Relativistic Quantum Mechanics
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Lecture - 45
Status of QED, Organization of perturbative expansion, Precision tests

In this concluding lecture, I am going describe the present status of the theory of quantum electrodynamics. QED has been tremendously successful as a perturbative quantum field theory. To explain this success let me first tell you how the results of perturbative calculations are systematically organized.

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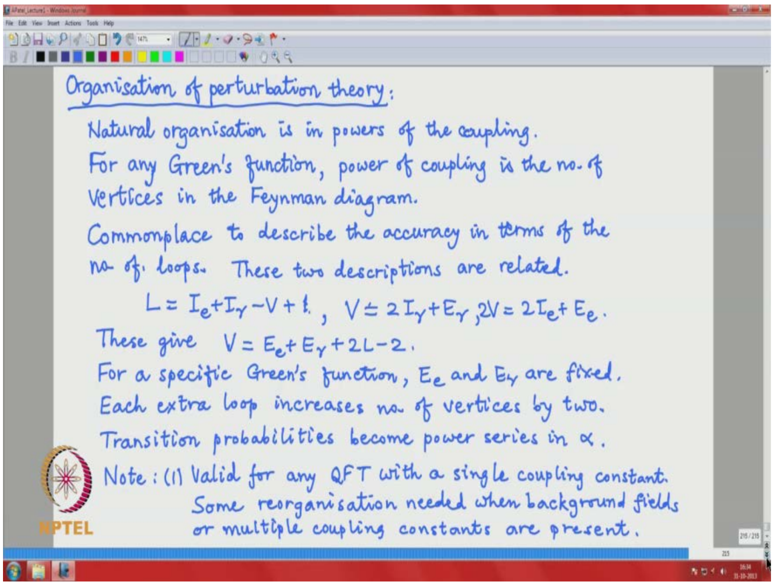

Organisation of perturbation theory:

Natural organisation is in powers of the coupling.

For any Green's function, power of coupling is the no. of vertices in the Feynman diagram.

Commonplace to describe the accuracy in terms of the no. of. loops. These two descriptions are related.

$$L = I_e + I_\gamma - V + 1, \quad V = 2I_\gamma + E_\gamma, \quad 2V = 2I_e + E_e.$$

These give $V = E_e + E_\gamma + 2L - 2$.

For a specific Green's function, E_e and E_γ are fixed.

Each extra loop increases no. of vertices by two.

Transition probabilities become power series in α.

Note: (1) Valid for any QFT with a single coupling constant. Some reorganisation needed when background fields or multiple coupling constants are present.

In any perturbative framework, it is natural to arrange all the contributions to a given process in powers of the small parameter, which in case of QED is the electromagnetic coupling. For any Feynman diagram contributing to a specific endpoint Green's function this power of the coupling is just the number of interaction vertices. For any Green's function we just have to count the number of vertices to find out to which order of perturbation theory it belongs to...

But on the other hand it has become quite common place to specify the accuracy of perturbation results in quantum field theories in terms of the number of loops. These two descriptions, number of vertices and number of loops are related, and let me show how. In your earlier lecture I had given the topological relations between the number of vertices, number of loops, number of external lines, and number of internal lines for any Feynman diagram, and these relations where the number of loops is number of internal lines minus the number of vertices plus 1.

The number of vertices can also be related to the number of photon lines and also to the number of electron lines. If we eliminate the number of internal photon and electron lines from these relations, we obtained a result which relates the number of vertices to a number of external lines and the number of loops. For a given Green's function the number of external liner are fixed. So, each extra loop increases the number of vertices by 2. As a result the transition probabilities and cross sections which are obtained by squaring the amplitude can be expressed as power series in the fine structure constant alpha which is the square of the electromagnetic coupling.

This is one relation which says that one can label the perturbative contributions either by power of alpha or by number of loops. I should point out two specific points in this framework that this analysis is valid for any quantum field theory with a single coupling constant. When there are multiple coupling constants in the theory or there are background fields some reorganization of the series is needed depending on the relative strengths of different terms. So, we have to watch out for that.

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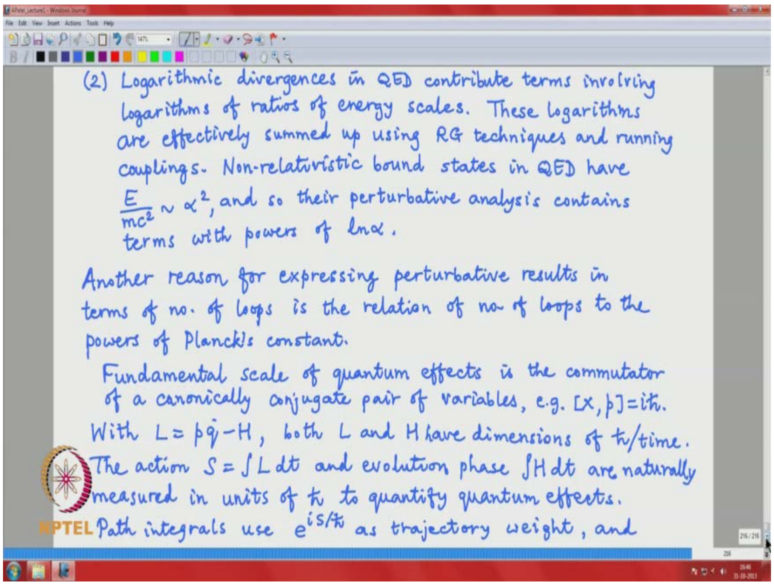

(2) Logarithmic divergences in QED contribute terms involving logarithms of ratios of energy scales. These logarithms are effectively summed up using RG techniques and running couplings. Non-relativistic bound states in QED have $\frac{E}{mc^2} \sim \alpha^2$, and so their perturbative analysis contains terms with powers of $\ln\alpha$.

Another reason for expressing perturbative results in terms of no. of loops is the relation of no. of loops to the powers of Planck's constant.

Fundamental scale of quantum effects is the commutator of a canonically conjugate pair of variables, e.g. $[x,p]=i\hbar$.

With $L = p\dot{q} - H$, both L and H have dimensions of $\hbar$/time.

The action $S = \int L\,dt$ and evolution phase $\int H\,dt$ are naturally measured in units of $\hbar$ to quantify quantum effects.

Path integrals use $e^{iS/\hbar}$ as trajectory weight, and

The second point to note is that there are divergences in the calculations carried out by perturbation framework, and these divergences give physical contributions after they are regularized properly. In case of QED the divergences are logarithmic and so there appear terms involving the logarithms of different energy scales. These logarithmic terms can be controlled essentially summed up to a large extent by the techniques of renormalization group evolution, which replaces the fixed coupling constant by a running coupling. Once that is done remaining logarithms are relatively small; they do not disturb the hierarchy of terms in the perturbation series. Sometimes these logarithms can be converted to logarithms of alpha itself from logarithms of ratios of energy scales.

That is the situation in case of non-relativistic electrodynamics where the ratios of the scales are essentially powers of the couplings. So, sometimes the series in perturbation theory can involve terms that are log alpha, but one should realize that they are coming from ratios of energy scales. Given this two caveats we can easily rearrange the series where successive terms decrease in magnitude and the perturbative machinery makes reasonable sense. There is another reason for expressing the results in quantum field theory in terms of the number of loops, and that is their connection to powers of the Planck's constant.

The fundamental scale of quantum effects is given by the commutator of economically conjugate pair of variables. The simplest example in quantum mechanics is the commutator of position in momentum is equal to i H cross. From the definition of the conjugate momentum and the relation between the Lagrangian and the Hamiltonian we can easily see that both L and H have dimensions which are those of Planck's constant divided by time. The theory actually indulges time integrals of the Lagrangian in the Hamiltonian.

They are the action which is the integral of the Lagrangian and the evaluation phase which is the integral of the Hamiltonian. Both these time integrals are naturally measured to quantify quantum effects in units of the Planck's constant. As a matter of fact the path integral formulation uses e raised to i as i H cross as the weight of the trajectories, and the time evolution operator is an exponential of minus i by H cross integral H dt. When these powers of Planck's constants are explicitly included in the formulism, remember that we had chosen a notation at some stage setting H bar equal to 1, but now we are putting back the factors of H bar explicitly.

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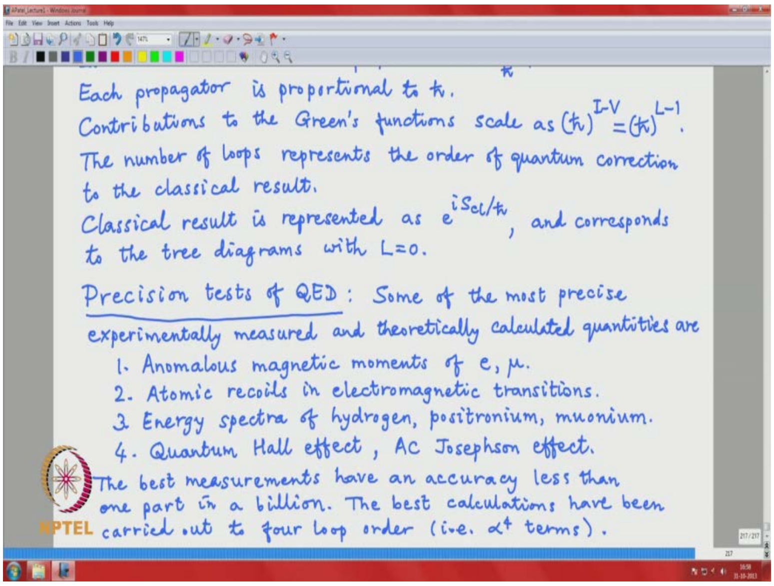

Each propagator is proportional to $\hbar$.

Contributions to the Green's functions scale as $(\hbar)^{I-V} = (\hbar)^{L-1}$.

The number of loops represents the order of quantum correction to the classical result.

Classical result is represented as $e^{iS_{cl}/\hbar}$, and corresponds to the tree diagrams with $L=0$.

Precision tests of QED: Some of the most precise experimentally measured and theoretically calculated quantities are

1. Anomalous magnetic moments of e, μ.
2. Atomic recoils in electromagnetic transitions.
3. Energy spectra of hydrogen, positronium, muonium.
4. Quantum Hall effect, AC Josephson effect.

The best measurements have an accuracy less than one part in a billion. The best calculations have been carried out to four loop order (i.e. α^4 terms).

Each interaction vertex is proportional to 1 by H cross and each propagator which is given by the reciprocal of the differential operator appearing in the Lagrangian or the Hamiltonian is proportional to H cross. So, the contributions to the Green’s functions scale as h bar to the power of I minus v; I counts the propagators and v counts the vertices which is equal to H bar the power of L minus 1 by the topological relations which we have already seen. So, the number of loops represents the order of quantum correction to the classical result. Note that classical result corresponds to extermination of the action, for example, and so it would be represented as exponential of the classical action divided by H cross, and it corresponds to the tree diagrams with no loops.

This is the way perturbative results in quantum electrodynamics, and in general in any quantum field theory is organized. Once we have a systematic description of the results we can go and compare them with experimental observations; that is indeed done in many important checks in case of QED. Let me list some of them. These are called the precision tests of the theory. Some of the most precise experimentally measured and theoretically calculated quantities are the anomalous magnetic moments of the electron and the muon. Atomic recoils in electromagnetic transitions energy spectra of hydrogen positronium and muonium which is the bound state of an electron and a muon. And also condensed matter phenomena like quantum hall effect and AC Josephson effect.

The best measurements have an accuracy that is less than one part in a billion. The best calculations have been carried out to the order of four loops. These two have similar accuracy to give a familiar perspective to this high precision measurements and calculation. I can say that such measurements to one part in a billion amounts to measuring the distance from Kashmir to Kanyakumari to accuracy that is less than a millimeter. So, these are indeed phenomenal results and the achievement of the theory is that the experiment and theory agree to this level of accuracy.

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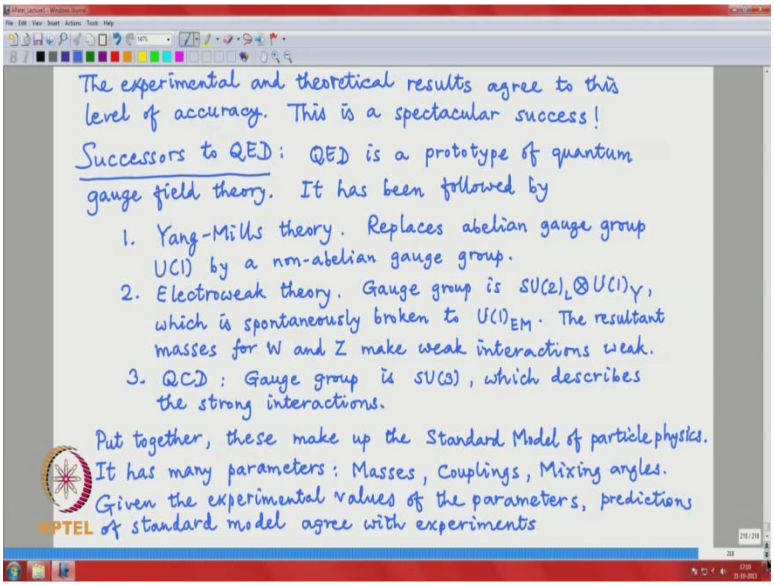

The experimental and theoretical results agree to this level of accuracy. This is a spectacular success!

Successors to QED: QED is a prototype of quantum gauge field theory. It has been followed by

1. Yang-Mills theory. Replaces abelian gauge group U(1) by a non-abelian gauge group.
2. Electroweak theory. Gauge group is $SU(2)_L \otimes U(1)_Y$, which is spontaneously broken to $U(1)_{EM}$. The resultant masses for W and Z make weak interactions weak.
3. QCD: Gauge group is SU(3), which describes the strong interactions.

Put together, these make up the Standard Model of particle physics. It has many parameters: Masses, Couplings, Mixing angles. Given the experimental values of the parameters, predictions of standard model agree with experiments

This is a spectacular success of QED; no other theory in physics has been so stringently tested and has proved to be so stunningly accurate. Obviously, once we have found formulation of quantum field theory I am showing that it works very well. It is natural to apply it to different phenomena, different observations and build theories having similar structure. That has indeed followed the developments in quantum electrodynamics. We have many successors to QED. QED is a prototype of quantum gauge field theory, and it was followed by other quantum gauge field theories. The first one to combine was Yang-Mills theories which replace the abelian gauge group U 1 by a non-abelian gauge group.

The next one is the electroweak theory which combines electromagnetism with weak interactions. The gauge group is SU 2 left cross U 1 hyper charge which is spontaneously broken to the U 1 gauge group of electromagnetism. The masses for the weak bosons w and z are the result of this spontaneous symmetry breaking, and they make the weak interactions weak. The next one to follow was the theory of quantum chromo dynamics which uses the gauge group SU 3 and describes the strong interactions. All these ingredients put together make up the so called standard model of particle physics.

This standard model has many parameters essentially the masses of the particles, the couplings of the gauge interactions and the mixing angles which parameterize the mismatch between mass Eigen states and gauge interaction Eigen states. So far we have no understanding of where these parameters have come from, but given the values of these parameters from experimental observations. The standard model can predict a large variety of phenomena observed both in low energy systems as well as high energy experiments. The situation is such that we have not so far seen any significant deviations from the predictions of the standard model in all the experiments that have been carried out. So, in that sense the standard model is a highly successful effective field theory.

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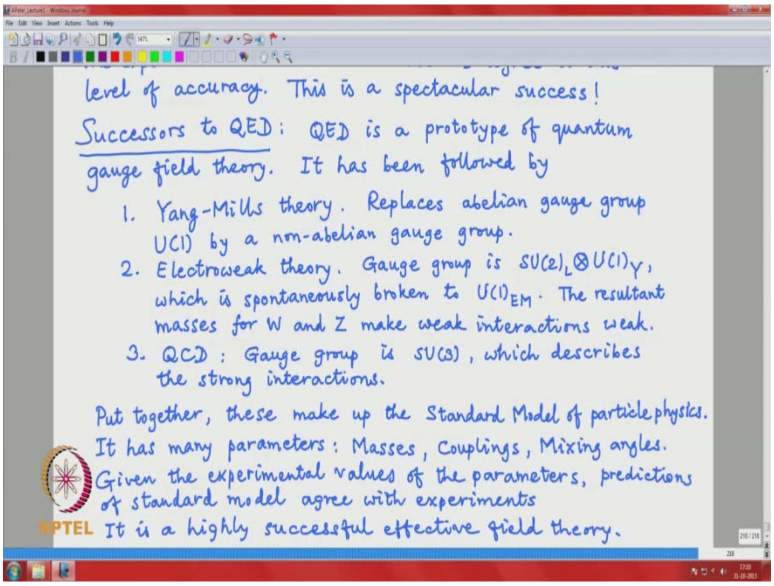

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It is a highly successful effective field theory.

The merit of any effective theory is that it is not going to go away as long as you remain within its domain of application. You can of course, explore beyond, and maybe there are deviations in the region outside the boundary of the effective theory. Maybe that will produce a generalization of the theory where you able to reach such regions outside the boundary. But if you are going to work within the boundaries the theory will always be useful. That is essentially the status of high energy physics these days.

That has not stopped physicist from working hard; experimentalists have being trying to set up new devices, new accelerators, new situations where they can measure or hope to measure deviations from predictions of the standard model. And theories have been busy trying to make a more general theory, sort of a unified picture which can explain the values of the parameters which appear in the standard model. And also at some stage include the theory of gravity which is not part of the standard model, but right now both these features, experimental searches as well as theoretical developments are outside practical regions where tests have been carried out.

Let me recapitulate where all these subjects started. It was a successful merger of the theory of special relativity and quantum mechanics which produce the formulism for quantum field theory. The role played by quantum electrodynamics has been a pioneering one in these developments. Nowadays the subject of quantum field theory has expanded. It has applications in high energy physics, statistical mechanics, condensed metal physics covering a variety of phenomena involving different type of fields and also different number of space time dimensions.

But the formulism has a common thread which can be backtracked to all the developments which took place in building the theory of QED. The role of QED in this sense cannot be overemphasized. In summary I can just repeat the closing words of Richard Feynman in his Nobel Prize lecture. “So, what happened to the old theory that I fell in love as a youth? Well, I would say that it has become an old lady that has very little attractiveness left in her, and the young today will not have their hearts found anymore when they look at her. But we can say the best we can for any old woman that she has been a very good mother and she has given birth to some very good children.” So, that is the way it is.