Relativistic Quantum Mechanics
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Lecture - 40
Helicity Properties, Bound State Formation

The restrictions of helicity on the annihilation process can also be seen in terms of the Lorentz group properties of the various operators involved, and also the properties and classification of various states, which are included in this process. And that can be easily seen by performing tensor products of all the factors involved and looking at what that produces. So, when the parities, not parities…

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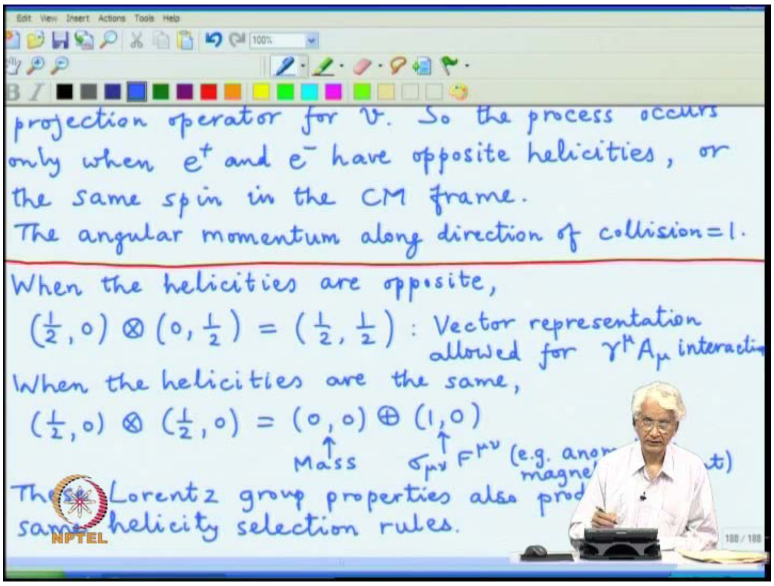

When the helicities are opposite, we have the combination of the spinners say half, 0 tensor product with 0, half, which produces the representation half, half. And this is the vector representation and which is an allowed representation for the interaction given by gamma mu A mu. That is the coupling of the electromagnetic current to electron and positron. On the other hand, when the helicities are the same, one will have the tensor product, which is a combination of 2 spin half producing 0 and 1. And this 0, 0 is a representation for a scalar, which is the mass term. And this 1, 0 is a representation for F mu nu. So, the corresponding operator will be sigma mu nu F mu nu. And this for example, can be the anomalous magnetic moment coupling if it exists in the theory.

In our case, the anomalous magnetic moment does not appear in the tree level calculation, which we have done. But, the mass term is still there. And that does contribute. The only thing is that, that contribution will be subleading compared to the vector contribution. And the ratio of these 2 terms is basically just the ratio of energy to mass, which will be large in the particular situations we are dealing with, because the energy is the mass of the muon or larger; and the electron mass is much smaller compared to that. So, these Lorentz group properties also produce the same selection rules. And we will see these features explicitly in the computation of individual terms. So, now, let us calculate the various contributions explicitly.

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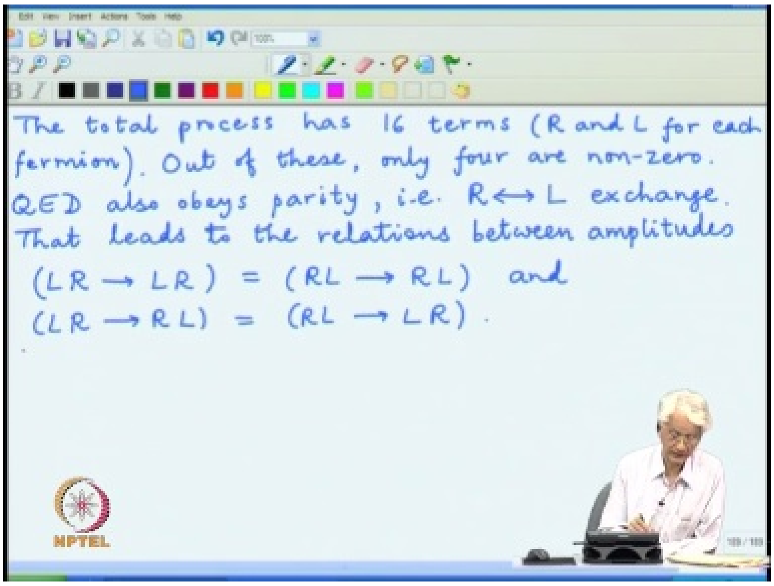

The total process has 16 terms, which arises for 2 choices of helicity for each of the 4 fermions involved. And out of these, because of the helicity selection rule, only 4 are nonzero. And even the 4, which are nonzero are further related because of the property of parity, which is obeyed by the theory of quantum electrodynamics obeys that role exactly, which is in terms of the helicity; means the processes, where R and L are interchanged, will be related; rather they will be exactly the same terms, which will appear twice. And that means that leads to the relations, where we interchange R and L; and the answer looks the same.

So, the process, which takes LR to LR will be the same as one which takes RL to RL and analogous one for the LR to RL being the same as RL to LR. So, these are the only 4 nonzero amplitudes; and parity cuts down the calculation to again only 2 of them being nonzero. And that is the 2 pieces, which we will calculate explicitly by evaluating the traces.

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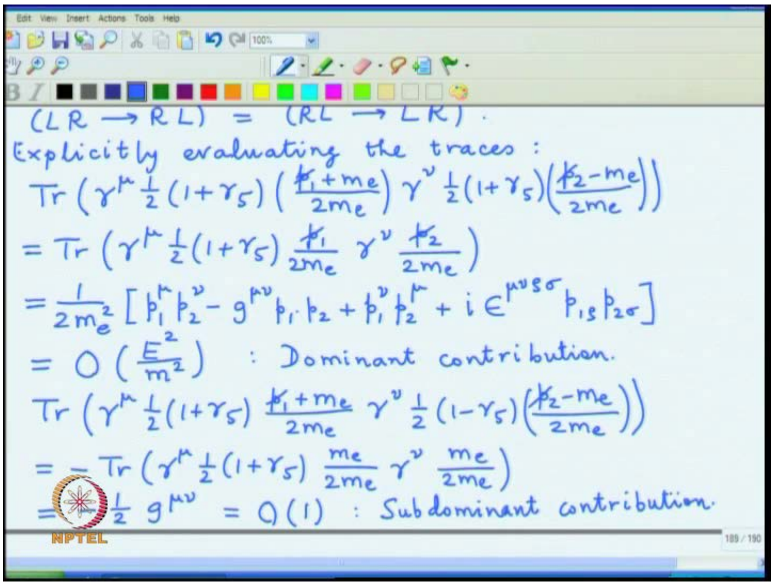

But, now, the traces will include the factors of 1 plus or minus gamma 5 exactly. So, the case, where both the u and v spinors have the same factor of 1 plus gamma 5, which is an allowed process, produces the combination, which is now this particular form. I have just introduced the half 1 plus gamma 5 both for the electron wave function as well as for the positron wave function. And this actually corresponds to the helicity, is being opposite for the electron and the positron in terms of the u and v wave functions of projection operator is the same on half 1 plus gamma 5. And that is according to the convention, which we have set up. And now, this can be simplified by noting the fact that, every time 1 plus gamma 5 goes through a gamma alpha. The 1 plus gamma 5 sign changes to 1 minus gamma 5. And if 1 minus gamma 5 and 1 plus gamma 5 comes next to each other, they are just analyte themselves.

So, one has to just consider the cases, where they come next to each other by commuting the operators. For example, whenever this can be taken through the mass terms, then there is only one gamma matrix in between. And 1 plus gamma 5 will turn into 1 minus gamma 5 when commuting with that gamma matrix giving zeroes. So, the mass terms in both these momentum plus mass factors do not contribute; we just have p slash and no m surviving. And so… And the projection operator when get squared, it gives a same result as well. So, we only have the factor that, the projection operator can be counted once and the mass can be deleted and it becomes smaller.

And now, this can be explicitly evaluated. There is a factor of 4 from the trace of the gamma matrices, which end these denominators, can be combined and pulled out. And now, one just have to combine the dot products of various gamma matrices together. Just first evaluate the one part of the 1 plus gamma 5 and then the gamma 5 part. So, the one part is the trace, which we had evaluated earlier. And it just produces mu dotted with p 1 first; then mu dotted with nu; and then nu dotted with p 2 giving this particular result. And the gamma 5 is now contracted with 4 gamma matrices mu p 1 slash nu p 2 slash. And that has to be evaluated by its own trace definition. And the result is the epsilon symbol multiplied by the 2 momenta with the corresponding indices. So, this is the dominant part of the contribution. It is in terms of magnitude. It is order E square divided by m square. And this is the dominant contribution.

Let us also evaluate the other part, which has the wrong helicity combination and see what comes out of that. So, in that case, we had the same factor. It is just one sign of gamma 5 is different. And now, when commuting this 1 plus gamma 5 factors through the gamma matrices, a different combination from the propagators survives. The mass term does contribute, but the p slash plus part produces combinations of 1 plus gamma 5 and 1 minus gamma 5, which annihilate each other.

So, that result is just this. Again, the 1 plus gamma 5 projection operator when squared, gives the same thing again and we have taken this out. And now, this is a trace of only 2 gamma matrices in one case of the identity. And when gamma 5 contracted with only 2 gamma matrices, produces no trace at all. So, the result is actually quite simple. It gives minus half g mu nu. This is in terms of magnitude; it is order 1. And this is subdominant. So, in doing this explicit calculation, we see little more physical behavior than just counting the helicity as the leading term. And that is, we see actually the magnitude of separation whenever the helicity conservation is violated.

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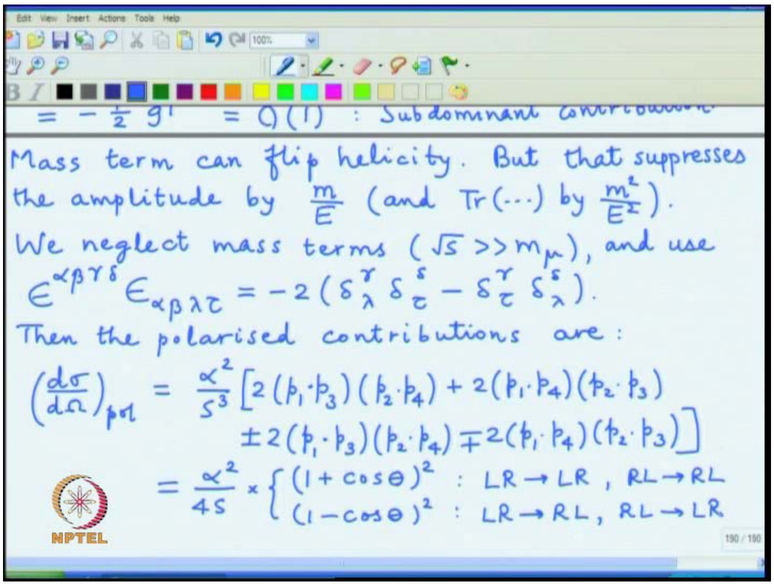

So, in general, mass term can flip helicity. But, that suppresses the amplitude by a factors of m over E. And thus, square of the amplitude, whichever is in part of this trace gets suppressed actually by m square by E square. And we see that explicitly in the calculation, where the allowed helicity combination is larger by a factor of order E square by m square over the unfavored contribution, which comes from the mass term flipping the helicity. So, that much for the selection rule. And to do the leading order calculation, we will ignore the contributions, which are suppressed by factors of mass. So, we will only do the high energy part, which is characterized by the relation that, square root of S is much larger than the muon mass.

And now, we square the total T f i square, which has product of two of these traces; all the indices contracted. And in particular, we need a contraction of 2 epsilon symbols. And the explicit result, which is required, is a contraction of 2 of the indices of the epsilon symbols. And it can be easily seen that, all the pairs have to match completely to produce a nonzero result. And that produces chronicle deltas in these contractions – gamma equal to lambda and delta equal to tau. Or, if the combinations is switched around, then epsilon symbols are antisymmetric.

And so there is a minus sign. And the overall factor of 2 comes because this alpha beta can be contracted in 2 different ways because of the permutation possibilities. And overall sign is negative, because we are dealing with Minkowski space-time. So, the epsilon with all the upper indices is negative of the epsilon with all the same indices in the lower place; 3 of them produce a minus sign and the fourth one has a plus sign. So, this is the result.

And then one has to now just contract 2 of these traces together. The part of the trace, which does not involve the epsilon actually has the same structure as what we dealt with before. It is symmetric under interchange of p 1 and p 2 and it has either the indices on p 1 and p 2 or a ((Refer Time: 23:02)) term. So, we can just take the previous result and write it down in this particular context.

So, that gives the polarised contributions to the cross-section, are first just the phase-space factors explicitly; and then the contraction terms one by one. So, it gives p 1 dot p 3 and then p 2 dot p 4. These are the result for the symmetric terms. And the antisymmetric epsilon terms produce result, which depends on the sign of helicity projection. So, say the plus or minus or plus or minus is the same sign, which is there in 1 plus or minus gamma 5. So, I am just writing both the results in the same formula. And the interesting point is this Kronecker deltas produce a result, which has the same structure as the first line apart from the signs. And here the combination just produces this result.

Now, it can be simplified in case of the two independent amplitudes, which we have. And that can be easily calculated to be the factors, which are nothing but 1 plus cos theta whole thing square or 1 minus cos theta whole square, where the first option is valid or LR going to LR or RL going to RL. And in this particular case, one has to keep in mind that, we have assumed a perfectly polarised state. So, there is no averaging factor to be included in the sum over the spins; we have made a specific choice explicitly. And this one is valid for LR going to RL or RL going to LR.

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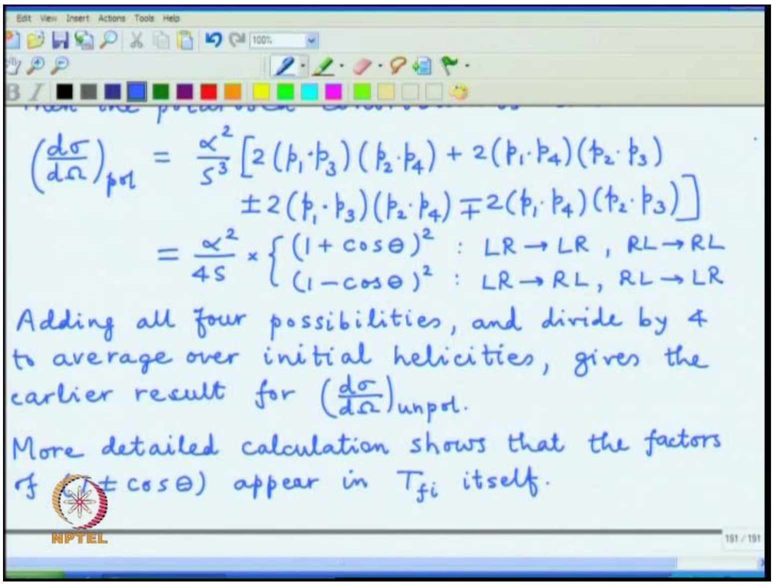

And, one can easily see that, if we add up all the 4 possibilities and divide by 4 to average over the initial helicities of electron and positron, gives the earlier result for d sigma by d omega unpolarised. So, that is the explicit calculation. We learnt a little bit more about identifying individual terms and seeing the feature of helicity explicitly appearing. One can actually see the features by doing a more elaborate calculation instead of just evaluating the traces, which I have done. So, more detailed calculation shows that, these factors of 1 plus or minus cos theta, which are appearing as squared in the cross-section; they appear in the calculation of the amplitude itself. And they are the consequence of these helicity properties. And when you square the T f i, of course, you will get the squares of this particular amplitude. So, that is one feature.

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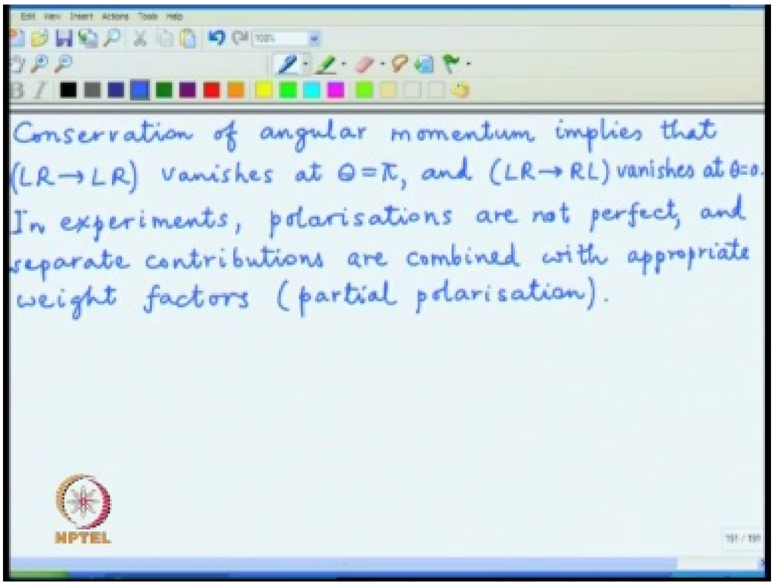

And, other feature, which is buried inside this factor, is also quite easy to notice. And that is a consequence of the total angular momentum conservation. So, if we have an amplitude LR to LR, it is perfectly OK. When theta equal to 0, the angular momentum is just carried once straight over from the initial state to final state. But, if we look at the situation for theta equal to pi, then the momentum direction is reversed. And since we are looking at helicities here, the angular momentum has to be taken into account including the direction of momentum.

And so this will vanish at theta equal to pi, because the angular momentum would not be conserved; it is conserved exactly at theta equal to 0. Without any separation factors for other angles, there will be something in between. And similarly, in the flip case, LR to RL amplitude vanishes at theta equal to 0. So, that explains, where this 1 plus cos theta and 1 minus cos theta appears as well. So, that is as much as can be done in this simple fashion in terms of polarization annihilation process.

And, experiments with polarised e plus e minus beams have indeed been carried out. The experimental polarizations are not perfect. And in that case, one has to include these various terms with appropriate weight factors. So, one can actually go all the way from unpolarised case, where there are no weight factors or rather weight factor is uniform for all possibilities to partially polarize, where some contribution is more than the other to total polarize, where one particular contribution dominants over everything else. So, that is as much as I want to say about calculation of polarised cross-sections.

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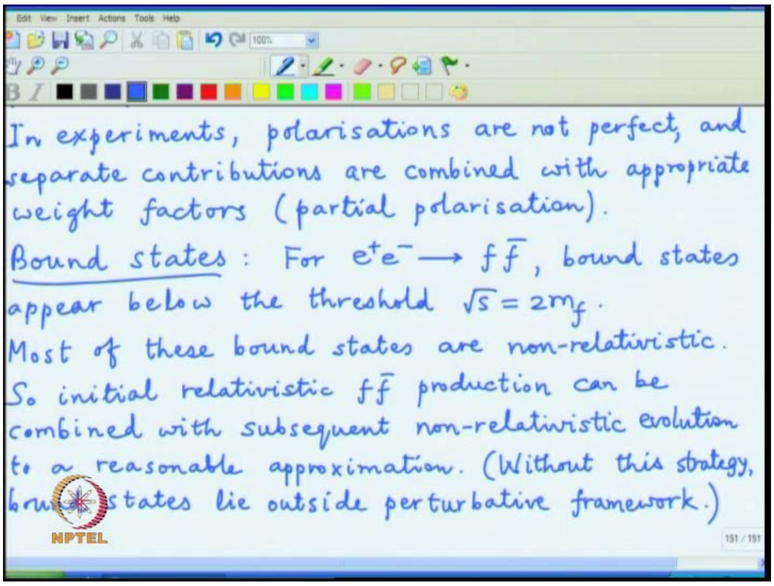

Now, I want to illustrate another process, which can be obtained by the same machinery, which we have set up. And that is a question of analyzing production and decay of bound states. And generically, for processes involving a particle and an antiparticle, there are bound states just below the threshold, because particles and antiparticles with opposite charges – they end up attracting each other. Question is, can the formulism, which we have set up be used to predict some features of this bound state. The answer is, yes, if we are little bit clever, in particular, we need to include the features of bound states on top of the perturbation theory. The perturbation theory per se cannot produce a bound state; it is a natural formulism for dealing with scattering. And we need to sum up the infinite series in scattering if we really want to see the bounds states.

On the other hand, we can use our experience from non-relativistic quantum mechanics to combine this relativistic scattering theory with the corresponding wave functions described in the non-relativistic language. And the non-relativistic wave functions have to be used explicitly. They will be beyond the perturbation theory. But, provided, we combine them properly, we can actually predict something for production of bound states. So, what we will assume is that, most of these bound states are non-relativistic. And we will assume that, that is indeed true for the calculation, which we are going to do. And so we will use initial relativistic production of the fermion-antifermion pair can be combined with subsequent non-relativistic evolution to a reasonable approximation. And without this strategy, bound states lie outside perturbative framework.

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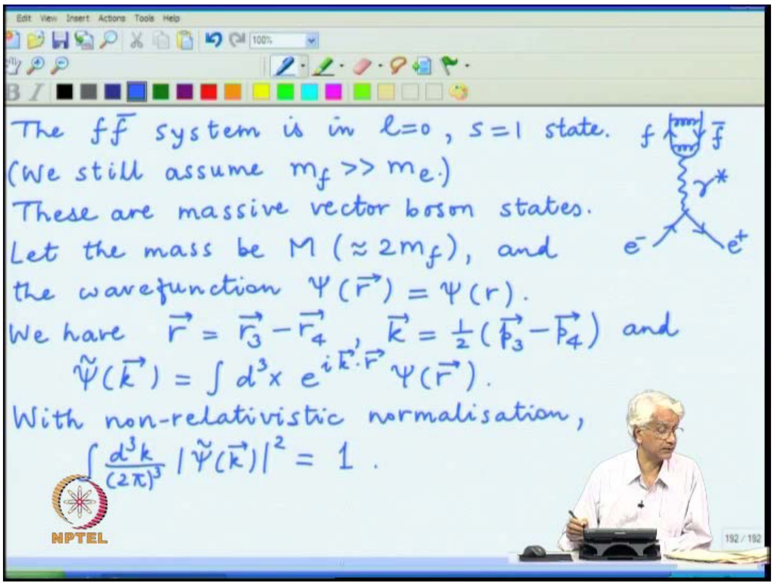

So, now, let us just explicitly write down some formula, which show this structure. So, let me draw the fermion-antifermion bound state diagram. Once again slightly different than what I drew earlier. So, there is electron and positron, which annihilate into a virtual photon. But, now, the f and f bar are not flying away from each other, but they remain together constrained by whatever the force that may be bounding the ferimion-antifermion pair with each other. And in this case, the f f bar system is in l is equal to 0, s is equal to 1 state. We have assumed that, the system is non-relativistic. So, l and s can be specified separately.

We already saw in the previous analysis that, the cross-section just above the threshold was isotropic; which meant that, the angular momentum involved in that particular case was 0. And the helicity selection rule says that, the state will be produced s is equal to 1. And in particular, we still assume that, the fermion mass, which is produced is much larger than the electron mass. So, the electron helicity rule holds to an excellent approximation and then the spin state has s equal to 1. And these are called massive vector boson states. And these are the ones, which are often the first states to be seen whenever a new particle-antiparticle threshold is crossed. They are the easiest to observe because of this various conservation laws involved. And we are looking at the dominant contribution in that particular case.

So, let us now give the explicit formula of the wave function, which we are going to deal with it. So, let the mass be M. And since we are non-relativistic, this M is approximately 2 times the fermion mass, and the wave function psi of r… In particular, we have already seen that, l is equal to 0. So, this object is going to depend only on the magnitude of r and not the direction. So, the various rules for non-relativistic description in terms of this centre of mass frame of 2 identical mass objects specify that, we have this relative coordinate is the separation between the f and f bar. The corresponding conjugate momentum – it will be the half of the difference; rather it is just the momentum of one of the objects if we want to use the centre of mass frame. And the relation between position and momentum coordinate is the standard rule in non-relativistic quantum mechanics of for fourier transform. So, this specifies the wave function and we have assumed that, it is a bound state of these 2 objects.

What we also need to calculate explicit numbers is a particular convention for normalization. And here we will chose the non-relativistic normalization convention for this wave function, which is the mod psi square integrated over the whole space gives 1. In relativistic normalization, this was proportional to energy. And we are going to use a different convention. One has to check it out that, all these conventions are accompanied by appropriate factors of a volume. The box normalization, which we had used earlier for using plane waves versus this overall normalization, which corresponds to a bond state. And once that is done, consistently, all the factors of the box normalization has to cancel out of the final formula.

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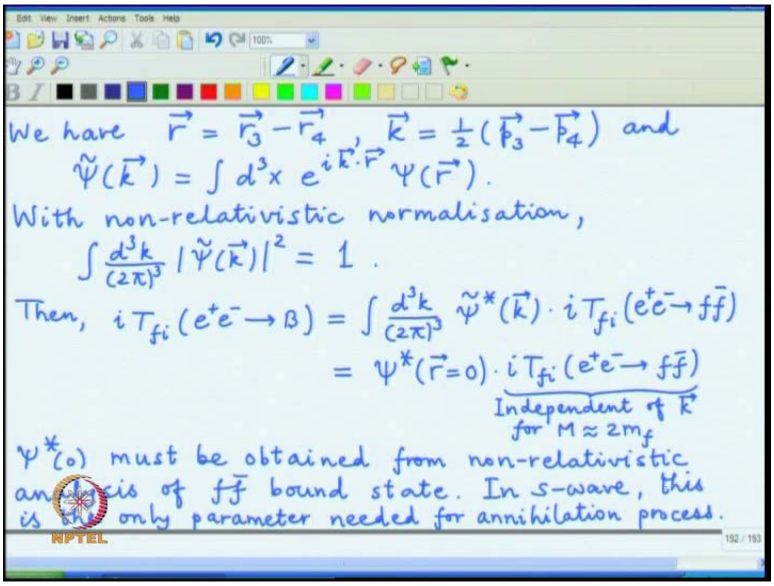

And then now, we can put together the transition matrix elements combining the 2-stage process. First, the relativistic production followed by formation of a bound state. And that gives the amplitude or e plus e minus going to some bound state denoted by B, which will be the overlap of the bound state wave function with whatever is produced by the relativistic annihilation process. And I will take this overlap in the momentum space. So, in the first process, e plus e minus to f f bar produces this matrix element T f i. And then the next one basically just projects that part of it, which can form a bound state. And this result turns out to be reasonably simple in the special case, which we are dealing with, because close to the threshold, this T f i is essentially a constant; it does not even depend on the momentum vector.

The factors of various momenta, which we saw in the calculation of cross-section – all of them actually came from the phase-space integrals of the delta function in this near threshold situation. And so if T f i is going to be independent of k, it is isotropic; say it does not depend on direction of k, but it is also independent of the magnitude of k near threshold. Then we just have an integral of psi star over the whole space. And that is nothing but the inverse fourier transform with the phase factor corresponding to 0. So, this gives nothing but psi star at origin and the position space multiplied by this object, which is independent of k. So, we have this rather simple rule. So, all we have is now just the wave function at the origin, which is characterized by the type of bound state in the non-relativistic calculation and the relativistic perturbative factor, which we have already calculated before. And just put them together.

So, this psi star 0 must be obtained from non-relativistic analysis of f f bar bound state. And that will depend on what is the force holding the f and f bar together. One has to solve the Schrodinger equation and calculate what is the wave function at the origin. And that is not calculable in the machinery, which we have introduced so far. But, treating it as a parameter, we can calculate everything else. And so the total cross-section will depend on whatever this wave function at the origin is in the actual system. Or, turning it around, one can use the measurement of the total cross-section to estimate the wave function at the origin of the bound state.

And, in s-wave interaction, this is the only parameter needed for the annihilation process. All the features are tied together; annihilation means the particle and antiparticles have to come on top of each other. That can open only in s wave; the other waves have no amplitude at the origin. And so the amplitude at the origin psi star 0 actually dictates the probability of f and f bar coming together on top of each other. And l equal to 0 dynamics of the process guarantees that, that is the only parameter, which should be present in the calculation.

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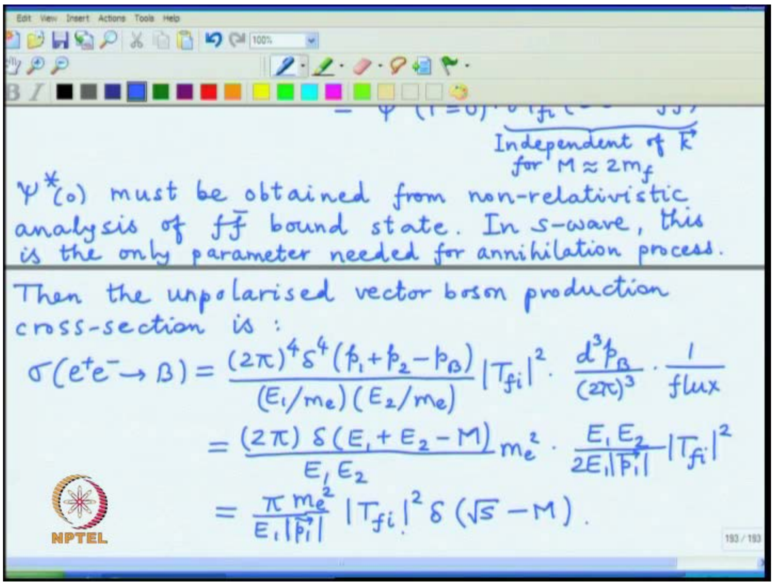

So, now, we can put together the various factors once again. Again, we will treat the unpolarised vector boson production cross-section. Here it is no longer the differential cross-section; there is only one object in the final state. And we just get directly the total production cross-section by putting in all the constraints and doing the delta function integrals. So, that result is the overall normalizations of 2 pi raise to 4; then the normalization of the initial state electrons, which are still relativistic, the matrix element square, which we have written down above. And now, there is only one differential element in the final phase-space. And then there is an overall normalization dividing by the flux. And this can be now very easily simplified. The momentum integral is rather trivial. So, we will just get 2 pi delta function of energy alone. Then there is a factor of E 1, E 2 and m e square. And then the value for the flux; we calculated it earlier and the result was E 1 E 2 divided by 2 E 1 magnitude of p 1. And that is the expression.

The momentum integral is already gone, but the energy delta function still remains. And one can keep the various things together in the form, which are and just write down the final answer. These various factors are easily taken care of the energy factors cancels; E 1 p 1 survives. I forgot here this T f i square. And so the result can be written as… Cancels out various things. So, there is pi divided by E 1 p 1. Then there is m e square. And then the matrix element square and the delta function can now be written as square root of s, which is a center of mass energy minus M. And this is the condensed result for the cross-section. Now, one can calculate this T f i explicitly in terms of the results, which we obtained earlier and reduce these two explicit formula in ((Refer Time: 01:01:33)) of the fine structure constant and the wave function at the origin.