Relativistic Quantum Mechanics
Prof. Apoorva D. Patel
Department of Physics
Indian Institute of Science, Bangalore

Lecture - 37
Klein-Nishina Result for Cross-Section

In the previous lecture, I derived the expression in case of Compton scattering for the probability of, and initial photon with momentum k i going to a final photon with momentum k f. And now to convert that probability to a cross-section, we have to sum over all the degrees of freedom, which are not actually observed in the experiment. So, we have to integrate over the electron degrees of freedom and see what structure comes out and then evaluate the relevant gamma matrix traces, which is a significant amount of algebra in this particular case, but it can be carried out using several tricks. So, the expression for the probability of scattering involved various delta functions and well-use of the flux, which we have to now calculate. The flux actually is easy to…

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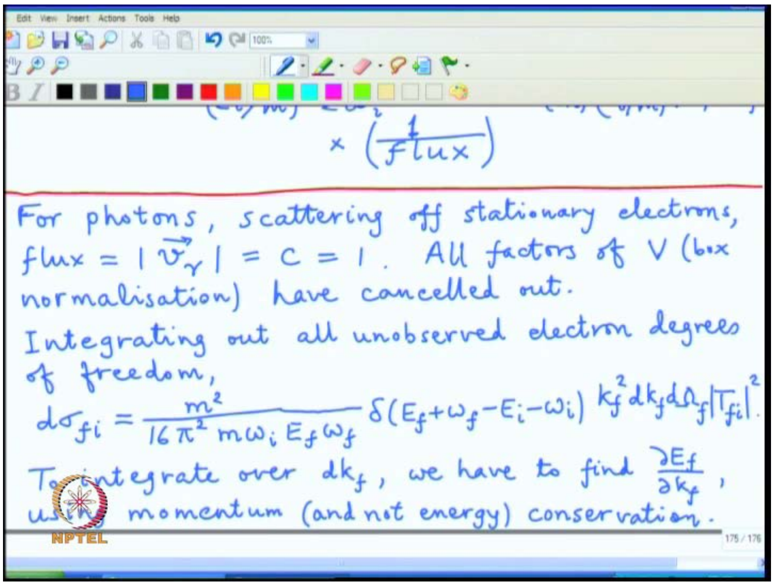

See in this particular example, because for photons, scattering from stationary electrons, the flux is nothing but the speed of the photon, which is c or 1 in the units, which we are working on. So, the flux part does not give any more complications, and for the expressions, which I have obtained, I have already canceled off all factors of V, which is the volume for box normalization. I have not bothered to write them down, because they already cancel out as they should in any physical observable quantity. So, we just have to tackle this particular delta function. So, what we do is integrating out all unobserved electron degree of freedom; we get rid of the space part of the delta functions rather easily by removing d cube p f integrals. And then the energy part will remain as a delta function; but that has to be removed by integrating over d cube k f or rather the radial part of the d cube k f integral. The angular part survives and that converts this cross-section to a differential cross-section in terms of scattering angle.

So, let me just write the first step, which is just integrating over the electron degrees of freedom. So, it becomes d sigma f I – overall factors of m square I can put in the numerator; all the various 2pi's and 2 etcetera simplify to 16 pi square; the energy and momentum becomes m times omega i. And, for the final state, it is E f omega f. So, these are the normalization factors. Then there is a delta function of the energy, which remains from the 4-dimensional delta function. And then the integration over k f can be separated into radial and angular part. All the 2 pi cubes, etcetera, have been already taken care of. And finally, there is the matrix element square.

So, this is the expression; and which we now have to integrate over the radial part of k f. And, in doing that, we need to take into account that, in the delta function, not only omega f is equal to k f, which is one dependence. But, E f also depends on k f due to the energy momentum conservation. So, we need to calculate; we must find the derivative of d E f with respect to k f. But, this derivative must be calculated using only the imposed conditions so far, which is the conservation of momentum. And, only after we have integrated over the energy delta function, we can put the energy conservation back into the equation.

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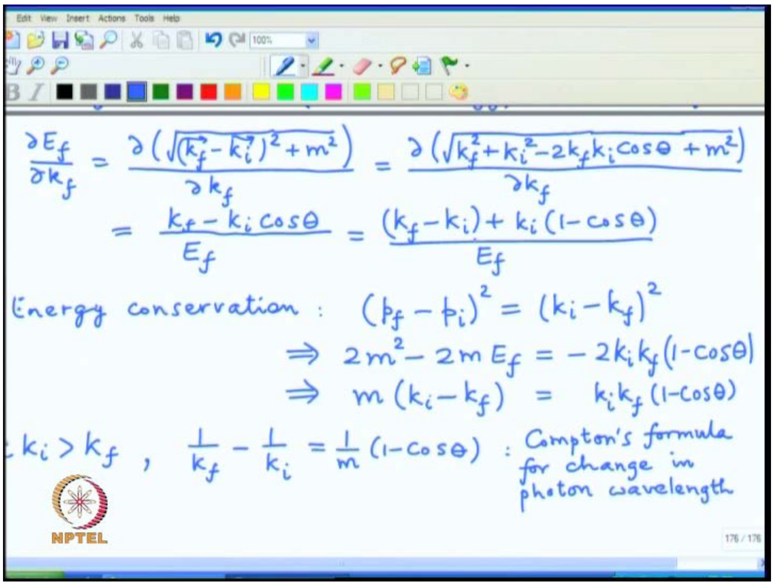

So, now, let us just do that. So, what is this quantity? It can be written as derivative of this huge square root coming from the momentum conservation. And then one can trivially expand this and write out the full expression; expand it. So, let us see; it is a square root of k f square plus k i square minus 2 k f k i cos theta, which is a dot product of k f and k i and plus m square by d k f. So, now, the derivative is calculated in a rather straightforward manner; square root goes one power less; it becomes in the denominator and that value is nothing but E f. So, taking care of all the factors of half etcetera, we have an E f in the denominator. And, the derivative of the numerators inside the square root now gives k f minus k i cos theta. So, this is a simple enough expression, but we have to now simplify it more, because we do not know cos theta right away. So, we can rewrite this object as k f minus k i plus k i into 1 minus cosine theta divided by E f.

And now, since a derivative is already calculated, we can use the various tricks of energy conservation and simplify this expression further. So, what does energy conservation tell you? And, writing in the 4-vector notation, it can be expressed in a various different ways. But, one way of writing the conservation law is p f minus p i, which is the momentum gained by electron must be the momentum lost by the photon. And so this identity has to hold. And, what this implies now is we work it out explicitly; p f square is m square; p i square is also m square. So, there is 2 m square; p f dotted with p i – since p i is just the rest mass energy, is 2 m E f. Right-hand side gives similar expression; but k square is 0. So, it is just minus 2 k i dotted with k f and writing explicitly in terms of energy and momentum part. The dot product of energy just gives k i k f in terms of magnitude. And, dot product of space built just gives the same thing, but with an angle in between. So, that is the result.

And, one can re-express this in several different ways. One way of writing it is write everything in terms of the photon variables. So, the right-hand side is rather simple; even that 2 can be removed. So, leave it out. And, on the left-hand side, you have m as a common factor and then E f minus m; but E f minus m is the same as k i minus k f by energy conservation of just the zeroth component.

And so this is an identity, which follows and which now can be used to simplify the expression for the derivative, which we wanted. And so let us see… There are several other consequences as well. So, let us see what all things are employed. One thing is k i has to be greater than k f; then there is a second formula, which is often related in the form, which Compton wrote down by dividing this equation throughout by k i k f. And, this is Compton's formula for changing photon wave length. And, the third thing, which we need, is calculate this d E f by d k f; for which we must substitute all these cosine theta's and get rid of various unwanted quantities.

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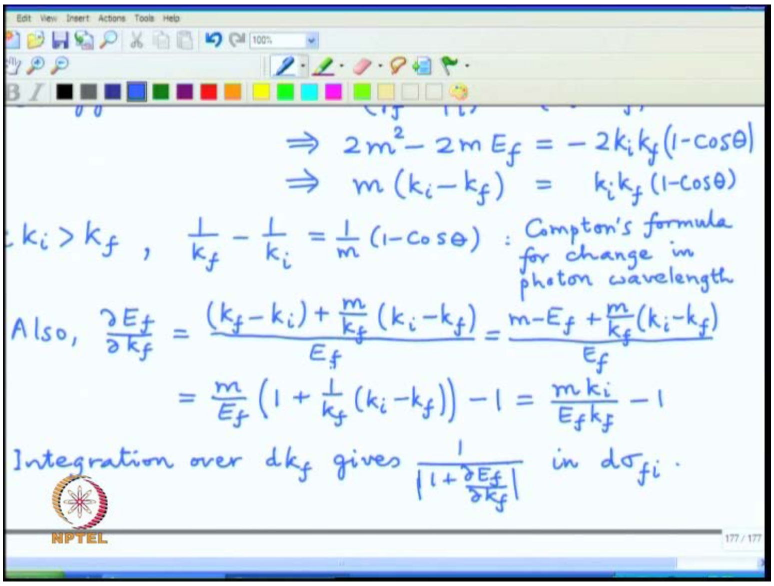

So, let us just do that. So, d E f by d k f… k f minus k i will… Just let it write down. So, it is k f minus k i plus – k i into 1 minus cosine theta is this object divided by k f… And, this can be simplified. You need a factor of 1. So, this is m minus E f; I have to subtract out basically 1; which I do not want – plus m by k f into k i minus k f divided by E f. And so this is m by E f is a common factor; here then the various terms are 1 plus 1 by k f k i minus k f. That is one part. And then there is minus 1. So, the net result is valid. This k f, nominator, denominator also cancels with the 1 on the other side. And so the total result at the end of this whole jugglery is m k i by E f k f minus 1. And, this is in a form, which can be now used to integrate over the delta functions of energy, which was written earlier. And, that delta function basically produces the factor of 1 plus d k f by d f in the denominator. And, that is where the whole simplification occurs. So, integration over d k f gives 1 divided by magnitude of 1 plus del E f by del k f in d sigma f i.

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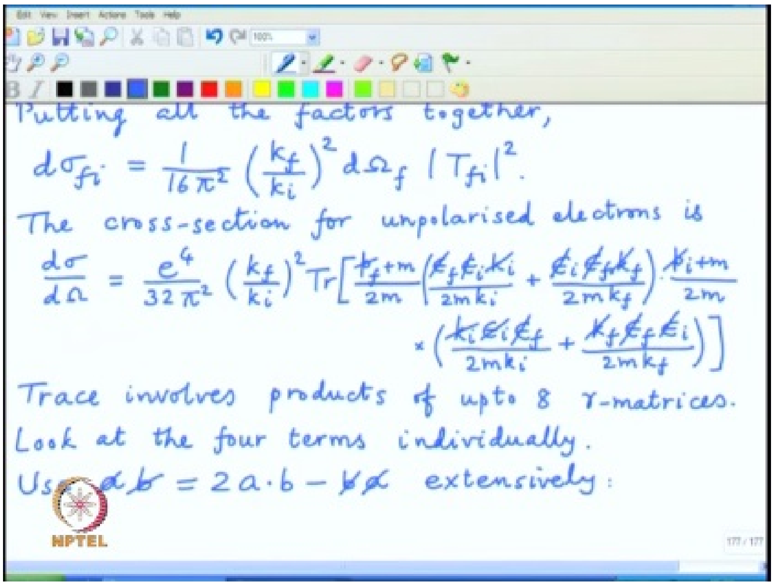

And finally, once you put in all these various factors together, many things simplify this 1 over 16 pi square, which is fine. Then this E f k f parts essentially all cancels out; m also will cancel out to some degree. And, the net result is with omega f and k f being equal, etcetera in a rather simple looking form, where the differential cross-section is just proportional to k f by k i whole square multiplied by the matrix element. And now, our job is to calculate this matrix element. And, we will use the same tricks as we used in the case of Coulomb's scattering. They have to be written in terms of u bar and u and then rewrite the u bar u factors by cyclic arrangement into the form, where they become all traces. And, that expression so depends on the initial and final states.

But, if we use the same ideas that, the electrons are unpolarised; then the only projection operators, which appears are the p slash plus m for the electrons after summing over the spin states. So, the result looks like d sigma by d omega. We had written down t f i earlier. So, I am just going to count those things explicitly. This extra factor of half comes in, because you are averaging over the initial spin, but summing over the final spin. So, there is a half in there. And now, there is a big trace and where all the various projectors appear.

And, here is the explicit expression, where there were two terms in the scattering corresponding to crossing symmetry of the two diagrams; then another projector; and then the final factor, which has the capital gamma bar, which means all these things written in an opposite order as per all the slash combinations are concerned. So, this is the expression, which requires now all the tricks of the gamma matrix algebra, calculations of traces, which we had summarized before.

And, there are… So, the trace involves products of up to 8 gamma matrices. You must simplify those products to numbers. So, it can be done in a rather ((Refer Time: 25:12)) force; but one can work out the various terms individually. In particular, there are four terms, which arises from these two diagrams multiplied by its conjugate combination. And, there is certain symmetry – the crossing symmetry between these objects. And, it helps to look at these four terms one by one. It simplifies the calculation somewhat.

And, I will give a brief derivation of what happens and the individual steps can be filled in a rather straightforward manner. So, look at the four terms individually. And, what helps is using all the various combinations, which kind of vanish. So, what we will use is the generic trick that, use a slash b slash by anti-commuting one through the other to give 2 a dot b minus b slash a slash. So, these various factors can be commuted together and to take them into a form where the last combination kind of vanishes extensively.

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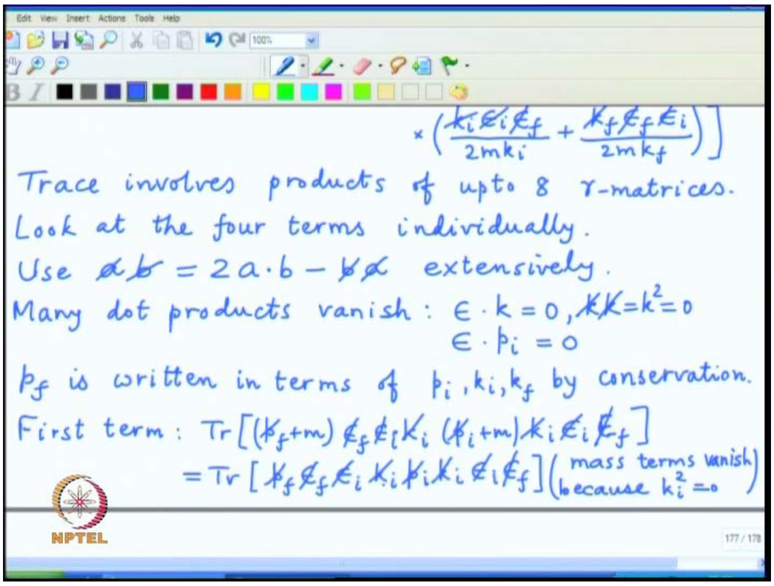

And, in particular, many dot products vanish. Obviously, some of them are epsilon dot k for a photon, which is always 0. The photon is a transverse polarized object. Then k slash k slash is equal to k square is also 0 for the photons, because they are on shell. On top of that, the initial momentum is such that it is a rest electron state. So, the transverse photon – either of them – initial or the final also happens to be 0. And, all those objects have to be used in this commuting these operators, where this a dot b will land up to be some of these combinations and they can be immediately eliminated. And, there is no simple combination to involve p f. So, p f is written at some stage or the other in terms of p i k i and k f. And, that is true by conservation law of the 4-momentum. So, then the dot products of p i, k i and k f with the other objects have simpler forms. So, p f does not have a simple form; we will eliminate them, then this particular problem.

So, let us look at the first term. It has a form of trace p f plus m slash; then these various factors of epsilons; then the second; and then the second factor. And, that closes the trace. So, now, we want simplify this object. First, we see that, the mass term has this k slash on either side. Mass is just a number and k slash is… k slash square is 0. So, the mass part does not contribute.

So, this p i slash plus m is just equivalent to only p i slash. And, once that mass is gone, the other mass also disappears, because that now corresponds to a product of 7 gamma matrices, which happens to be 0; the number have to be even to give a nonzero trace. So, the first object is just to get rid of all the mass to write this as a simpler form, where we have used that mass terms vanish, because k i square is 0. So, now, we have an explicit product of 8 matrices. Again, the thing to do is use this k slash square equal to 0. We have to anti-commute through one of them to get the result in the correct form. And, that requires anti-commuting k through p; which gives only the dot product between the two.

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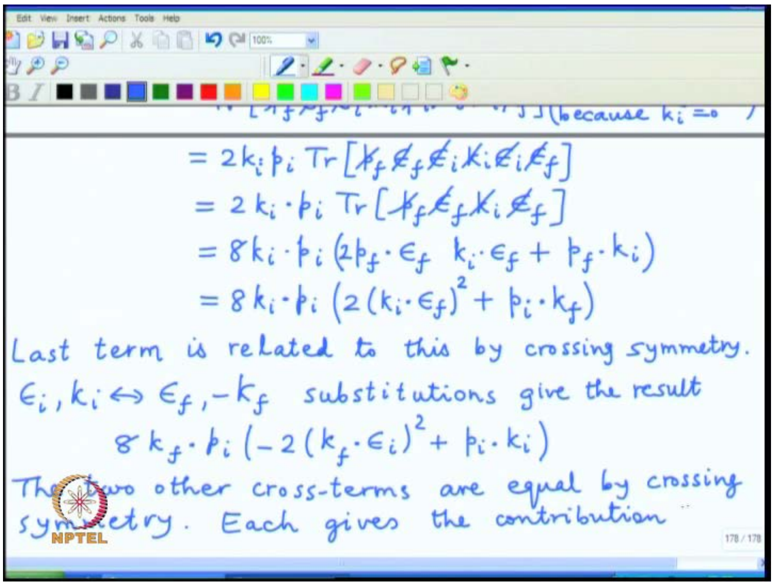

So, we can rewrite this object is a 2 k i dotted with p i and then the trace of p f slash epsilon f slash epsilon i slash. This has gone through already and then there are…

Whatever is remaining – one k i slash is still there; epsilon i slash epsilon f slash. So, now, we have a product of 6 gamma matrices. Again we can reduce it further by noting that, these epsilon and k together – they have a 0 dot product. So, we can write it in the opposite order. So, it becomes minus k i slash epsilon i slash. But, then there are 2 epsilon i slash together. So, that also uses an identity, which I should have mentioned earlier like epsilon dotted with epsilon is equal to minus 1; that also can be used. So, one anti-commutation and the minus 1 2 epsilon i slash disappear. And, the result then is now k i dotted with p i trace p f slash epsilon f slash k i slash epsilon f slash. So, now, we are down to 4 gamma matrices.

And now, this can be just explicitly written down in terms of the cyclic rule, which we have used. So, the result is 4 comes from the trace; and then there are various combinations, which appear in these products. So, the first thing is the dot product of epsilon with k. So, let us just write down. It is p f dotted with epsilon f. And then the other two produce their own dot product, that is, k i dotted with epsilon f. So, this is one term. The same term occurs twice, because when the p f is dotted with the last one, it gives the same result. So, this object has actually an overall factor of 2. And then there is a dot product of p with k and the epsilon with epsilon. And, there is a minus sign that with p f dot with k i. And then epsilon dot epsilon is equal to minus 1. So, that can be just included in a straightforward fashion. And then this is a final form, which can be written maybe in a little bit different form.

Again using energy conservation, we will get rid of all the factors of p f. So, k i dotted with pi is outside. And then and you substitute p f as p i plus k i minus k f; in both these places, things simplify, because epsilon f dotted with k f is already 0, cannot change much about that. k i dot E f is appearing here. So, it gives two times the contribution. So, two times k i dot E f square. And then there is a third term, which is p i dotted with epsilon f; that also happens to be 0, because p i is just the rest mass.

So, first term can be rewritten as this. Same trick for the second term here again. The energy momentum conservation implies that k i minus p f equals k f minus p i in a rearranged form. Squaring this expression, k square equal to 0 and p square equal to m square are eliminated from the two sides of the equation giving p f dot k i equal to p i dot k f. This is the final form of this first term, which is useful; all specified in terms of the observable constant. p f is the one, which is not observed and you have to get rid of it from all the expression. So, this is the term.

Now, the last term in this product of four objects is related to this by crossing symmetry. So, the substitutions in that particular case as epsilon i k i got interchanged with epsilon f and minus k f. And so the contribution is now easily written down. It is 8 times k f dotted with p i. There is a negative sign, but we are going to absorb it inside this combination. So, it is minus 2 k f dot epsilon i square. And, here the sign remains plus and it is p i dotted with k i. So, this is the explicit evaluation of two of the terms. The two other cross-terms are equal by crossing symmetry there. There are the sign changes; do not do anything explicitly. And so we only have to evaluate one of them.

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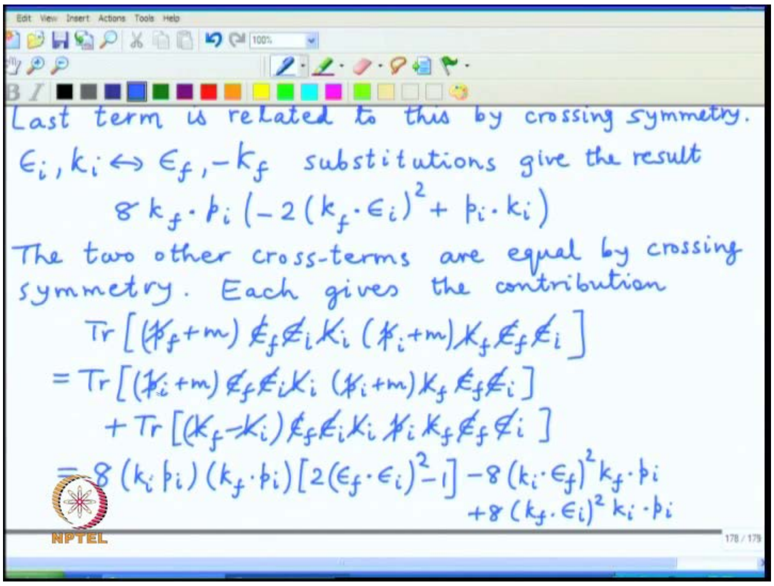

And, each then gives the contribution, which can be written as trace p f slash plus m epsilon f epsilon i k i; then p i slash plus m; and then the other object – k f slash epsilon f slash epsilon i slash. And, here nothing simplifies as easily, because there are no two factors of k I, which will square to 0. But, we just have to use the other part, which is well get rid of the p f; and write everything in terms of p i. And, that gives the combinations only in terms of… And then the remaining part, which happens to be k f minus k i. And, k f minus k i is in a good position, where thing can be used to simplify together with all the epsilon. But, let me just write down that thing first. So, k f minus k i together with epsilon f slash epsilon i slash k i slash p i slash… And, here the mass can be dropped, because that gives an odd number of gamma matrices with 0 trace; the other part, the mass still survives. So, this is the first step.

And now, the point is to get this various objects simplified in some combinations or the other. Various things can be tried in terms of epsilon and k. For example, this contribution of epsilon i slash k i slash can be written in the opposite order with k i slash epsilon i slash with a negative sign. And then epsilon will have 0 product with p i slash again, etcetera. And, that procedure can be now continued till all the traces basically come down. And, I will only write down the final answer; the rest of the stuff is an exercise.

So, that result is k i p i into k f dot p i into 2 epsilon f dot epsilon i square minus 1 minus 8 k i dot epsilon f square k f dot p i plus 8 k f dot epsilon i square k i dot p i. And, these terms still explicitly obey the crossing symmetry. And, it can seen by the same interchanges. And, one just has to work them out. Important part is that, the mass again drops out completely from the calculation. And, here it is kind of consequence that, p slash square becomes m square and then it cancels with the other quadratic term of minus m square for the trace. So, that is the summary.

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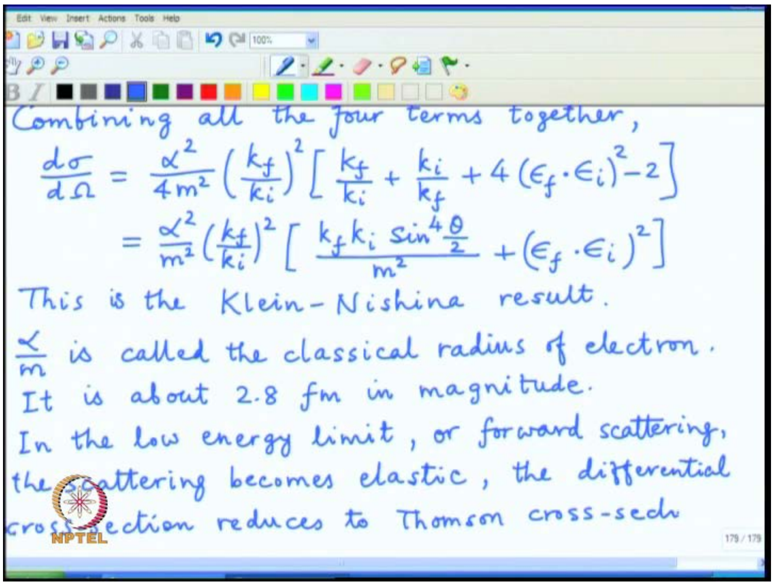

And now, I can write down just the final expression. So, we have much simpler result, where the overall constant is alpha square by 4 m square. After taking into account all the factors of 2m's in the denominator, k f by k i square survive. And, the interesting part is that, the remaining contribution coming from the trace is much simpler than all the intermediate algebra would have suggested. And, it looks as this particular form. And, this is a result, which can also be written in a slightly different form using the angles explicitly. So, if one simplifies this k f square and k i square, they both happen to be 0. And, the expression can be rewritten in terms of just the last part. So, let me just do that. And, this object is k f square plus k i square minus 2 k i k f, which is k f minus k i the whole thing square. And, that can be rewritten as cosine theta simplified to sine 4 theta by 2 in terms of the energy momentum relation, which I wrote down. It said k f minus k i was proportional to k f k i times 1 minus cosine theta. And, 1 minus cosine theta becomes 2 sine square theta by 2. So, all these jugglery is just playing around with the magnitudes.

And, that brings the angle explicitly inside it and then the remaining part of the polarization contributions still survives. So, this particular object was calculated first by Klein and Nishina. And so it is known after them. It is a differential cross-section calculated explicitly in terms of the property of the photons. The electron stuff does not expire explicitly anywhere, except for just this factor of electron mass; nothing else is important. The charge of course is there in parameterization of the strength of the coupling. And, ((Refer Time: 53:01)) various things can be inferred from this particular result. In particular, this object alpha by m is called the classical radius of electron. It is essentially not 1 over m, which will be the Compton wavelength, but a Compton wavelength reduced by a factor of alpha. And, here one can look at it that, the size of the cross-section is basically characterized by this particular parameter. So, it is just a shadow produced by some object of this particular radius up to the other factors, which we have all going to be order 1.

And, the magnitude of this object is roughly 2.8 fermi in magnitude. One get this result explicitly that, this becomes essentially the cross-section in the sense that, the first term in this expression can be neglected in various situations. So, in the low energy limit, which means k f and k i are much smaller than the mass, that part can be just thrown out. And, in that particular case, k f will approximately become equal to k i; it becomes an elastic scattering situation. Or, one can go to the same result of dropping the first term in case of forward scattering. Again, in that case, k f becomes equal to k i by energy conservation, because 1 minus cosine theta, which appeared over there is 0. And so k f is equal to k i. So, the scattering becomes elastic. And then one has a simple form, which is known as Thomson cross-section.

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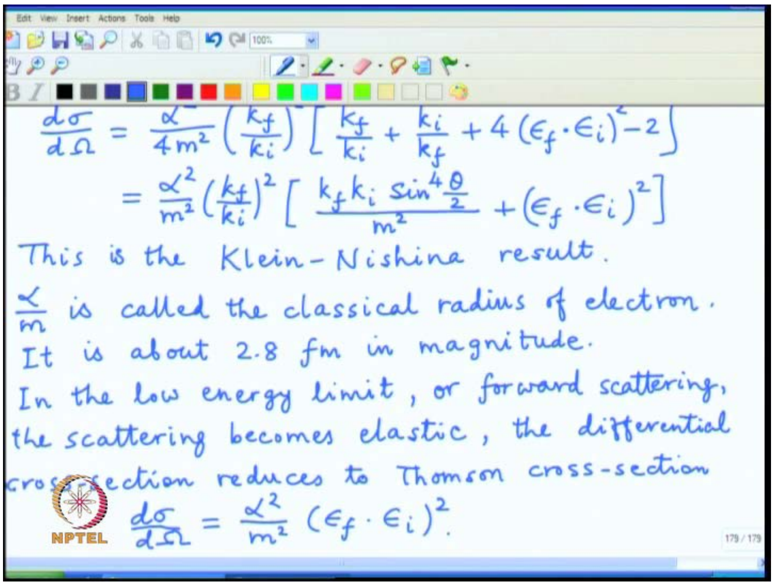

And, the form for that object is d sigma by d omega is equal to alpha square by m square times epsilon f dot epsilon i whole thing square. So, not only the scattering is elastic, but the polarization also is maintained in the sense that, there is no cross-section for change in polarization to an orthogonal direction. And, the magnitude of the Thomson cross-section is one way of defining the classical radius of electron. It is exactly the size of the shadow created by this scattering in the low energy limit. Next time, I will mention some further properties of this cross-section in case the polarization effects are also not observed. So, we have to sum over polarizations to derive some related forms of this particular cross-section.