factors of h cross c by i, just cancel out, what is the accompanying the definition of covariant derivative and proportionality with the vector potential. So, this is now the equation.

We can now see the extra terms compared to the Klein Gordon equations. So, the first this term and the m c h cross square, these are the terms in the Klein Gordon equation. And the new contribution is essentially this sigma mu nu F mu nu. Now, one can break it up into 2 different parts corresponding to the electric and magnetic field because the space component of F mu nu gives magnetic field, and space time component of F mu nu gives the electric field.

And, one can break up this object into an explicit notation. One has to evaluate the corresponding commutators of gamma mu gamma nu. When there are 2 of them are spaced like we get the exactly the spin operator which we defined earlier. And when one is space and the other is time, it gives gamma 0 gamma i which nothing but the alpha matrix as we had defined earlier.

So, this object simplifies to 2 times what I had defined as, sigma dot B, and then alpha dot E. And the convections I have used was, the spin operator was h cross by 2 times sigma. So, again you see the effect directly coming in. This is the Zeeman coupling appearing to do a spin. It is not part of the Klein Gordon equation. And then there is this extra term. So, this is the spin dipole.

And, this alpha dot E term appears many times in analysis of perturbation theory of centrally symmetric potential, and it can be rewritten as it has the spin part buried inside alpha, but it is off diagonal. And the electric field which appears here can be related to the magnetic field in the frame of the moving electron. So, the static electric field in which electron is moving becomes a magnetic field in which the electron is addressed. That is the usual Lorentz transformation. And this term basically leads to the coupling between the spin and a motion of the electron, and that is often labelled as the spin orbit coupling.

So, both these effects involving spin, directly a dipole interaction with the external field, and also a coupling with the orbital angular momentum. They appear inside this Dirac equation. They are not part of the Klein Gordon equation themselves, and we have not used any non relativistic expansion so far in this analysis. And both these terms are quite general. One can still do a non relativistic expansion, and simplify them further if necessary, but it is already written in a quite a general form.

So, this is the covariant version of Dirac equation simplified, so that you can directly identify the effect of electric and magnetic field. Now, one can play around with a various dynamical equations based on this algebra, how does the particle move; if it is spin then it will certainly evolve under the external magnetic field, how does it change and so on and so forth. And these things can be easily worked out. Let me illustrate 2 very simple cases.

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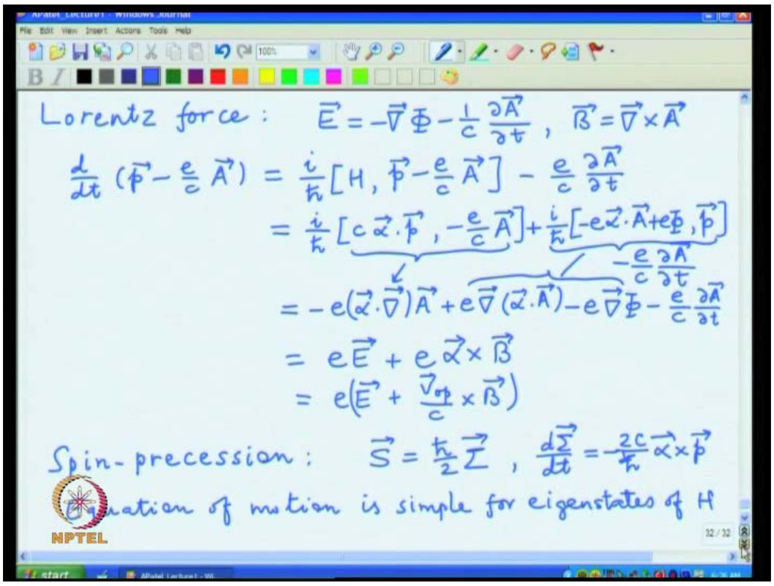

So, one is the so called Lorentz force for a particle moving in some external electromagnetic field. And it can be obtained as an equation of motion as a rate of change of momentum. Except that the momentum which we have to consider now is the canonical momentum, or equivalently the one corresponding the covariant derivative. And the electric and magnetic field we will define with the standard convention that E is equal to minus grade phi minus 1 by c del A by del t, and B is equal to curl of A.

So, then the rate of change of momentum is easily evaluated by working out the commutator with the Hamiltonian. So, it is this expression. And now, different terms in the Hamiltonian may or may not commute with the terms which are involved over here. And we have to just work it out one by one. In particular, this operator does not have any structure in the internal space. There are no alpha beta matrices involved over here.

So, the only term which produces a non 0 answer is a, when there is a operator inside the Hamiltonian and which acts non trivially on the vector potential. If there is nothing else involved then we do not have any extra contribution. So, just look at it term by term. So, the part which acts nontrivially on this vector potential is just alpha dot p. The remaining part does not act on A at all, and so it does not produce any contribution. So, this is one part.

And, the other part is the terms which depend on space because it involves the commutator with the gradient operator buried inside here. And those terms can now be taken again from the Dirac equation. There is this alpha dot A term coming from the momentum part, and then there is also a static potential which can get differentiated with the momentum. The momentum part p itself commutes with p, and does not contribute over here.

And then to consider a most general situation where you have the complete time derivative acting one is here, instead of just being in a time independent field you have to add a extra term, and that term is just a partial derivative directly acting on this term. So, it actually happens to be; this is the time derivative, explicit time derivative of this vector potential. The gradient operator is not a function of time, and so we add this extra term to the equation of motion. So, this is a total result.

And now you have to evaluate the various commutators, what happens with gradient acts on this term versus that term, etcetera. And that now can be done in a straight forward manners; just let me do that; this is a gradient acting on A. So, it produces alpha dot gradient acting on A. The nontrivial part of the commutator is when p acts on one of these terms; p acting on anything which follows it does not contribute to the commutators. So, the first commutator actually is just this.

The second commutator is the same way where this p will be acting on this particular terms, and that produces the term which is here plus p acting on phi. So, it is actually this part which gives this; this part which produces these 2 terms; and then we are still left with the last partial derivative. So, now, we have evaluated the commutator. And it is convenient to put back the electromagnetic field, inside of the vector potential in this notation. So, these last 2 terms easily produce the electric field.

And the, these first two terms gradient acting on A can be now rewritten as a triple product which is involves alpha gradient and A, and that can be written as alpha cross B. So, if you write alpha cross, curl cross A, and expand the triple product it will be alpha dot A, the gradient acting on that, and then alpha dot grad the whole thing acting on A. So, this is what becomes of the equation.

And this indeed describes the Lorentz force with the identification which we have seen before, that this can be written as the velocity operator cross B because velocity operator was indeed this matrix alpha. So, the Lorentz force does not survive in exactly the form where the velocity was, but velocity has to be replaced by its appropriate operator. And then the equation works; and then one can construct the trajectory of the particle by solving this equation given in certain external field.

So, this is the modification of what happens to Lorentz force. There is a equation of motion not for the coordinates, but for the internal degree of freedom which is the equation corresponding to the motion of the spin, which is actually the spin precession. And the operator here involved we have already seen before, that it is this matrix sigma. And we derived the equation also for this in constructing a angular momentum operator. So, the total angular momentum combing L and S was conserved for a free particle, and that gave a equation of motion of this object sigma, which was equal to minus 2 c divided by h cross times alpha cross p.

So, this object does not look anything like the effect of a external field, even when you substitute this p by covariant derivative. It does not take a appropriate form required for the magnetic field, but this alpha is already the velocity operator which we have seen before. And to be able to get the equation of motion which is of the same structure as in case of classical electrodynamics.

Instead of looking at this operator d sigma by d t, we will look at its anticommutator with the Hamiltonian which will have a definite value in eigen states of Hamiltonian or equivalently eigen states of energy. So, that is the object. So, equation of motion is simple for eigen states of the Hamiltonian not in case of a general Hamiltonian, or a mixture of various states.

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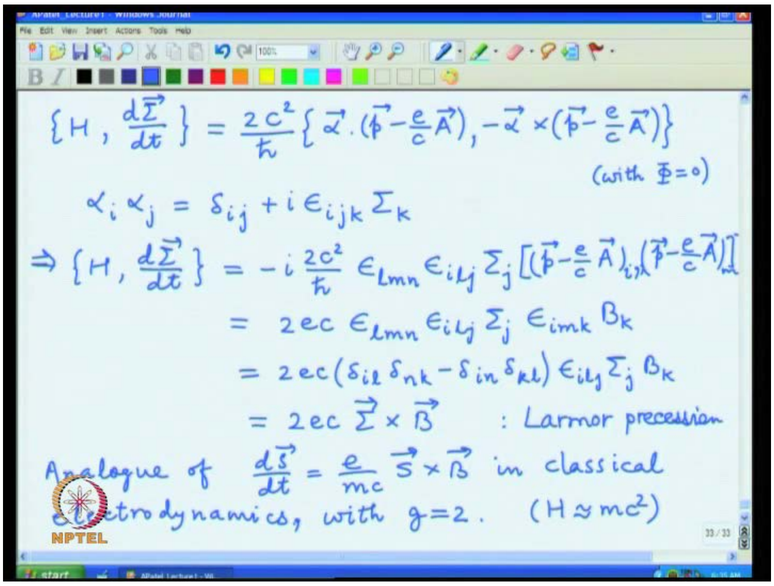

So, we will see this derivation what happens, and the object which is convenient to use in this particular case, it is the anticommutator we did this same trick in understanding the velocity operator earlier. And this now can be written down by inserting the expressions of the Hamiltonian as well as whatever happens in sigma dot b n. We have the 2 terms. The alpha part commutes, the alpha part produces a non 0 answer with the rate of change of the spin operator. And for the beta part these anticommutator of alpha and beta is 0, so that, the rest mass term is not going to contribute.

And, we are only going to look at the effect of the vector field. So, the electrostatic field we are going to keep 0. And now, this can be evaluated by writing down the products of these alpha matrices. And since alpha has the Pauli matrices, they obey the same kind of structure when you have a products of 2 alpha matrices. So, this alpha i alpha j happens to be delta i j when the 2 ones are equal, and if they are not equal then we get the Pauli term. Now, the Pauli term is diagonal and that is what we have defined as this matrix sigma k.

And, with this the product of 2 alpha can be written in a simple fashion, and the cross products can be written again with a epsilon symbol. So, this object then can be simplified to all the various factors of i are going to appear from this commutator as well as this term minus sign is from this part. And the 2 products of alpha are now quite explicitly written. There is a cross product here, which I am going to use this epsilon L m n to expand.

And then there is a epsilon here in the products of alpha which I will write using epsilon i l j sigma j. And then now we have the remaining part which is these 2 terms. One of them has the same index as i which comes from the product of alpha, and second one comes from these cross products. And it will have the index m. And the combination of these 2 terms becomes a commutator instead of an anticommutator. And the reason for being that is the cross product over here. So, when you change the order of the alpha matrices and do the simplification, you have to flip around the order of this indices as well, and that can be taken inside by relabeling, and this object then reduces to this.

Anticommutator, this object is quite familiar and one can work it out as has been the case many times in non relativistic quantum mechanics. It is just a commutator of 2 covariant derivatives, and we have seen it what it produces. It produces the f mu nu or electromagnetic field. In this case these are all space components. So, it produces a magnetic field. And the result of that simplification is now written down explicitly in terms of all these various tensors; the one more epsilon symbol from this commutator and the magnetic field.

So, now, the whole structure is in the form where the only nontrivial operator it has the spin operator and the magnetic fields. And these products of all epsilon can be simplified in terms of chronicle deltas. And once one does that, for instance, they are product of the first 2, I can just write it explicitly. So, this is gives delta i l delta n k, and the one which is the reverse order. This middle one leave it as it is, so it involves this sigma terms in the quite the appropriate fashion.

And, now this chronicle delta simplify the various indices put everything in place. And this term becomes sigma cross B. And this is indeed the expression for spin precession which is also referred to as Larmor precession of a magnetic dipole in an externally applied magnetic field. The factor of 2 came here because I took the anticommutator, but did not divide by a factor of 2. So, there is a, this term actually represents 2 times energy multiplied by d sigma by d t.

And, if you take all these thing, this is a analogue of the classical equation of a d s by d t. You have to take out these 2 from here, and insert the factor of h cross by 2, but that is the sigma is there on both sides of equation. It just does not matter. And h will give a non relativistic limit. It will become close to m c square. So, this produces the usual Bohr Magneton constant, well upto h cross and S cross B. So, with, again the important point is the gyromagnetic ratio has a value 2. And we have approximated the value of the Hamiltonian by m c square in taking out the factor of the energy. So, this is the way the spin precision also appears with the correct gyromagnetic ratio in the external field.

So, not only we have the energy eigen states, we also have dynamical evolution equations in presence of electromagnetic field. And they can be used in many different situations to calculate many kinds of effects. Of course, all these analysis was nice, and it helps to understand the effects which are mostly seen as a small corrections to the dominant part. And in general, they are evaluated using perturbation theory and external fields.

The dominant part was actually solution of the coulomb potential problem, and that was the crucial component of the atomic physics, and all these things were extra corrections. What we have seen so far is that the extra corrections do work out as expected, the spin degree of freedom is there it has the correct normalisation, etcetera. But the benchmark is the solution of the complete hydrogen atom problem. That is where everything has to be checked.

And, Dirac's equation got the approval once the hydrogen atom eigen spectrum came out correct. It differs from the spectrum which I derived in earlier lectures from the Klein Gordon equation, and it matches the formula which was obtained by Sommerfeld without knowing anything about the spin. Sommerfeld's formula agrees with experiment, and so does Dirac's answer. And we will work this explicit solution out in the next class.

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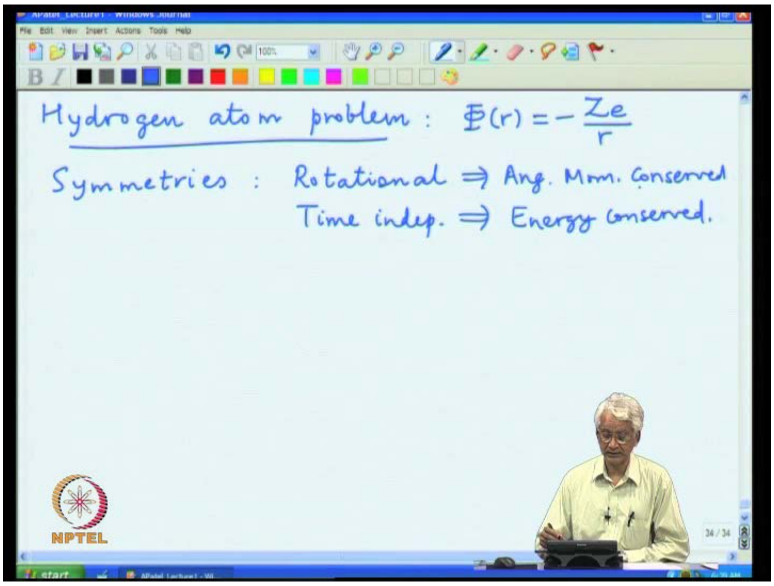

And, this is the hydrogen atom problem, sometimes also referred to as the Kepler problem, the one over r potential. And we have to use a standard normalisation which is a conveniently taken to be z e by r, and this solution has certain symmetries. So, one is a rotational symmetry which means that angular momentum is conserved, the other is a time independence which means energy is conserved. And both these objects correspond to well known quantum numbers for the hydrogen atom, both in relativistic theory as well as the non relativistic theory.

And, we want to obtain the general solution in terms of those quantum numbers for the Dirac equation which means you have to solve it. Find an operators corresponding to these conserved numbers and then express the solution in terms of these conserved quantum numbers to get the complete expression for energy eigen states and the wave function, etcetera.