Relativistic Quantum Mechanics
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Lecture - 43
Infrared divergences due to massless particles, Renormalization and finite physical results

In the previous lecture, I described the Lagrangian formulation of quantum electrodynamics, how one can obtain the Feynman rules from that formulation by going through the procedure of deriving the equations of motions and treating each term individually. And then how divergences arise in the calculations of arbitrary Feynman diagrams. This diagrams typically involved loops over which momenta have to be integrated and the divergences appear, because the momentum integrals are unbounded. They go from minus infinity to plus infinity, and then I introduced a procedure called regularization which modifies this momentum integrals in order to give it mathematically well-defined meaning. I want to say a few more things about this procedure and then techniques to get rid of the divergences that appear, so that we can have finite answers for physically observable quantities. One thing I want to add to the regularization, description I mentioned last time is that.

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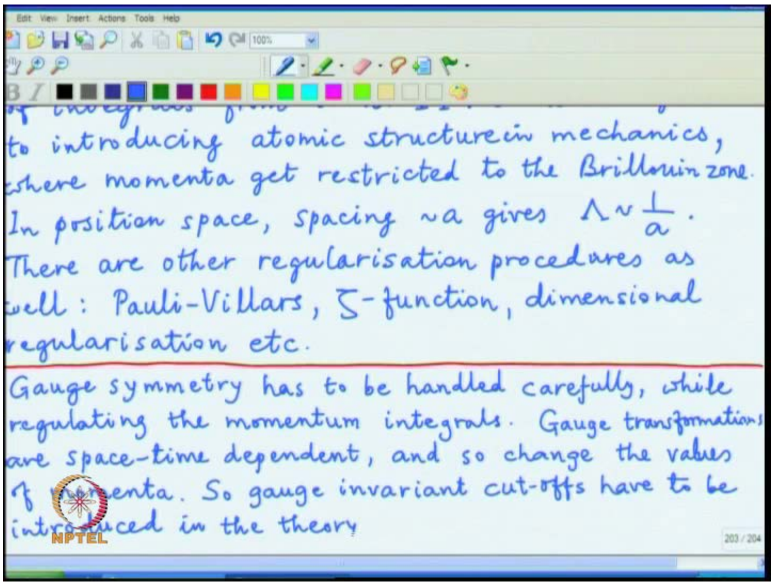

The gauge symmetry has to be handled carefully, while regulating the momentum integrals. This is because the gauge symmetry or gauge transformations are space time dependant and so change the values of momenta. And so if one is going to modify the definition of the momentum integrals one has to make sure that that definition is not specific to some particular frame, rather it holds in all different frames which are related by gauge transformations.

And that requires that one has to worry a little about the type of modification one does to the momentum integral. In particular just the knife cutoff of a momentum integral at a value of lambda which I mentioned last time is not gauge invariant. One has to introduce more sophisticated cutoffs. It can be done, but I am not going to go any detail. And this is indeed possible in the other regularization schemes which I mentioned poly villas, zeta function, dimensional regularization, etcetera as well.

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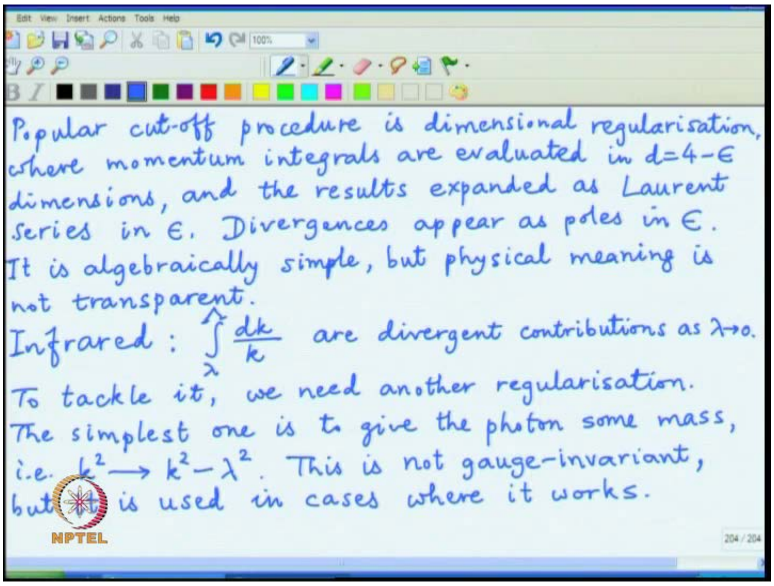

Next I want to also state that the popular cutoff procedure is dimensional regularization where momentum integrals are evaluated in dimensions analytically continued from the physical value f 4 to 4 minus epsilon, and then the results expanded as Laurent series in epsilon. Now this looks like a mathematical trick, but it works and in particularly divergences appears as poles in epsilon. This procedure is popular, because it is algebraically the simplest; one can even program computers to do the calculations. But mathematically why it works is a sort of mystery, because we do not understand the physical underpinnings of such a procedure very well. And so it is used because its work, but it does not have a simple interpretation like a brivion zone kind of momentum cutoff which says that there is some underlying structure to the theory. Here it is more of a mathematical prescription and it does the job.

So, it is algebraically simple, but physical meaning is not transparent. So, that much I would like to say in the case of ultraviolet divergences; I also want to say a few things about another kind of divergences which is called the infrared divergences. And that is a peculiarity of the theory which has massless particles, and the way it will appear is there will be integrals like d k over k are divergent contributions as the lower limit of the integral, which I am going to call here by lambda goes to 0. The part which I dealt with earlier was the upper limit of the integral where capital lambda goes to infinity, and this is a logarithmic divergence which actually diverges at both hands. It is the most common divergence occurring in case of QED; that is why I took this particular form.

But one can have different powers of k also appearing which will have infrared divergences when the lower limit goes to 0, and the question is that this also has to be tackled. But we need another regularization procedure now working with the lower limit of the integral, and the easiest one to think about is that this is a contribution coming from the massless particle and it had a propagator which had a negative power of k in the denominator. And so if one just modifies this propagator by giving it a mass and sticking with our Minkowski signature, it should be minus lambda square. Then the divergence certainly disappears, because one can put k equal to 0 and the denominator will be finite. This is not gauge invariant, because the photon mass will break the gauge symmetry, but it is still used in situations where it works and we will see exactly how this things work in the next step.

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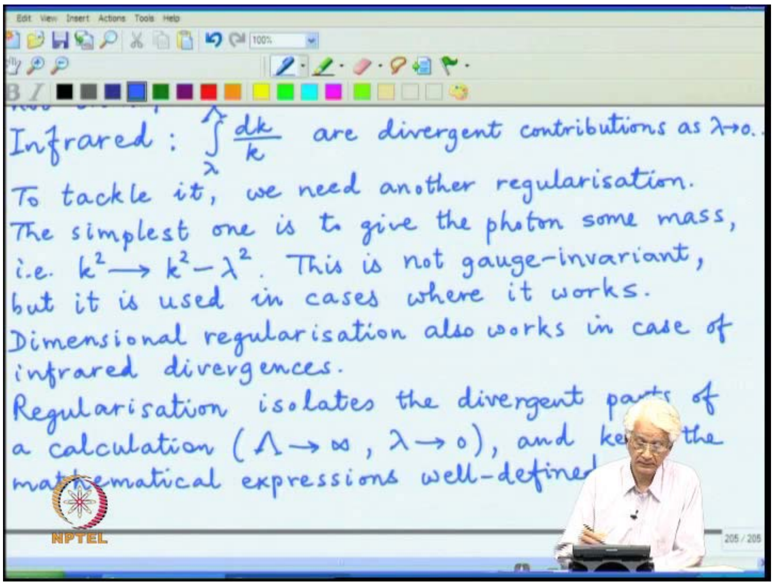

I would also like to say that the dimensional regularization also works in case of infrared divergences, and again it is often used just like it is used in case of ultraviolet divergences. So, the description of various kinds of divergences and the diagrams from which it is produced is necessary to understand the meaning behind the whole theory. But so far what we have done is taken the mathematical expressions, identified the singularities and introduced the concept of regularization.

What regularization does is isolates the divergent parts of a calculation. And one can explicitly do that by either taking lambda going to infinity and expanding in powers of lambda including logarithms, or similarly expanding in powers of the small lambda including logarithms when it goes to 0. And by doing that we keep the mathematical expressions well-defined and this is only the first part of the calculation in dealing with divergences.

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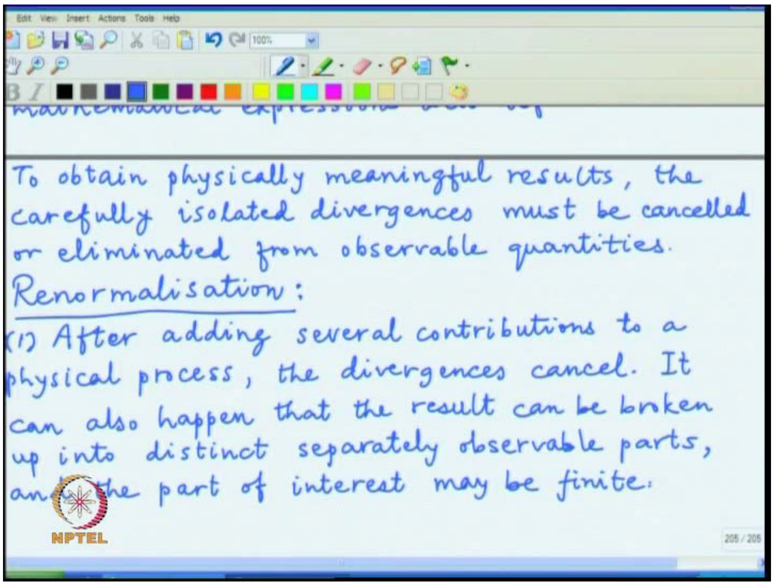

Now to give physically meaningful results the divergences which have carefully been isolated by the regularization procedure, they must be cancelled or eliminated from observable quantities. And this is an elaborate and also somewhat sophisticated procedure named as renormalization, and there are various different possibilities which can occur in dealing with the divergences; renormalization actually refers to only one particular part of dealing with divergences. But I would list all the various possibilities which one encounters and then elaborate on renormalization.

So, the first possibility is to after adding several contributions to a physical process the divergences cancel, and this can very well happen, because at a particular order in perturbation theory there may be several different fermion diagrams contributing. And we add up them together, and even the individual contributions might have divergences in the total sum things cancel out. If this happens then there is no problem at all with the calculations, and one can take the final answer as a physical result. It can also happen that the result can be broken up into distinct separately observable parts, and some parts may be infinite, but some part may be finite. And in particular the part of interest may be finite, and even in this case one can leave aside that some other part is infinite, but the part which of physical interest is okay and that can be considered as a valid result of the calculation.

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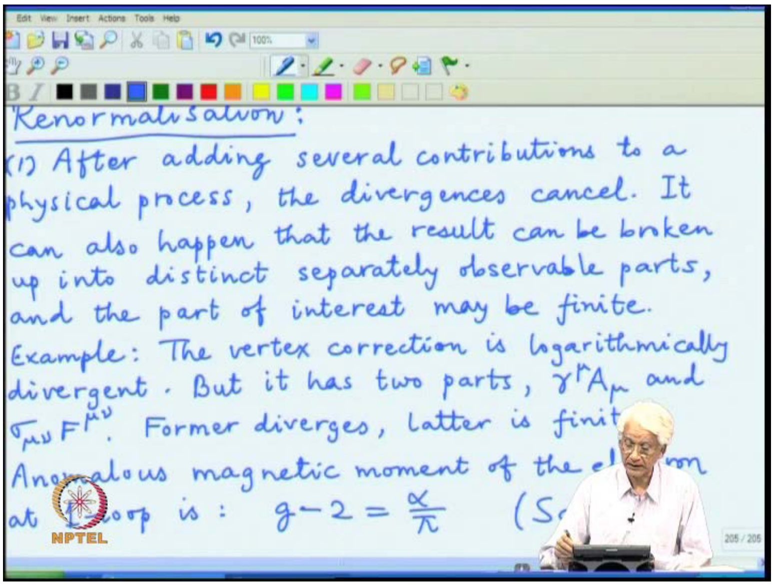

And I would list an example that the vertex correction which I described diagrammatically earlier is logarithmically divergent. It has superficial degree of divergence equal to zero. But this has two parts; one when expands out the whole coupling up the photon with the external electromagnetic field, and they can be written as one is the coupling of the vector potential as in case of a current gamma mu A mu, and the other one is the contribution corresponding to sigma mu nu F mu nu which contributes to the anomalous magnetic movement of the electron, and the former diverges, the latter is finite. And so one can take the expression as far as the calculation of anomalous magnetic moment of electron is concerned gets a finite correction, and it is immediately is physical. One does not have to worry about that it came from a calculation which are some part which was formally divergent.

Because the divergence is not relevant to the particular process which is being calculated, and this two different parts are separately physically observable. So, this was a calculation which was performed by Schwinger, and it gives the anomalous magnetic moment of the electron at one loop is g equal to or rather its written as g minus 2 is equal to alpha by pi; explicit calculation of a one loop diagram, and this was a calculation performed by Julian Schwinger. So, this is one possibility; one does not have to undergo any complicated part to remove the divergences, because in the part of interest the divergence is absent and then the result, obviously has physical meaning

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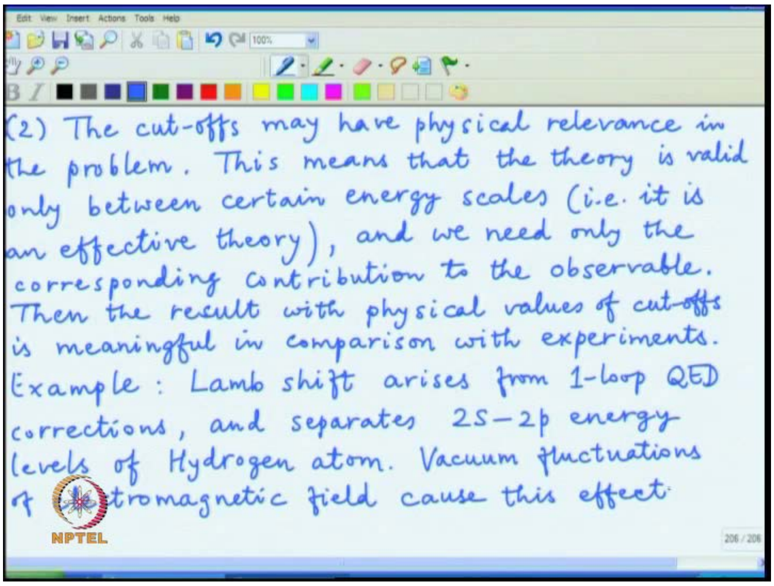

Now let us go to the second possibility which is that the cutoffs which are formally introduced in the regularization procedure. We may have physical relevance in the problem, and by this I mean that the theory is valid only between certain cutoffs, and we want to get only the contribution from the theory between those particular cutoffs. And this means that the theory is valid only between certain energy scales. Such theories are often referred to as an effective theory in contrast to a fundamental one and we need only the corresponding contribution to the quantity which has to be observed.

And in such scenarios, obviously, there are other features beyond the cutoff which we do not really know, and so there will be certain parameters which will appear in the result which will say that that momentum integrals have to be cutoff at certain values. And the answer then will depend on those values specifically, but those values are no longer mathematically defined objects. But they are actual physical scales, and again I will give an example then the result with physical values of this cutoff is meaningful in comparison with experiments.

The example I want to give is again a landmark calculation in the early days of quantum electrodynamics which is known as Lamb shift arises from 1-loop QED corrections, and separates 2s and 2p energy levels of the hydrogen atom. Now this correction is a vacuum fluctuation of the electromagnetic field. We have dealt with vacuum fluctuations of the electron field which were typically labeled as Zitterbewegung where the trajectories are not well-defined but undergo very rapid oscillations. The same thing can happen in case of electromagnetic field as well and in the standard treatment of even the relativistic hydrogen atom the electric field was a background field; it was not allowed to fluctuate.

But when we do that it gives a correction which shows up in QED calculations at one loop and this vacuum fluctuation of electromagnetic field. They influence the effective potential felt by the electron just as in the case of Zitterbewegung, and the contribution is largest where the field is largest. And that happens to be true, close to the nucleus of the atom and the states which are affected the most are therefore, the s wave states or l equal to 0. And so the difference of l equal to 0 and l not equal to 0 states gets connected to the size of this particular fluctuation, and that is what can be explicitly calculated in QED.

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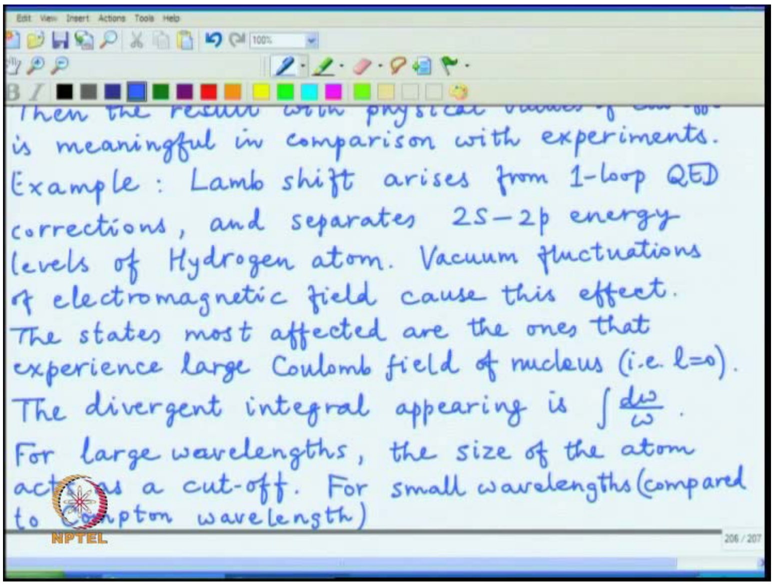

So, the states most affected are the ones that experience large coulomb field of nucleus which are the l equal to 0. So, now we want to calculate this particular effect; there are integrals. The divergent integral appearing in this particular case is a logarithmic singularity, but now we are dealing with a specific bound state problem, and we have reasons to cutoff these integrals at the two hands. So, for small value of omega or large wavelengths the size of the bound state acts as a cutoff, because the atom is not going to fill wavelengths which are much larger than its size, and for small wavelengths the meaning of small is compared to the Compton wavelength.

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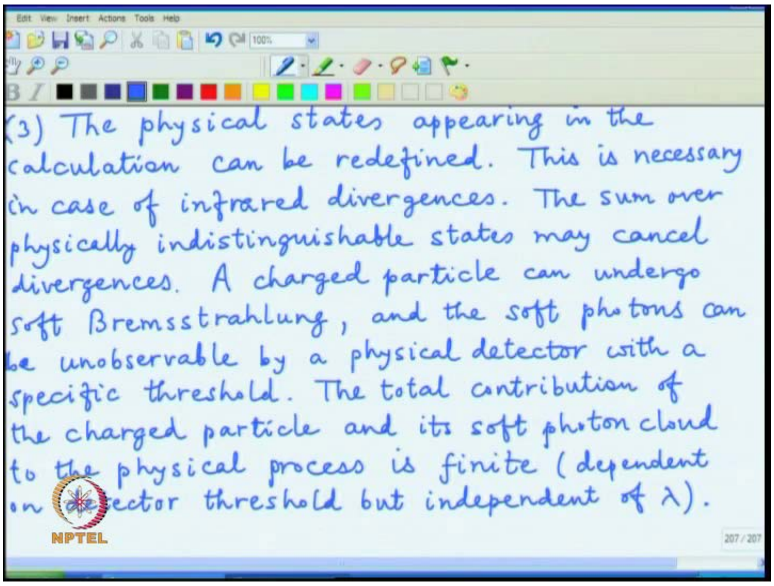

The non-relativistic natures of the problem enter the picture, and so it makes the scale of pair production which is mc square a cutoff. And so this integral is divergent, but the contribution which is important is only ranging from the size of the atom to the threshold for pair production, and that particular part of the integral is what contributes to the physically observable splitting of 2s to 2p energy levels. And this indeed so match with experimental observations and the theoretical justification arises from doing the integral within specifically justified limits. And the in-between contribution matches nicely with experimental data. This was again a famous calculation which was carried out by Hans Bethe.

So, this is a second possibility. Again we have bypassed the dealing of divergences, because the integrals were truncated, and there was a reason for truncating the integrals to finite answers. Now there is a third possibility and which is actually helpful in dealing with infrared divergences, and that is the physical states appearing in the calculation can be redefined. And this happens to be necessary in case of infrared divergences, because as I will explain there are certain states which cannot be distinguished from one another. And then as in any quantum theory if you cannot identify the states separately we must sum over them.

And when that summation is carried out again there is a cancellation between various contributions, and the final result will be divide of the divergences. And the way this occurs in case of infrared divergences is that a change electron can emit soft photons, and this is a process which is called soft Bremsstrahlung. And if the photon energy is really small that is the reason for calling it soft, then the detector may be not able to see them. And this is always true, because each detector will have specific threshold or sensitivity beyond which only it can detect a certain signal. So, now in calculation of the process we must sum over the contribution of not just the electron but the electron accompanied by these soft photons, or one can say that it is a cloud of photons surrounding an electron.

The number of how many sets of photons is there in the cloud we do not really know, but we cannot observe them, and so we must add up all the pieces together. So, the total contribution it can be shown that the result is finite in particular the cutoff small lambda as I have been calling in case of infrared divergences cancelled out of the expressions. And this finite is dependent on detector threshold but independent of lambda, and if this happens then again we are through; the problem has been solved, and the final result has physical meaning and the calculation which demonstrated this whole procedure exactly in case of infrared divergences was carried out in case of QED.

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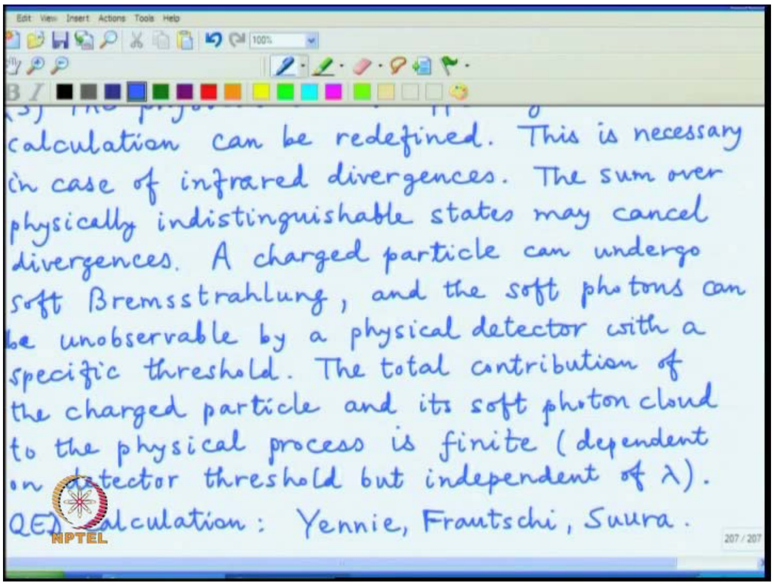

So, by Yennie, Frautschi and Suura; so that is the third possibility, and now we come to the last possibilities. And this is actually the one which is strictly referred to as renormalizations and that is by all the tricks which I have mentioned before; we cannot get rid of the divergences from the results.

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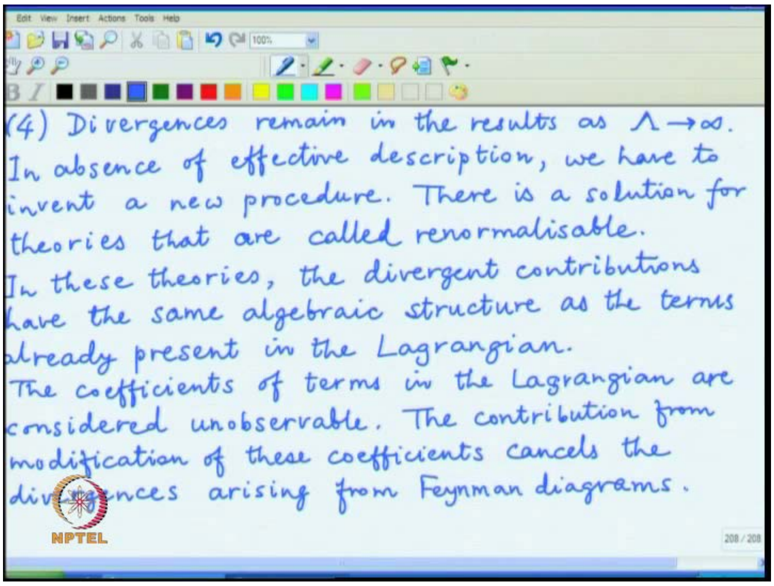

So, I should mention that this problem occurs only for the ultraviolet divergences; in case of infrared divergences all of them can be eliminated by redefining the meaning of physically observable states as I just mentioned. And so what does one do? Well, we have already gone through a separate case of an effective field theory where the lambda add a physical meaning and we could justifiably truncate the momentum integral and compare with experimental results. But in this particular case there is no effective description; we must invent something new, and this procedure can be invented only in certain classes of theory.

And QED happens to be in that particular cast, and there is s solution for theories that are called renormalizable. And this solution or rather this characterization is possible because the so called renormalizable theory have a property. The divergent contributions have the same form; I can say algebraic form or structure has the terms already present in the Lagrangian. And so one applies a particular trick of modifying the coefficients of the terms in the Lagrangian and then gets two contributions. One from doing the usual Feynman diagram calculations, and another because the terms directly coming from the Lagrangian which has now a different coefficient.

And these two contributions can be cancelled against each other, because they have the same algebraic structure. So, the coefficients of terms in the Lagrangian are considered unobservable, because they are just some mathematical parameters. And we must tune them so that we get some physical result at the end in that philosophy. So, the contribution from modification of these coefficients cancels the divergences arising from Feynman diagrams.

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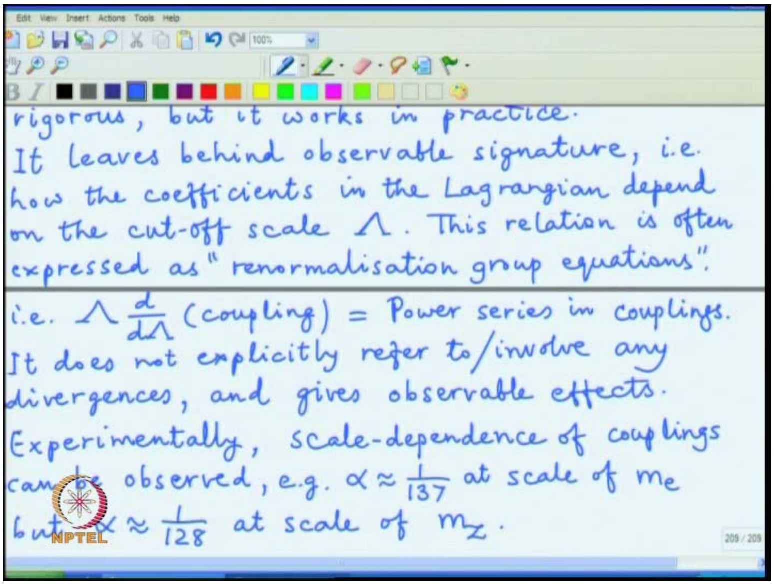

And that procedure is what is called renormalization and is not mathematically rigorous, because you are cancelling one infinity against another, and at the same time talking about perturbation theory where the term is supposed to be small in a certain sense, and infinity mathematically speaking can never be small. So, this is not mathematically rigorous, but it works in practice, and in particular it leaves behind one specific signature which is useful in tests with experiments. So, it leaves behind observable signature, and the signature is i.e. how the coefficients in the Lagrangian depend on the cutoff scale lambda which appeared in the divergence explicitly, and of course with a different value of lambda you will require a different value of the coefficient because the two have to mutually cancel.

And so they have a specific relation between the two, and this relation is often expressed as what are known called renormalization group equations. Again this is a specific label for a certain behavior, but what it gives is a particular form which is say lambda d by d lambda appearing acting on some coefficient in the Lagrangian, say, a particular coupling is equal to a series expansion in one or many of the couplings which may be present in the theory. And this equation is quite powerful in the sense that it does not explicitly refer to or involve any divergences and gives or predicts observable effects.

And this feature what it observes or can be verified experimentally is that the various couplings are predicted to have scale dependence, and it indeed is predicted and verified. In case of QED the well known number is that alpha is 1 over at 137 at the scale of electron mass, but its value changes to about 1 over 128 at scale of the mass of the z goes on that characterizes weak interactions. And this can be calculated in QED and matches with experimental observation and in particular this so called renormalization group or RG framework was put in a strong foundation by the original work of Gell-Mann and Low and later work by Wilson.

And that basically concludes whatever I want to say regarding all the types of divergences, how they can be handled, and what are the physically observable effects that can be seen, and they have let to many important tests of QED over the years from its early formative years to moderned technology where the calculations have been carried out to not just the one loop which was the starting point but to much higher loops like fourth and fifth order loops in the expansion, and the accuracy matches the precision of experimental observation. So, it indeed is a theory on a very strong foundation because of experiments, and theory is both proceeding hand in hand. So, this much I should say about the theory of electrodynamics, and there are certain other features which you might read later in going through the books which I mentioned in the very first lecture as reference material.