Relativistic Quantum Mechanics
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Lecture - 5
Dirac matrices, Covariant form of the Dirac equation, Equations of motion, Spin, Free particle solutions

So, in the last lecture I introduced the Dirac equation, and showed how Dirac could linearize the dispersion relation by introducing coefficients from Clifford algebra. And today I will discuss some elementary properties of that equation which immediately follow from it is a peculiar structure. So, first thing is to just write down this equation in covariant form, and that is easily done by converting the time derivative to have the same structure as the space derivatives are...

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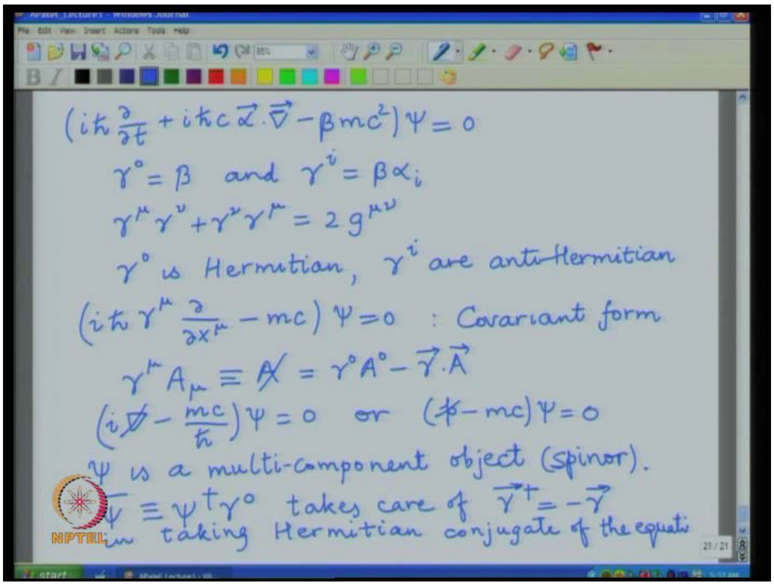

So, first we write the energy and momentum in terms of the corresponding operators which make the equation look like i h cross del by del t, which is the operator corresponding to time, and then the particular form for the momentum, and then the risk mass the energy acting on the wave function equal to 0. Since the matrices are from Clifford algebra, the wave function also will be a column vector of the same dimension as the size of the Clifford algebra, and so it will be a multi component quantity. And we will come to the interpretation of what these various components mean, very soon.

Now, this equation can be written by combining the time derivative and the space derivative into a 4 vector. And to do that we have to get a corresponding matrix, also for the time derivative and which is easily done by the following convention. Let me define a matrix gamma 0 which is identical to beta, and we define matrices gamma i, these are equivalent to beta times alpha i. And, in terms of this matrix, this is the Clifford algebra, can be reduced to the commutators which now produce the Murkowski signature. And also, simultaneously we can see that because of the anti commutation rules of beta and alpha, the gammas will also have slightly different properties, in particular gamma 0 is Hermitian, while gamma i are anti Hermitian.

Given this convention, the equation can now be written as; this particular form which is explicitly covariant. And gamma, the 4 components of the gamma matrices can be treated as a Lorentz 4 vector, and they will transform similarly under the corresponding Lorentz transformation. We will come to the details again later. Often this convention is simplified to write means something simpler and that is another notation introduced by Feynman, which for any 4 vector A mu when it is contracted with gamma mu represented by a slash, and which is in explicit form will be written as space minus the time products negative sign coming from the Murkowski metric.

So, in this slash notation, the equation takes its most familiar form which is del slash minus m acting on psi is equal to 0, or in Fourier Hook space the momentum values can be used as the Eigen values of the derivative operator, that is why it becomes p slash minus m acting on the vector equal to 0. And here, you can see the explicit scale of relativity appearing; this m c by h cross psi is just the reciprocal of the Compton wavelength. So, this is a common form which is used, and you have to remember that psi is a multi component object, in particular it has a specific name, it is called a Spinor.

Now, one can write the complex conjugate equation again in a covariant form, but one has to now keep track of the Hermiticity of gamma 0 and anti Hermitian nature of gamma i, to change the sign and to take care of that particular quantity, it is convenient to define yet another notation; I introduced an analogous notation in discussing the Klein Gordon equation which is taken by putting a bar on top of the spinor wave function and that is to define this object, psi bar is equivalent to psi dagger gamma 0. And this extra gamma 0, once we take a Hermitian conjugate takes care of the extra sign which appears for the anti Hermiticity of gamma i, because this gamma 0 will anti commute with gamma i.

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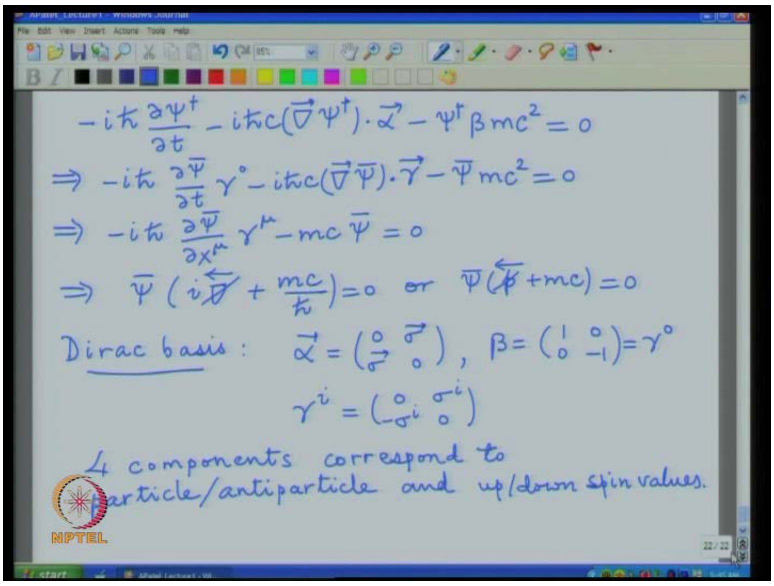

So, we can now explicitly see the Hermitian conjugate part of it which is the time part, the space part. Now, introducing the bar notation, the equation changes to this which now can be rewritten as 0 in this combination, and then psi bar can be factored out now on the left and with the explicit convention in that this derivative is going to act on the left as well; in terms of momentum I can state it the same as. So, this is the complex conjugate form of the Dirac equation, and we can easily see that the only thing which has changed with respect to the original equation is the derivative. Now, exchange the opposite signs and the sign of the derivative is flipped which comes from the complex conjugation of the whole structure.

So, these are the various forms of the Dirac equation as it is written. And one can study its various properties now in this covariant form which are most easily derived, and one can construct equations of motion or various observables, and one can also take a non relativistic limit and study the corrections coming from relativistic effects, etcetera. And for many of this purposes, it is convenient to choose a now specific representation for the gamma matrices, or equivalently the Clifford basis. Now, many times this basis choice depends on the particular up location which is in mind and various choices are made.

The most common one is, of course, the one made by Dirac which is the most useful when trying to take the non relativistic limit of this Dirac equation. So, you get the straight forward separation of the particle and antiparticle components, or equivalently the positive and negative energy modes. And that is achieved in 4 dimensions by choosing the basis for the matrices which is written in terms of poly matrices as this particular combination where each entry here corresponds to a 2 by 2 matrix, and the overall matrices are 4 by 4 matrices, and the 0s are conveniently placed. So, one can decouple the components in taking the non relativistic limit.

In the same notation, the beta is the same as gamma 0, of course, and one can also constructs, gamma i, in this structure which will now written as sigma i and minus sigma i on the of diagonal. And one can now write the 4 component Dirac equation in terms of this 2 by 2 components in poly matrices. We will use this decomposition extensively to identify the various modes of the solution, and in particular the particle and antiparticle component as well as the spin degrees of freedom.

So, there are 4 components correspond to particle, antiparticle, and up and down spin values. We will come to this interpretation very soon. But, now let us first look at the same current conservation relation in terms of this new notation, and that does give us some new idea about what all happens in case of Dirac equation. So, the strategy is just simple. We have the Dirac equation. We will use it twice - one the original equation, another the complex conjugate one, and then multiply one from the left by psi bar and the second from the right by psi, and then subtract the 2 to get a relation which will contain only the derivative operators while all the constant terms drop out.

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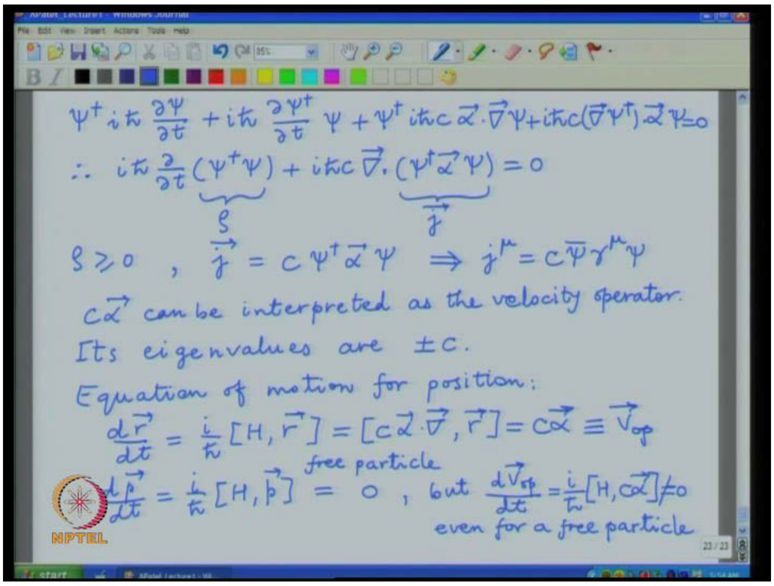

And, that relation can be easily seen now to be the combination. This is the time term, then you subtract its complex conjugate version; similarly, this is the space term and then you subtract its complex conjugate version. And of course, the risk mass term just cancels out in this particular procedure, and this derivates now is can be combined into the form. And this is the form which Dirac was looking for. In particular, he will linearize the equation in time derivatives.

So, the quantity which appeared in this equation had only the first derivative in time, and that is the derivative of the charge density on the particular procedure which you use. It produce a charge density which is just psi dagger psi, and so it is positive definite. And then the corresponding object, which appear together had to be identified with the current to convert this into the current conservation equation. The positive definite character of the density now makes its identification convenient; that you can call it a probability density or a number density. And that saves the trouble which was the inconvenient feature in case of Klein Gordon equation.

And, Dirac thought here sub mounted the problem of the negative density and its interpretation in terms of charges instead of numbers, and also the associated ambiguity in the signs of the energy where it turns out that that is not really true; the ambiguity in the sign of the energy still survives in a different form; in particular it survives here in the form of multi component nature of this wave function; it is no longer a single complex number; and that nature has the peculiarity which ultimately Dirac had to resolve in terms of its new name; nowadays it is called the whole theory or antiparticle interpretation.

But, in terms of this continuity equation, rho is positive definite, but the current now looks very different. In particular, it does not look anything like the velocity multiplied by the charge density. The operator which appear here not related to momentum or derivative of space, rather it is a operator from the Clifford algebra. And one has to now reinterpret that operator in terms of the velocity.

And, we can choose to do so that they will immediately notice that it has its peculiar properties. Its Eigen values are just plus or minus c n naught and arbitrary real number bounded by speed of light. And so it will have its algebra which is distinct. And one has to now see how the various components of psi contribute in defining a state which has a particular velocity. One can define a covariant notation for these vectors as well which can be written as, j mu is equal to c psi bar gamma mu psi, which takes into account both space and time component, and the matrices gamma mu, now get associated with the indices of the current 4 vector.

So, now to understand this velocity in a little more detail, we can now compare it with the equations of motion. And the simplest equations of motions are can be just worked out in analogy with calculation in case of non relativistic equation. And so that object is, d r by d t, is the commutator of r with the Hamiltonian. And now one can stick in the explicit form of the Hamiltonian.

The only part of the Hamiltonian which does not commute with position vector is the one containing gradient. And one can easily evaluate this quantity which does produce the same operator which we identified in the current component. And indeed we can now take this to be a definition of the velocity operator. It can be interpreted just as a standard classical mechanics, the rate of change of position, but it is no longer related to the momentum operator, and that is the peculiarity of Dirac's constructions.

For instance, d p by d t, will be equal to commutator, the same way with Hamiltonian. And if you take a free particle which is a eigen states of momentum then this commutator is 0, but on the other hand if you construct the derivative of the velocity operator, and then it will become a commutator of H with alpha, and this will be non 0 even for a free particle. So, this is a peculiarity of this new velocity operator, that it is related to the position operator in the usual sense of time derivative, but it is not related to the momentum operator in any simple minded fashion.

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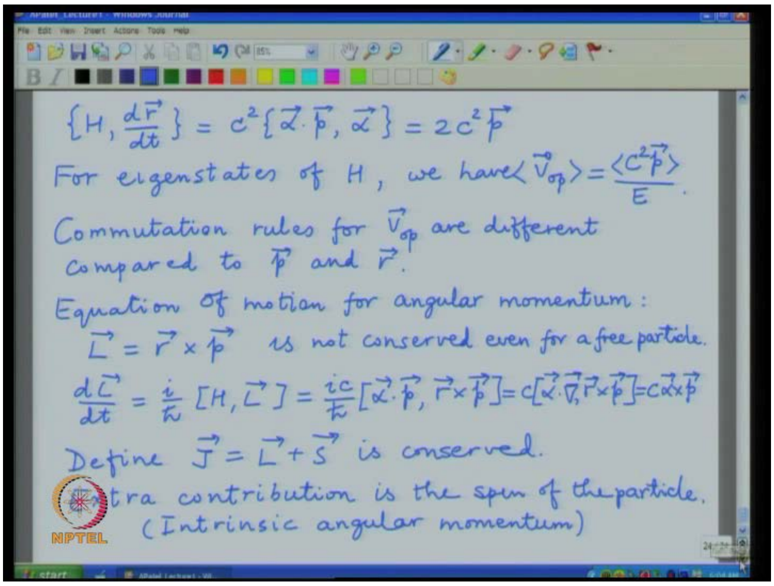

One can try to construct some other combination which will work in the sense of relating this particular operator to the momentum. And after some trial and error it can be figured out that one can construct the anti commutator of the Hamiltonian. And this derivative of the position operator which can be now written as anti commutator of alpha dot p and alpha, the risk mass term gives 0 commutator; and now the here the term which survives is just the momentum part.

So, now, this can be interpreted as an equation which relates the velocity operator to the momentum. But only in some specific situation, in particular to get rid of this explicit factor of the Hamiltonian, one can use eigenstates of the Hamiltonian which will make this into hectares of energy. And then this v operator, the expectation value is equal to the expectation value of p divided by E, and that is essentially our definition of what we mean by velocity in relativistic dynamics.

The only thing, it is necessary to associate the expectation values of the velocity operator to the momentum, and not the operator directly itself. So, this is one of the caveat which appears in Dirac’s equation, and we just get used to it. One can also note at this point, that the commutation rules for the velocity operator are different compared to the commutation rules of p and r.

In particularly, the individual components of momentum commute with each other, individual components of position commute with each other, but individual components of velocity are not going to commute with each other. And this has a nontrivial consequences in terms of smearing of the wave functions over a certain region, no matter how hard you try to localize it. It is just not possible to do it in relativistic mechanics. So, this was one particular equation of motion, the one for position.

One can now look for equation of motion for another important quantity which is the angular momentum. And one notices another new feature appearing in that particular case. Now, we have a well known operator which is, r cross p, both in classical mechanics and quantum mechanics. And we can now try to see how this particular object evolves under time. And we discover that even for a free particle without any forces this object is not conserved.

See this explicitly, just workout the corresponding commutators; simple algebra. The parts which do not commute with L are of course the parts, which involve the gradient operator; the m c square is a constant part and that does not have any effect on the commutator. And in this particular case, the term which gives non 0 commutator is at p x and r. And so it can be now written as alpha dot gradient acting on r cross p, and which now easily simplifies to alpha cross p.

And, as we saw, this object is not going to be 0 even for a free particle which will have some fixed value of momentum. One can now again see that the, what is appearing here is a cross product of the velocity operator. And the momentum, if these 2 things had been identical it would have been 0, but they are not the same in; velocity operator is actually little different than the momentum operator; and so this angular momentum is not conserved in general.

One can ask the question, what do we do? Because this is not a easy thing to give up. Not only angular momentum is conserved in all the classical formulations, but we know that experimentally it is conserved in quantum situations as well. So, there must be some way to generalize this equation, so that the well known properties of angular momenta are not destroyed. And Dirac indeed found a way to generalize these objects. He defined a new operator which is different than the classical L. And that extra contribution was added to L, so that the total angular momentum is conserved.

And, nowadays we call this extra contribution is as the spin of the particle. Often it is referred to as a intrinsic angular momentum, which is not due to the motion of the particle in some space time, but it is internal property of every particle. That angular momentum has to be added to the L which is called the orbital angular momentum. And the combination of spin and orbital angular momentum together define the total angular momentum denoted by J, and that object is conserved. So, now the job is to define this as, whose time evaluation exactly compensates the time evaluations of L. And with some amount of working backwards one can quickly figure out what the corresponding algebraic structure of this spin angular momentum is...

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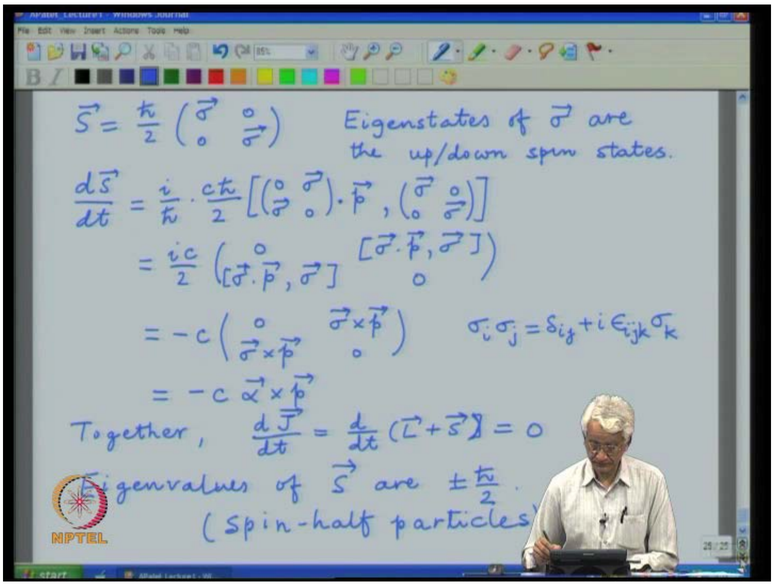

And, that turns out to be the form is h cross by 2, multiplied by a poly spin matrices on the diagonal. And we know that this particular bases choice is again chosen conveniently as the up and down spin states. To see that, this indeed does the required job; we can again calculate some simple commutators. So, this object is now; the first term comes from the alpha dot p part of the Hamiltonian, the matrix beta is diagonal and it commutes with s, so it does not contribute over here.

And, so all we have is this commutators of poly matrices, and then can be now simplified by simple 2 by 2 matrix products. And the products of poly matrices, and in particular the anti commutators are easily evaluated by the well known identity of the poly matrices which generates this particular structure. The identity which we need is essentially the product rule for poly matrices.

For the commutators, only the epsilon term contributes, and I use the epsilon term together with the poly matrices to write the whole thing into a cross product notation. And that is exactly the term which was present in the time evaluation of the angular momentum operator, d L by d t, give c alpha cross p; and d S by d t gives minus c alpha cross p. So, once you add those things together, the combinations cancel to conserve this total angular momentum. And one can also see that these eigen values of S are, plus or minus half in units of Planck's constant; and that is why the particles which obey Dirac equation are often referred to as spin half particles.

So, this is something which emerged out of the Dirac equation. It was not there in the Klein Gordon equation. There were no multiple components for the wave function, and so there was no extra degree of freedom. But, in the Dirac equation, the extra degrees of freedom were already there. And by looking at angular momentum we can identify that this contain the degrees of freedom corresponding to spin half particles. This degrees of freedom take care of a factor of 2 out of the 4 components, and the other factor of 2 for the 4 components remains associated with the particle and antiparticle equations. And that is something which is identical compared to the Klein Gordon equation.

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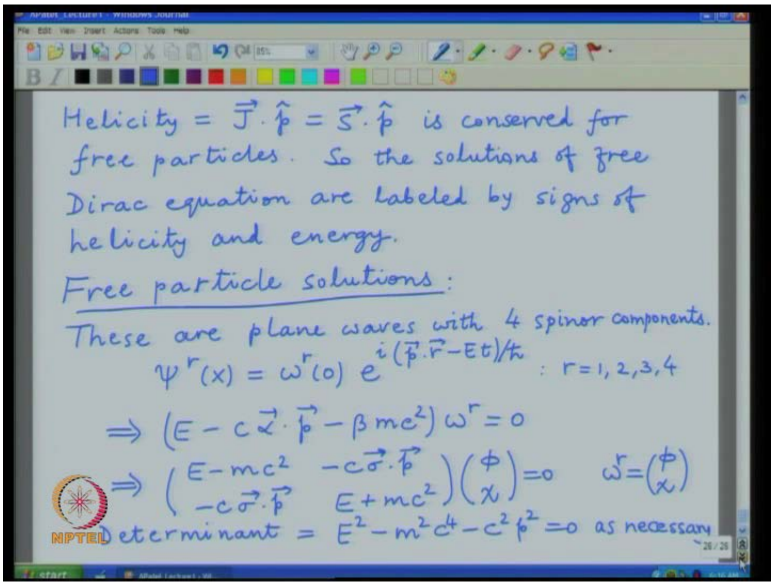

It is also useful to define a object which is called Helicity, is the projection of the angular momentum along the direction of motion. And since orbital angular momentum is always perpendicular to the linear momentum, this automatically reduces to the projection of the spin along the direction of motion. And this quantity is conserved whenever the total angular momentum is conserved, and in particularly it is conserved for free particles.

So, the solutions of free Dirac equation are labelled by signs of Helicity and energy. Each one has 2 different possibility for the sign; and so together they count all the 4 components. To see these things explicitly, let us write these solutions down. In particularly, we have all the ingredients are already present, and we just have to choose a notation which makes things rather explicit.

And, one thing which you already know, that these solutions are just plane waves with 4 spinor components; and we will label them as psi (x) as omega with some index. This index is the spinor index, multiplied by the explicit space times, dependence which we know for plane waves is p dot r minus E times t divided by h cross, and r has now the 4 values. So, once this space time degrees of freedom is factored out we want to see the interpretation of these 4 components, omega r. And here is where the convenient choice for the basis made by Dirac turns out to be helpful. The equation now has the form where E and p are now the eigen values. They are no longer operators. And one can simplify it little more by using a 2 component notation, where all I have done is replaced this omega r by this 2 component spinor phi and chi.

And now, the explicit choice of the coordinates allows us to solve for what is phi, and what is chi, in a specific situation. To have a non trivial solution, the determinant of this linear set of equations must vanish. And determinant is nothing but the usual dispersion relations. So, it is E square minus m square c fourth minus c square p square is equal to 0, as required for a non trivial solution to exist.

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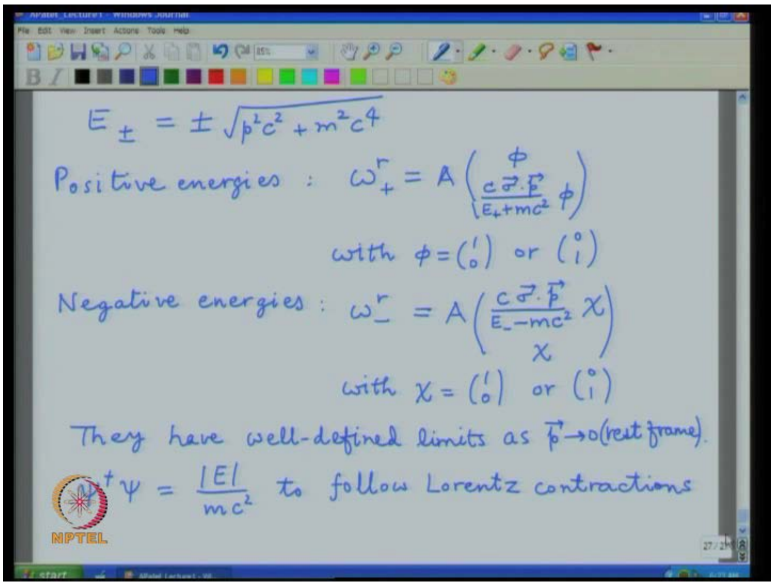

So, now let us just see what the solutions are. The determinant tells you that the 2 signs of the energy come out as 2 separate eigen values. And I will denote them as E plus or minus, as square root of p square c square plus m square c raise to 4. And this is a part which Dirac could not eliminate, even after linearizing the equation, that if you want to calculate the energies, eigen values, they will come out with 2 different signs.

And, now one can solve the equation in terms of this specific eigen values. And it is convenient to separate the 2 solutions, certain ones corresponding the positive eigen values, and certain one corresponding to the negative eigen values. So, for positive energies we write the solutions, omega plus is equal to some normalization constant, and then the 2 component spinors. Replacing chi in terms of phi is convenient in this case, because the denominator which appears here, E plus m c square is atleast twice m c square. And so this particular object is somehow smaller compared to the upper part of the spinors. And one can choose the basis for phi, a rather trivial choice which actually corresponds to up and down values of the spin.

The analogous solutions for negative energies are written now in terms of the lower components, so that the denominators which are appearing are larger than the numerators. And one can take the non relativistic limit in a simple manner when dropping either the lower components in this case, or the upper components in this particular case. And again the 2 solutions for chi are 1 0 or 0 1.

So, now, we have got in the 4 components with direct physical interpretation. The up and down choice of the spin is buried inside the values of phi and chi. And then the sign of the energy now is chosen, that in one case the upper 2 components are dominating, and in the other case the lower 2 components of the spin are dominating. If you literally take the non relativistic limit the p goes to 0. And in that case, the upper 2 components will be associated with the particle, and the lower 2 components again in that p going to 0 limit will be associated with the antiparticle.

So, this is the complete interpretation of the 4 components. So, they have well defined limits as the momentum goes to 0 which is also called a rest frame of the particle. One can choose a convention to normalize this thing, and again it is convenient to choose a convention which obeys Lorentz transformation properties perfectly. And that convention is that the number density, which we associated with this object which is psi dagger psi. So, it has to transform properly under the normalizations of the spinors.

And, once you go from one frame to another, it will undergo Lorentz contractions in boost operations, and so the proper normalization of this object is psi dagger psi is E divided by m c square, the absolute value to follow the Lorentz contractions. And it is not normalized to 1 as it is in the non relativistic mechanics. But, picking this particular convention now, it is a trivial job to construct this. Of course, the space time dependent phase is just cancelled out directly. And the part which remains is now to square this particular function. And in this case it is a simple relation of what these quantities become.

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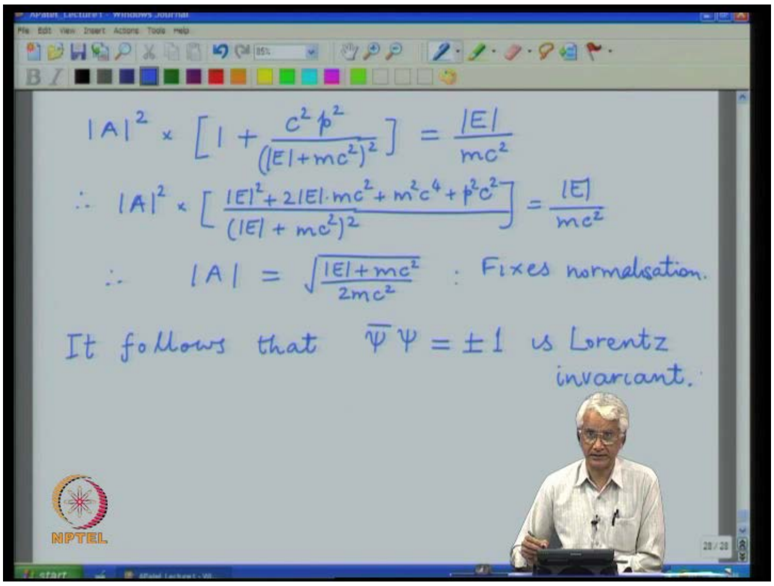

So, one gets this object A square, the square of the upper component say in the positive case gives 1, lower component gives p square. And one has to do a little bit of algebra combining 1 and p square to convert it into E and m c square. So, one can easily see that it is a 1 is there, square of the lower component will give c square p square divided by mod E plus m c square, whole thing square, sorry.

It is better to rewrite it in terms of the original equation this left hand side is psi dagger psi, right hand is E by m c square. And now one can do a little bit of manipulation, E square plus m c square combines if the c square plus q square to produce well known dispersion relation terms. There is one factor of E plus m c square which actually cancels with the, whatever comes in the numerator.

So, there is mod A square into; and then the same thing, and combine this thing into E square, and then one factor explicitly cancels between numerator and denominator. And one has the final result which is, normalization is given by this particular object. Of course, one can only determine the absolute value or normalization as is usual in quantum mechanics, overall phase is left undetermined.

So, and one can easily workout from this normalization an interesting fact that the quantity psi bar psi is 1 in magnitude. The only difference is instead of getting plus 1 here it will have a minus sign, but this objects turns out to be just 1 by this normalization convention, and that is convenient because this quantity is then Lorentz invariant, and we will have to make use of it in various applications later.