Relativistic Quantum Mechanics
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Lecture - 44
Symmetry constraints on Green's functions, Furry's theorem, Ward-Takahashi identity, Spontaneous breaking of gauge symmetry and superconductivity

Continuing the discussion of various kind of divergence is a little bit further, I want to mention a few more things which help in understanding the nature of the divergences as well as their actual type or size.

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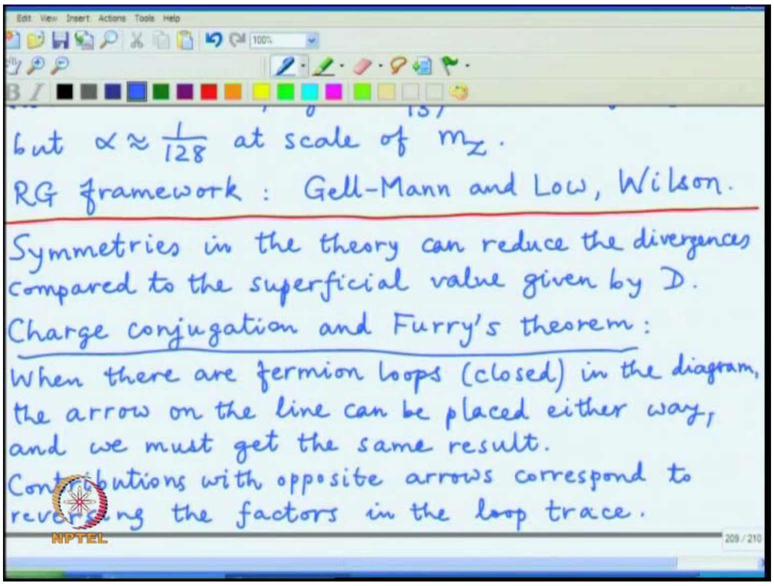

And that is the consequence of symmetries in the theory. They can reduce the divergences compared to the superficial value given by D that I defined earlier, and the two particular cases which I want to describe are the symmetries of charge conjugation and the symmetry of gauge transformations. First case is that of charge conjugation and which also is referred to as Furry's theorem and it applies to situations when there are fermion loops, by this I mean the closed fermion lines. We have a freedom of placing the arrow on the line either way, because it is a degree of freedom which is not fixed by external line, and because it is a genuine degree of freedom the answer must be the same.

Now the arrow on the line is associated with our labels of particle and antiparticle and what this symmetry says is that for close fermion loops there is no way still we can distinguish particle from an antiparticle, and the answer should be the same whether we labeled one state as a particle and the other as antiparticle or a vice versa. So, if you now want to compare these two contributions with opposite order of the arrow. So, the contribution will be made up of various propagators and vertices.

And they are multiplied in a certain order, and because it is a close loop there will be a trace; at the end of it if we reverse the arrows then all the factors appear in the opposite sans and there is still a trace. And one can do the mapping between these two contributions by taking the transpose of the matrix. The transpose basically reverses the order of the factors, and since there is an overall trace it is just a number and the transpose will not change the value of that particular number.

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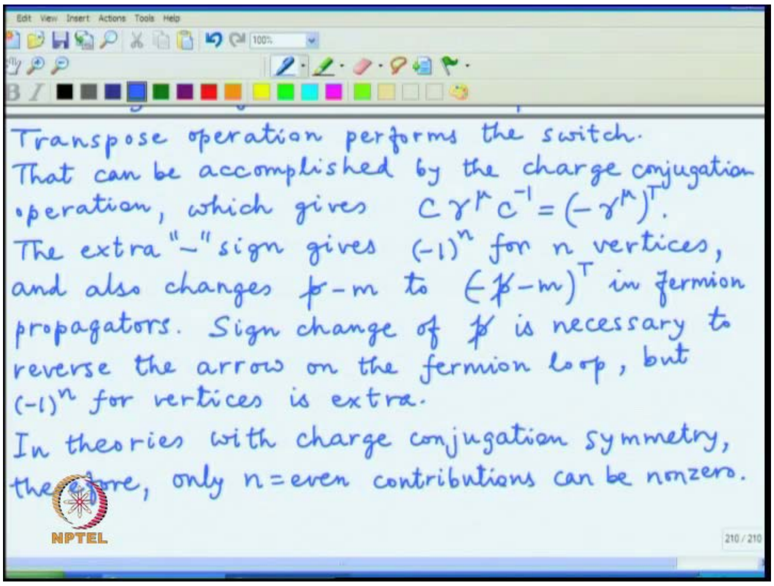

So, transpose operation performs the switch from arrow pointing one way or the other. And so now we have to find a relation which does this transpose mathematically, and that can be accomplished by the charge conjugation operation which gives the transforms which gives c gamma mu c inverse is equal to minus gamma mu transpose. So, one can take the complete product of various factor, and between each successive pair of factors introduce c and c inverse which is mathematically an identity and then transform every factor by this particular rule.

So, the transpose part is indeed the one which reverses the order of the factors and which is necessary to flip the sign of the arrow, but the minus sign in front gives an extra effect.

So, the extra minus sign gives minus 1 raise to n for n vertices which may be present in the diagram. Each vertex will just have gamma mu, and also the gamma mu is part of the fermion propagator p slash minus m to minus p slash minus m and should take the transpose for the fermion propagators.

And although all this sign changes this sign change of p slash is necessary to reverse the arrow on the fermion loop, because it indeed changes momentum to negative of its value, but the sign change corresponding to the vertices is extra. And as a result of this simple argument so in theories with charge conjugation symmetry; therefore, only n equal to even contributions can be nonzero. Otherwise, the result will be minus its value because of this particular symmetry and which will force it to be zero. And this has an immediate consequence on the divergent diagrams which I had drawn in discussing the superficial degree of divergences which are present at one loop.

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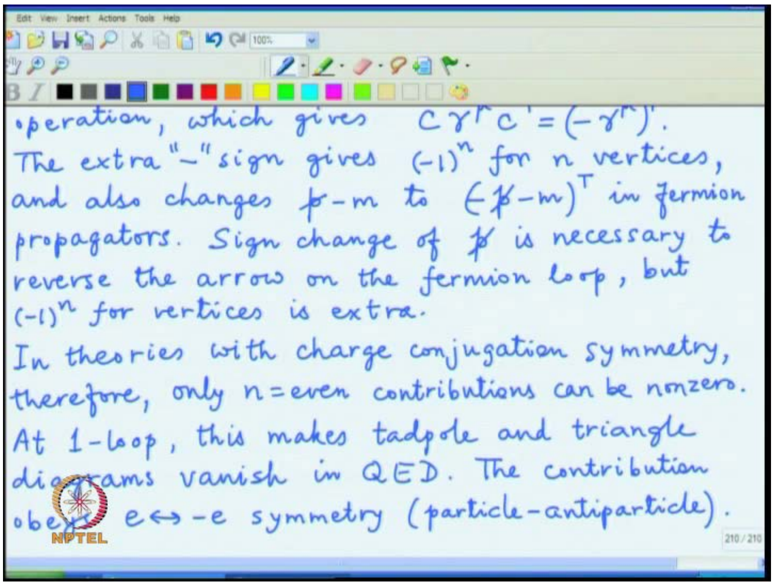

So, at one loop this makes tadpole and triangle diagrams vanish in QED. The diagrams which survive with the close fermion loop at one loop order are the vacuum polarization and light-by-light scattering. So, this is one particular consequence, and one can also say that the contribution obeys the so called e to minus e symmetry which in this particular case because we are looking at particle and antiparticle arrows and call it particle antiparticle symmetry as well. So, this is a genuine feature of the theory; it is symmetric between particle and antiparticle and does not, therefore, care whether which one we are going to call particle and which one antiparticle the result has to be the same and only even powers of the coupling e will appear in the final nonzero amplitude. So, this is one particular contribution of symmetry.

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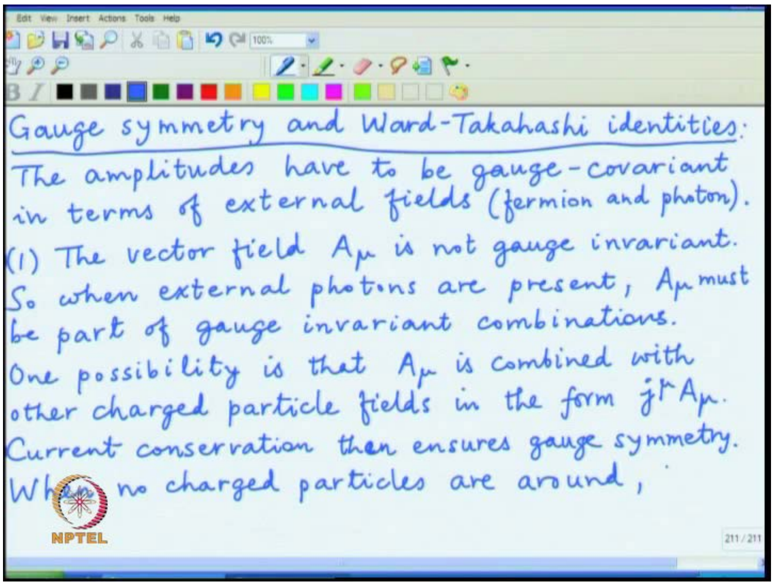

The other is the more familiar gauge symmetry, and its consequence for this result is called Ward-Takahashi identities. Now the amplitudes have to be gauge covariant in terms of external fields which may be present in the diagram either the fermion or the photon, and this puts several kind of restrictions which I want to elaborate. Again I will break it up into different categories which can be described in different language; one of them is the photon vector field A mu is not gauge invariant. So, if one want to extract the physical answers these have to be handled in only specific combinations when it is part of the external legs of the diagram.

So, when external photons are present then this A mu must be combined with other objects so that the total combination becomes gauge invariant. So, A mu must be part of gauge invariant combinations, and we have seen two examples of this type in our course. One of them is A mu is combined with other charged particle fields in the form j mu A mu, and if this is done then everything is okay, the current will be conserved and then ensures gauge symmetry. But it can be that the field is not combined with any charged particle currents at all or it may be that. There are no charge particles at all to combine A mu with and in that particular case when no charged particles are around.

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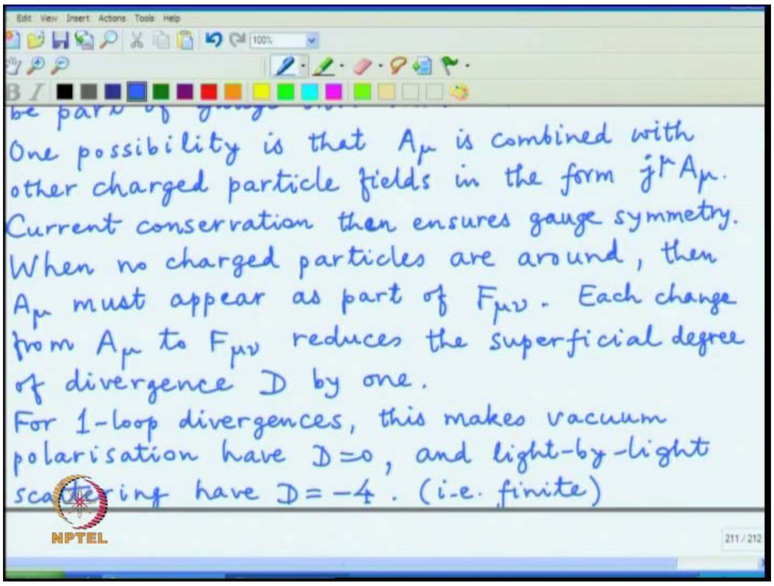

Then A mu must appear as part of F mu nu which is gauge invariants, but this is a particular reduction which brings in a factor of derivative or equivalently a factor of momentum. So, the result is that each change from A mu to f mu nu reduces the superficial degree of divergence D by one, because in calculating D we just counted powers of momenta and assumed that each one of them was going to infinity. In this particular case some powers of momenta will be part of f mu nu, and they correspond to external physical photon. They are not going to infinity; only the remaining parts of momenta will go to infinity and so every time A mu is converted to f mu nu. D will decrease by one, and this is the rule which again we can go back and look at the various diagrams listed in case of one loop divergences.

And we have diagrams with no fermion external lines in which A mu has to be converted to f mu nu; there is no current to take care of it. And in this particular case the vacuum polarization diagram which had superficially D equal to 2 had two external photons. And so converting them to f mu nu drops D to 0 which is a logarithmic divergence and not a much worse quadratic divergence. And the other diagram which I called light-by-light scattering; the value of D was 0 in that particular case, and it had four external photons. And so it becomes actually D equal to minus 4 which is a finite value, and in this way out of the four diagrams with only external photons we have only one surviving that has a divergence Furry’s theorem eliminated to, and this gauge invariance made one of them finite.

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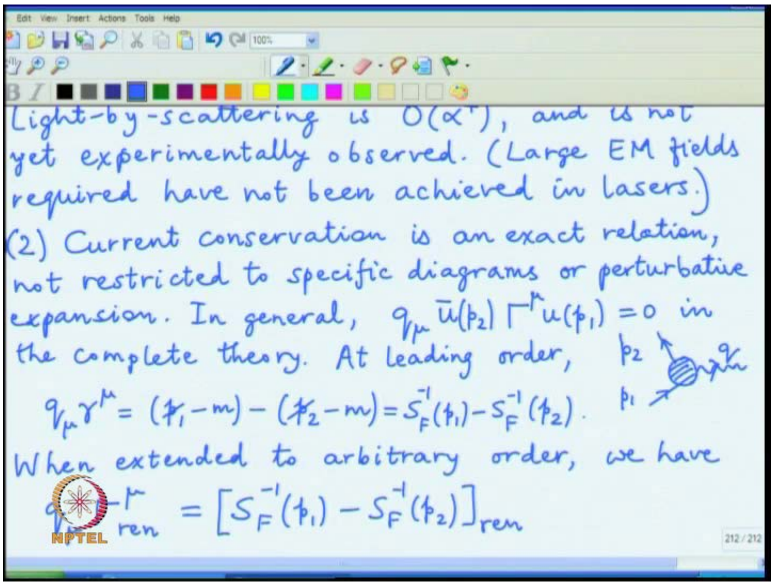

And in particular this light-by light-scattering is order alpha to the 4. There are four photons involved in the amplitude which gets squared in the cross section, and to observe it we need very large values of f mu nu very strong electromagnetic field, because alpha raise to 4 is nearly small. And though it is calculated explicitly and giving a finite answer we have not seen it by experimentally. So, large electromagnetic fields that are required have not been achieved, and the way to have very large electromagnetic fields actually use lasers, and we just do not have that powerful lasers which allows us to observe scattering of one beam from another.

So, this is one category where gauge variance is useful in reducing the degree of divergence. Now let me go to the second category, and that refers to the coupling of photon to the current. And this current conservation is an exact relation not restricted to specific diagrams or perturbative expansion, and so even when we carry out calculation by various techniques to any arbitrary order the consequences must follow. And this is actually the result which is most often called the Ward-Takahashi identity, and what it says is a generic interaction. So, in general you must have q mu contracted with the current where the vertex I denoted by capital gamma which can have many complicated terms and in the complete theory and to put specific notation I can generically denoted it as a blob.

There is a momentum p 1 coming in, p 2 going out, and q is the momentum of the photon that couples to the vertex, and the identity must hold irrespective of whatever complicated stuff may be going on in the blub. So, what does this identity imply is something which can be now easily written down from this specific structure, and one can immediately say that at leading order this capital gamma mu is equal to small gamma mu; one can rewrite this object as a q mu is p 1 slash minus m and minus p 2 slash minus m. Just rewriting q s p 1 minus p 2 1 m is just introduced as a dummy variable, because that is the way it appears in the propagator. And this result is nothing but the inverse fermion propagator at the two different momentum.

And this is quite straightforward at the leading order, but what it means is that there is a relation between the strength of the vertex and the normalization of the propagator. So, when one extends this property to we have the corresponding analog of q mu times this general vertex gamma mu. And I will put a specific labeled that it is a so called renormalized vertex and it is been applied all kind of renormalization procedure before we go to this particular structure. And that will obey the same relation, but the fermion propagator may also be renormalized by the same procedure.

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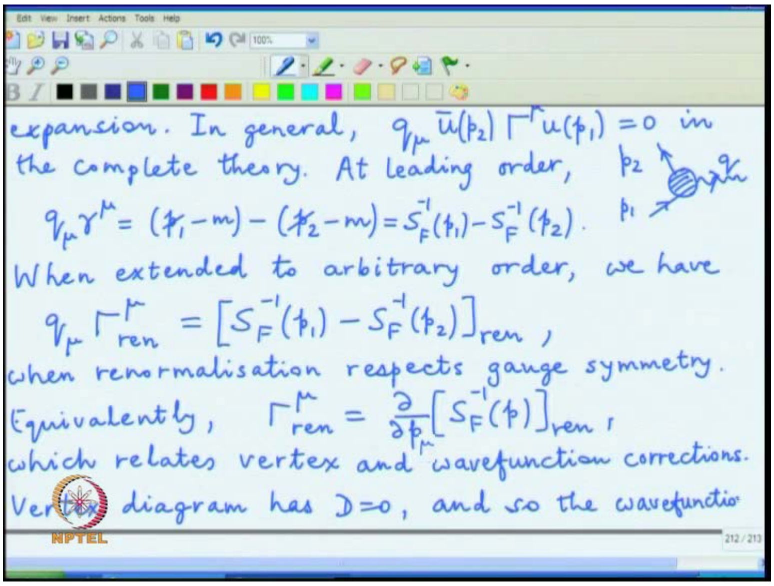

And this will be true only if the renormalization respects gauge symmetry, but as I said we will do not want to destroy symmetry by renormalization and so we will have a cutoff which will automatically ensure this particular relation. There is an alternative way to write the formula, and that is to note that q is just p 2 minus p 1. So, one can take it on to the denominator on the other side and construct a derivative, and that is helpful because it gets rid of unnecessarily dealing with the mass term and so which relates vertex and wave function.

Because left hand side will change the value of the vertex, right hand side will change the value of propagator which we labeled as wave function correction. And these are the two objects which we saw in the list of divergent diagrams involving two external lines, but they had different values of degree of divergence. And so this identity tells us that in a theory or the regularization procedure which respects gauge symmetry the divergences must be simplified and the vertex diagram had D equal to 0.

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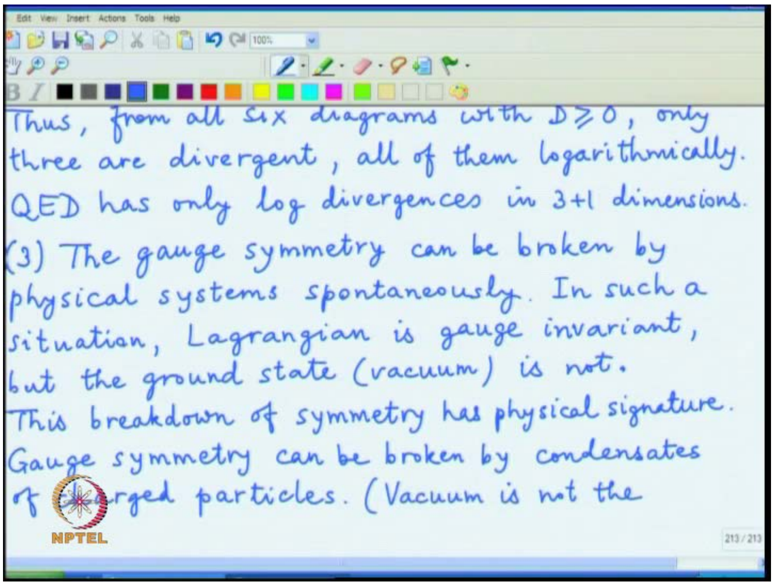

So, the nave value of the wave function diagram was D equal to one, and this restriction brings that D equal to one down to D equal to 0. And these are both log divergent, because that is included in the D equal to zero situation. So, this is the important restriction; of course, this procedure can be generalized to diagram which has many fermion and photon fields. But this leading term just involving one ferimon line and one photon is the important one, and that typically describes as the word Takahashi identity and many times in terms of the renormalization factors which are included in modifying the coefficient of the terms in the Lagrangian.

We have the relation which is often written as z 1 is equal to z 2 where z 1 and z 2 are the renormalization factors for the vertex and the wave function are the terms in the

Lagrangian. So, this is the important feature, and the result of this symmetry is then thus from all six diagrams with D greater than equal to 0 only three are divergent. The Furry's theorem eliminates three of them, and then the gauge invariance says that all of them are divergence only logarithmically, and not with any higher value of D they all correspond to D equal to zero. So, this is an important feature of QED. So, QED has only log divergences, and this is the characteristic feature of filed theories in 4 dimension.

So, this is a feature which completes the discussion of the various degrees of divergences and the restrictions coming from the symmetry. I would like to mention one extra feature which is not really related to what I have done so far, but it is related to the property of gauge invariance. So, the gauge symmetry can be broken not in the Lagrangian of QED which we explicitly constructed to be gauge invariant by physical systems, which we are dealing with may have some other interactions that can produce the breakdown which can be called spontaneous. And the specific definition of this spontaneous breaking of gauge symmetry is that in such situations the Lagrangian is gauge invariant, but the ground state which we typically call the vacuum is not.

And there are many features of this connected to the theory of phase transitions in statistical mechanics, and we can equally easily quote various examples which show this kind of feature that the ground state is not symmetric while the formal definition of the Lagrangian or the Hamiltonian is. One of them is the change in the rotational symmetry of a material when in changes from the form of a gas or a liquid to a solid, and in the fluid state the rotational symmetry is exact, but in the solid state there is crystallization and the rotational as well as translational symmetry gets broken; only certain discrete values of angles and the distances will shows symmetry but not any arbitrary values.

Another example is that of a magnet where the Hamiltonian can be completely isotropic in terms of interactions of individual magnetic moments, but at low enough temperatures the system will orient in one particular direction of magnetization spontaneously. And so this is an important feature where some external parameter controls the behavior of the theory. And in certain situations in the examples I just quoted that corresponded to value of the temperature when the parameter is changed beyond a certain value the symmetry of the system spontaneously changes.

This actually happens to be true even for gauge symmetry, and the example I want to quote is just an illustration that there is nothing sacrosanct about gauge symmetry compared to other kind of symmetries; all kinds of funny things indeed can in do happen, and that this breakdown of symmetry has physical signature. And what I want to illustrate that the gauge symmetry can be broken by condensate of charged particles that I mean that the vacuum is not the perturbative one.

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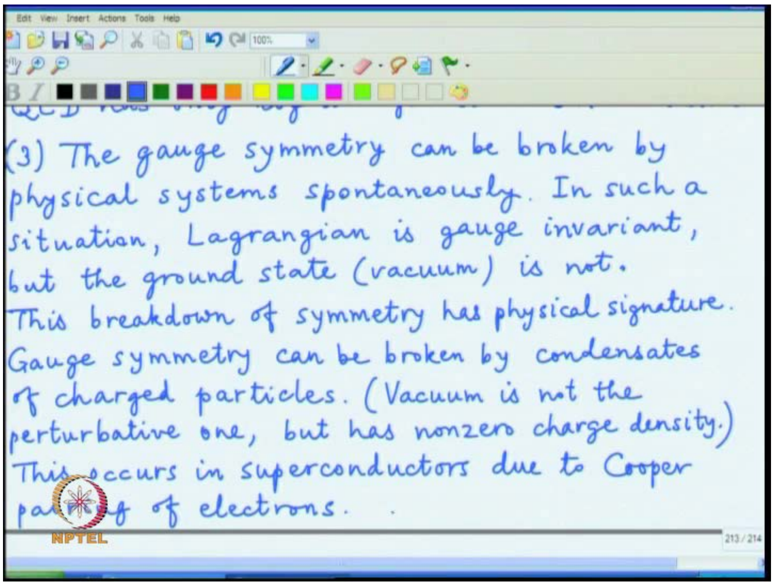

In which there are not particles at all but has nonzero charged density which is the meaning of the word condensate, and this occurs in case of superconductors due to what is known as Cooper pairing of electrons. The two electrons actually form a bound state and that is not neutral. So, in its presence the gauge symmetry is spontaneously broken because a nonzero charge object will transform under gauge transformation, and if that state is the vacuum itself then the vacuum is not gauge invariant.

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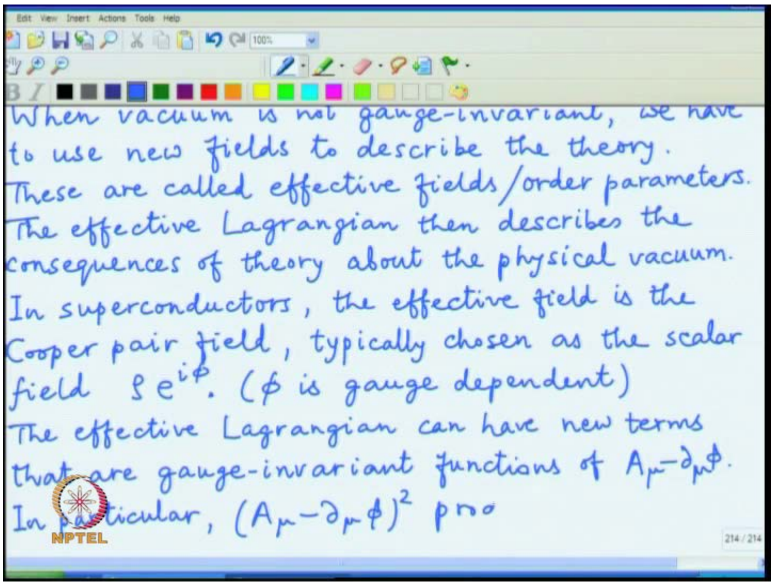

So, we cannot handle the description in the perturbative language anymore, because the vacuum is now completely different ones. Then we have to use new fields to describe theory, and these are called effective fields or the language as statistical mechanics; they are the order parameters. So, we have to completely now change the description of the theory in terms of these new variables which are some complicated combinations of the fundamental variables which with a perturbative framework was constructed from, but the physical descriptions must be now done in terms of these effective fields.

The effective Lagrangian then describes the consequences of theory about the physical vacuum and not the perturbative vacuum. Physical vacuum is the one which has the condensate. In superconductors the effective field is the Cooper pair field. It is a bound state of two electrons; if the bound state is a passive state, then the field is chosen as the scalar field with a magnitude and a phase. Now we have to write down Lagrangian in terms of this effective field. The magnitude remains unaffected by gauge transformations, because gauge transformations only change the phase, and that is the reason for writing this field in a polar coordinate.

The phi is the only variable which depends on the gauge, and then the effective Lagrangian in addition to all the other terms involving electrons and photons have new terms that gauge invariant functions of A mu minus d mu phi. Even A mu is not gauge invariant; the phi will also change under gauge transformation and so the combination A mu minus d mu phi will be gauge invariant.

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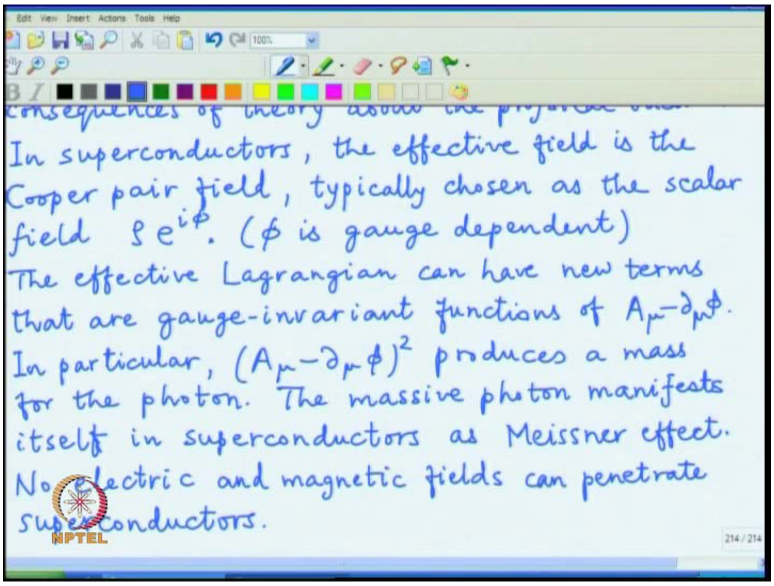

In particular A mu minus d mu phi square produces a mass for the photon, because it will be some number times A mu square term in the Lagrangian. It is not gauge invariant, but the gauge symmetry is already spontaneously broken, and now we can write down such a term. The massive photon manifests itself in superconductors as the so called Meissner effect. The photon is massive; it cannot propagate for a long distant, it just exponentially decays out. And so no electric and magnetic fields can penetrate superconductors. This is a drastic but very easily seen effect; magnetic fields do not go inside the superconductor, and the magnets will actually float on top of a high temperature superconductor in liquid nitrogen bath.

Electric fields do not go inside, because the material is superconducting. It cannot have any voltage setting inside it. Whatever voltage you try to put inside it will immediately cancel out by establishing a super current and neutralizing the charges. So, this is also another feature of gauge theory related to the symmetry, but the symmetry is realized in a spontaneously broken nature. And it produces very unusual phenomena of superconductivity starting from the same Lagrangians which we are dealing with, but in the presence of a condensate of charge particles. That is as much as I would like to describe in terms of symmetries and its consequences in case of quantum electrodynamics. This is just a brief exposure of the whole set of phenomena which can be deduced when more details can be found in various text books, again the ones which I had coated at the beginning of the course.

Thank you.