Relativistic Quantum Mechanics
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Lecture - 36
Mott Cross-section, Compton Scattering

In the previous lecture, I worked out the algebra for calculating the Coulomb cross-section for scattering of an electron. And we converted the amplitude to the differential cross-sections, simplified all the algebra coming from conservation of momentum and appropriate delta functions; included all the kinematics factors. And now the last step to put in is the appropriate structure of the input and output electron wave functions. And that is best done in the form of projection operators.

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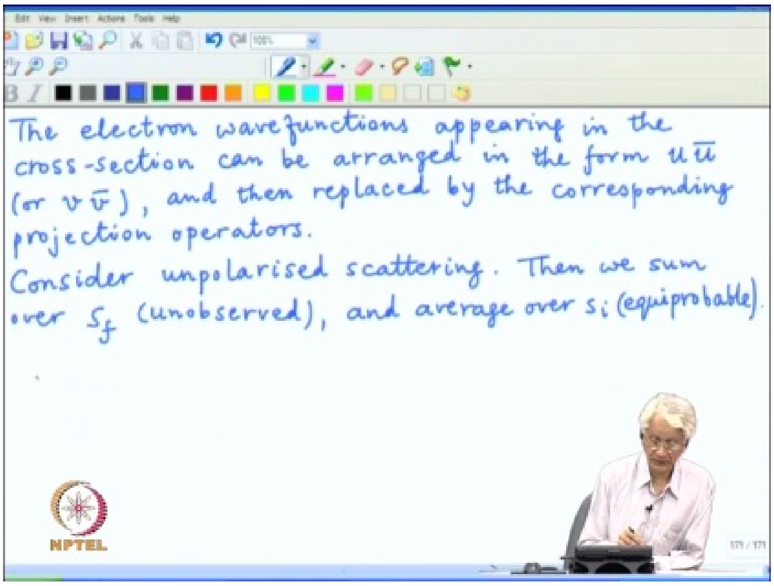

So, the electron wave functions appearing in the cross-section can be arranged in the form u times u bar or the positrons v times v bar, and then replaced by the corresponding projection operators. And this is the straightforward thing to do, because the arrangement in the order u bar u is not possible, because there is always a gamma matrix representing the interaction sitting between the two. And one cannot just take out that gamma matrix, but and the amplitude is squared; the squared form always have this u times u bar sitting together. Or if they are sitting at opposite ends, they can be brought together by including the cyclic trace functions, and then the whole structure simplifies, and one only ends up using projection operators and not the explicit form of the wave functions.

The projection operators can decide between positive and negative energy solution or also corresponding up or down spin whatever may be necessary. And, I will just illustrate in one particular case. And that happens to be the case of unpolarised scattering; in which case both the spin projections appear with equal likelihood. And so they are summed over; they sum up to the identity. And, the only projection operator remaining after that is the energy projection operator. So, if it is an electron, it will be p slash plus m divided by 2 m. And, if it is positron, it will be p slash minus m divided by 2 m. So, let us just work it out explicitly.

So, consider now the special case of unpolarised scattering; which means that, the initial value of the two polarisations are equally likely; and, in the final state, we do not observe the polarisation; we just sum over the two possibilities. So, then we sum over S f – that is, remains unobserved; and, average over S i – and, these are… because they are equiprobable. And, the only distance is you have to count the number of spin state to take out the average; otherwise, the structure is just the summation further two cases. And so we have the form.

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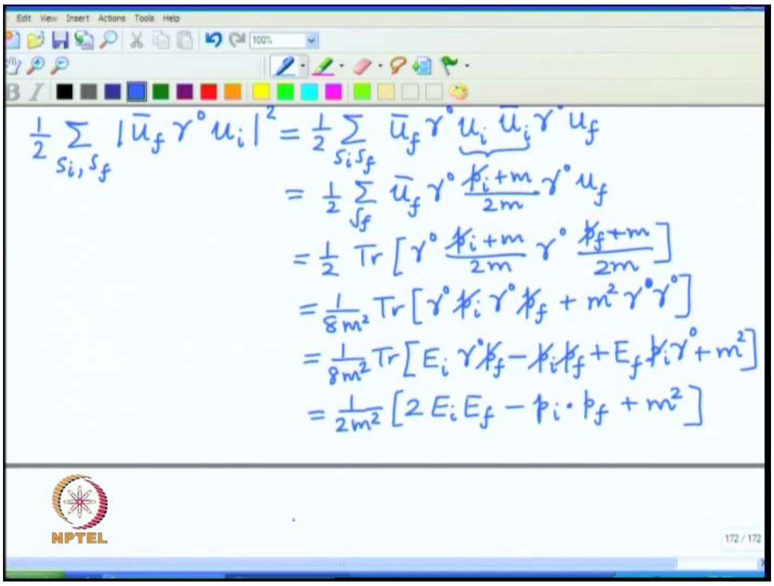

So, the required matrix element absolute value square – it becomes one-half coming from the average and the sum over both S i and S f; the matrix element, which we had was u bar f gamma 0 u i whole thing square. So, now, we use the rules for rewriting this square. And, the result is just the structure, which I described last time. It happens that gamma 0 bar is equal to gamma 0. So, nothing happens to that. So, we have now this object equal to half S i S f; then u bar f gamma 0 u i multiplied by u i bar gamma 0 u f. And, here we see the structure explicitly appearing of the u u bar structure, which can be replaced now by a projection operator. So, that becomes one-half sum over S f u bar f gamma 0 p i slash plus m by 2m gamma 0 u f.

Now, u bar f and u f are not next to each other in the form, which we want. But, we can see that, if these operators, the factors are permutated into cyclic manner, it will come into the right form; and, the cyclic permutation is something which is allowed if this product of these factors is inside a trace. Now, what we have is just a complex number as in the form written. And so trace is equal to the number and we can just add a trace outside without any trouble. And then the cyclic permutation produces another projection operator; and, the expression becomes one-half trace gamma 0 p i slash plus m by 2 m gamma 0 p f slash plus m by 2 m.

So, this is in a form, where we can now use all the algebra for developing traces of a product of gamma matrices and simplify this trace into just a pure number; and, it is easily worked out. First of all, the terms, which contribute to the trace are only which have an even number of gamma matrices sitting inside the product. So, we can just remove all the terms, which have an odd number of gamma matrices inside it. So, there is one term, which has all the four gamma matrices; and, there is another one, which has just two gamma matrices. And, that is it. So, this is the first step.

And now, we can use the recursion relation to reduce the trace of four gamma matrices to terms, which have trace of two gamma matrices. That is also kind of straightforward. You have to take a dot product of successive terms. So, the first is the dot product of gamma 0 with p slash i. So, that produces a term, which is E i; and then the remaining part remains gamma 0 p f slash. Then the next term appears with the negative sign with the dot product of 2 gamma 0, which happens to be 1.

And then it is p i slash p f slash. And then there is another with a dot product of gamma 0 with p f slash. So, that gives E f and then it is p i slash gamma 0. And, the last time, gamma 0 square is just identity. So, it is m square. So, we have now reduced everything to two gamma matrices. And, the trace of gamma matrices are in this particular trace of identity is easily done. There is an overall factor of 4. And, that gives the result that, there is a dot product – gamma 0 dot product with p f slash is giving E f; the dot product of p i slash p f slash is p i dot p f. The third term will give E f times E I, which is the same as the first term and then m square is as it is. So, the net result is 2 E i E f minus p i dotted with p f plus m square. And, this is the simplified form. It can be put inside still simpler form by expanding out this p i dot p f in terms of the various factors of beta and the angle cosine theta. We can do that.

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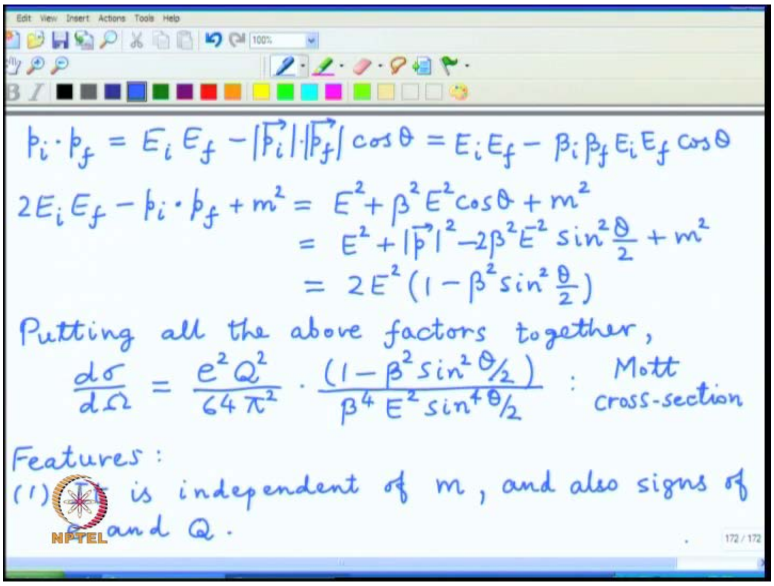

And so let us see p i dotted with p f is E i E f. And, the time component and the space component gives just the magnitude minus p i times p f times cosine theta. And, this can be written as E i E f minus beta i beta f E i E f cosine theta. Now, this can be further simplified by noting that, the process, which we are using, is an elastic process. So, the initial and final energies are identical. So, E i is equal to E f, beta i is equal to beta f. And, the whole structure then simplifies further.

And, one now has a simple looking form for the whole cross-section. So, first of all, the cosine theta part can be still be written in terms of sine square theta by 2; and, the various terms of E i and E f can be cancelled. So, let me work that one out too. So, this object 2 E i E f minus p i dotted with p f plus m square is now – 1 E i E f cancels, the two are equal; so it is E square. The second term becomes plus beta square E square cosine theta. And then the last term is plus m square.

Now, cos theta can be written as 1 minus 2 sine square theta by 2. So, this is E square; one will just give beta square E square, which is actually equal to magnitude of the momentum square and minus this beta square E square sine square theta by 2. And, there is a factor of 2, which goes with it. And then there is plus m square. So, again p square plus m square becomes E square; which can be taken out. So, there is a 2E square, which comes out as a common factor and then there is 1 minus beta square sine square theta by 2. So, in this form, now, everything is expressed in terms of the initial energy of the electron and the scattering angle; everything else is simplified.

And so finally, we put together all these factors calculated above to get d sigma by d omega. And, it has a form e square Q square divided by 64 pi square. There is a vector, which is just ((Refer Time: 17:33)) What we calculated is 1 minus beta square sine square theta by 2. And, there is the denominator, which ((Refer Slide: 17:46)) from essentially the normalisation coming from the flux. And, all these factors of E square have been included in the normalisation. So, this is the final result. And, it is often labelled as Mott cross-section, because it was calculated by Mott and it has a form, which differs from the classical or the non-relativistic formula by just this extra factor in the numerator 1 minus beta square sine square theta by 2. The rest of the stuff is identical to what Rutherford calculated and used.

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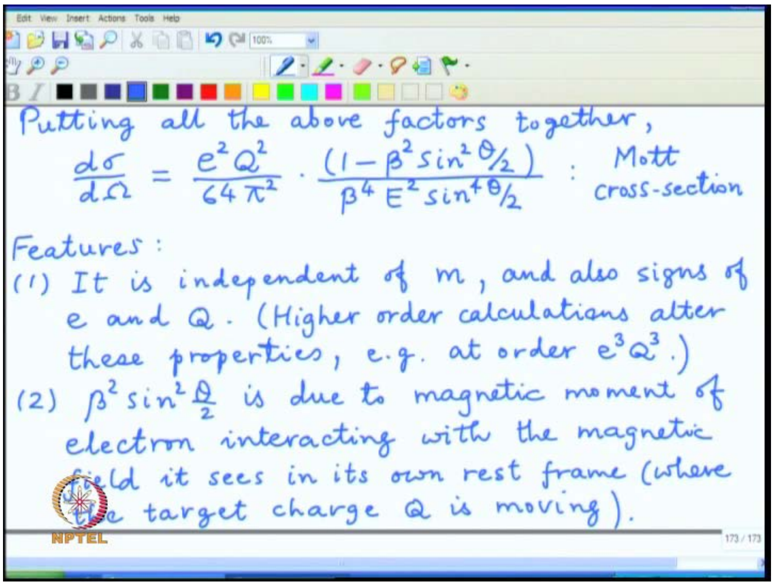

Now, let us look at some of the features of these cross-sections, which are useful to understand and also experimentally important in trying to fit this formula to the data. So, one feature is that, it is independent of the mass of the electron; just did not matter whether we use electron or we use some other particle. For example, Rutherford used alpha particles. It is all the same, the only thing which matters is the charge in this particular order and also signs of either the projectile or the target. The sign is irrelevant; positively charged particle will scatter with the same cross-section as the negatively charged particle although the trajectories, which are followed in the two cases, will look completely different, because one of them will have an attractive potential and the other one will have repulsive potential.

But, the cross-section when one sums over all the different impact parameters, turns out to average between the opposite signs of impact parameters and it ends up being independent of the sign of both e and Q. These properties are actually true only in these leading order calculations. So, higher order calculations – they end up changing it. But, the lowest order – this is kind of peculiar feature, which shows up. And, one of the consequences is that, whether we use electron or we use positron, the leading order cross-section is the same. So, for example, I can have these new corrections appearing not at e square Q square, but at higher order e cube Q cube, where things will change.

Let me make one more observation, is what is the significance of this beta square sine square theta by 2. So, it can be given an interpretation. So, beta square sine square theta by 2 – it is a consequence of the magnetic moment of the electron interacting with the magnetic field, which the electron feels in its own rest frame, because in the rest frame of the electron, the nucleus or the target charge is moving; it has therefore, a current and it will produce a magnetic field. And, the magnetic moment of the electron will respond to that magnetic field. And, that contributes to the cross-section.

And, this correction beta square sine square theta by 2 is actually the consequence of that interaction. If there was only coulomb field, we will just get the same as answer the Rutherford got. In other words, if the electron did not have the spin, this extra factor of beta square sine square theta by 2 will be absent – with the magnetic field, it sees in its own rest frame. That is because where the target charge Q is moving. And, one can see that, this is indeed true by doing a similar calculation, but not with an electron, but with the scalar particle like a Klien-Gordon field, where the magnetic moment is not there; and then this new feature kind of disappears. And so we can be sure that, it is a contribution of the spin of the electron.

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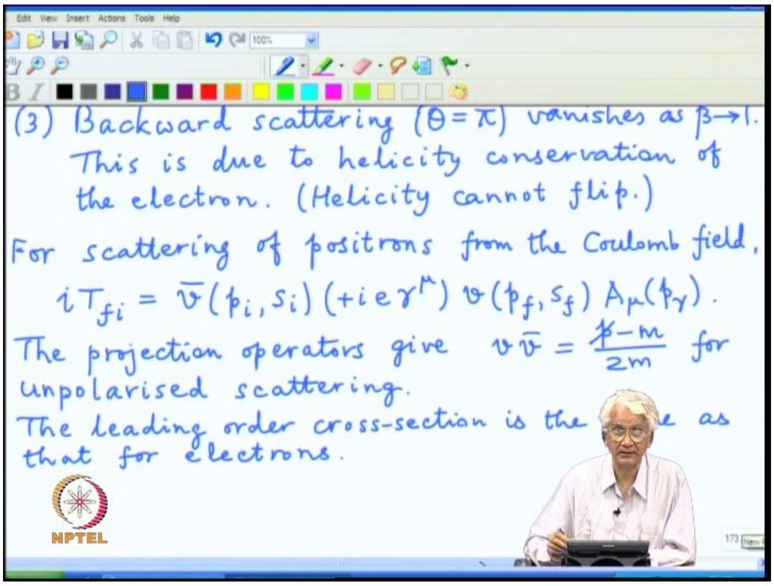

The third feature is that, the backward scattering, which means theta equal to pi vanishes as beta approaches 1. This overall factor 1 minus beta square sine square theta by 2 just becomes 0. And, this also has a meaning, because this is due to the helicity conservation of the electron. When beta goes to 1, the helicity is essentially decoupled from each other. There is no mass term, which connects the two; it is negligible. And, as long as the electron has a fixed helicity, it cannot scatter backward, because backward scattering means the spin will remain where it was pointing; the angular momentum is conserved. But, the momentum direction reverses.

So, a backward scattering corresponds to change in helicity. If the helicity is conserved, that contribution vanish, must vanish rather. And, this is also very useful feature to see not only in this Coulomb scattering, but many other processes, where if you have ultra-relativistic particle, the conservation of helicity produces restriction on backward scattering. As a matter of fact, we saw this feature in earlier in studying the properties of graphene, where indeed there is a massless excitation and the properties of it are governed by conservation of helicity rules. So, these are the some of the important features.

And, one more thing I can add is the description of the same process. So, for what will happen, just a little different notations. So, for scattering of say positrons from the Coulomb field, we have the same structure for all the kinematic properties, but the amplitude looks little different, because the wave functions are little different. So, I will write down only that particular part; where, now, we have instead of u v bar and v appearing; instead of minus i gamma mu, we have plus i e gamma mu for the change or flip of the charge. And then it is v p f, S f and combined with A mu with whatever photon momentum we had.

So, the same structure appears; the kinematics is the same. And, the only thing, which will come from this new wave functions is products, will be the order of v times v bar. And so the projection operators give this v times v bar equal to p slash minus m by 2m. And, this is again for unpolarised scattering. So, we have only the energy projection operator and not the spin projection operator. There is an overall sign from the actual projection operator, which is m minus p slash by 2m, but those signs we have already included in the Feynman rules. So, there was a minus sign for every positron in the initial state, which must be included in the cross-section. So, having done that, projection operator is easily written as p slash minus m by 2m.

And of course, as we mentioned earlier, the cross-section at the leading order gives the same result, because this minus sign does not matter everything as contributing only when the there are even factors of gamma matrices inside that trace. And, the minus sign will come only if there are odd factors. So, nothing changes. So, the leading order cross-section is the same as that for electrons. So, this as much as I can say for the Coulumb scattering problem. And, we have various important things to learn both in terms of the calculation techniques as well as the interpretation of final result. So, next step for me is to define a different process and do the calculation for that.

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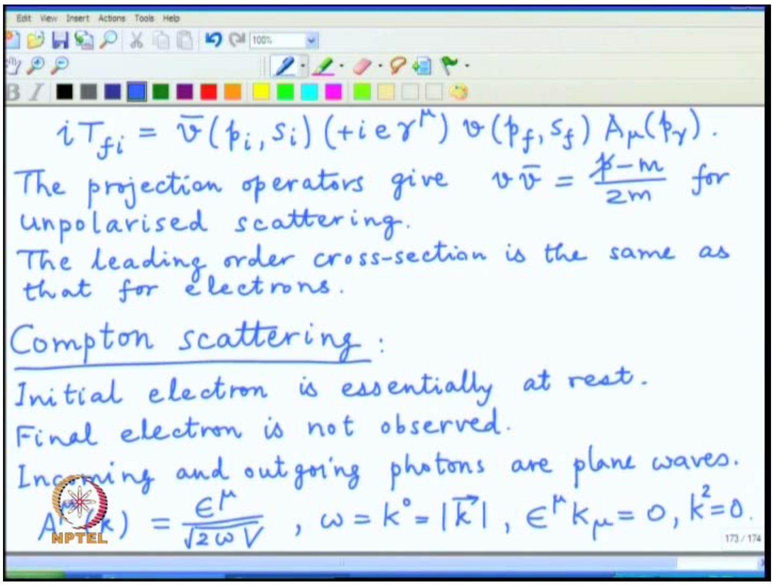

And, I will choose that process to be what is known as Compton scattering. We have seen that, in this case, there are two diagrams, which contribute. And, the process in laboratory is essentially observed as scattering from a material say a free electrons in a metal or even from atoms. So, there is just a target, which is essentially non-relativistic and you shine some light on it and you observed the final photon, which… And, that properties of the final photon is what we want to calculate. So, here the initial electron is essentially at rest. The final electron is not observed. And, the incoming and outgoing photons are plane waves. So, that describes the initial and final states of the process.

And, we want to now calculate various things in this particular normalisation. And so we can write down various rules at the photon wave function in the momentum space, is just given by the polarisation. And, the box normalisation in a volume V, omega is the energy of the photon. So, it is k 0; it is equal to magnitude of the k; and, the polarisations are physical. So, they will have the transversality property that, epsilon mu dotted with k mu is equal to 0.

And, we will normalise the polarisation vector to unity. So, these are all standard properties. And, we can write even k square is equal to 0 for the photons. And, these are the things, which we will use in simplifying the algebra, which comes out of it. The actual photon wave function will have an e raise to i k dot x factor. But, if when we transform to momentum space by a Fourier integral, that e raise to i k dot x becomes part of the Fourier integral and goes away and what is left is just this – overall constant, which is independent of the location.

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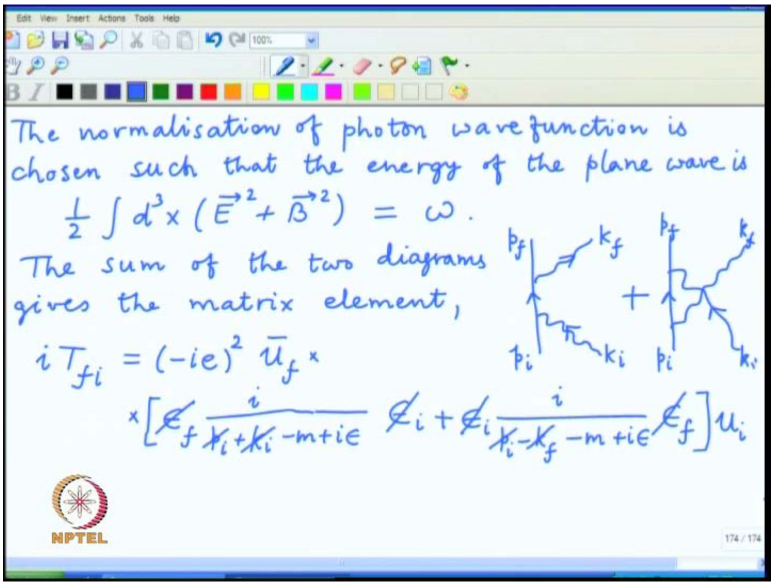

So, now let us go back and draw the diagrams for this particular process. And, those I had drawn earlier; I will draw them again. One thing I can mention before that, the photon normalisation of photon is chosen such that the energy of the plane wave, which we are familiar in electrodynamics form – it is one-half of E square plus B square in the units, which I have used. And, this is just a volume integral with this box normalisation and this energy is omega. In the units, again we have h cross equal to 1. So, this is the standard normalisation of the photon energy. And, the wave functions have been chosen to be consistent with it.

So, now, the Feynman diagrams I can draw. There were two of them. one of them is the ordering where the incident photon is absorbed, and then the final one comes out. And, the other diagram was the final one is emitted first and the incident one is absorbed little later. So, now we can put some labels. There is p i p f for the energies or the momenta of the electrons. And, k i k f is the same for the photons. And, as you have seen, the two diagrams correspond to just ordering of the vertices. In a different way, they are interactions with these bosonic photons and the contributions have to be added.

So, now, we have to write down the matrix element, which correspond to the sum of these diagrams. So, the sum of the two diagrams gives the matrix element, which I am going to write in this form of the transition matrix element. So, the overall delta function for energy momentum conservation is taken care of. So, there is a factor of minus i e square, which comes from the two vertices. And then there is a u bar f for the initial wave function. And then there is a huge product of various gamma matrices, which we have to now explicitly write down. So, the gamma matrices of the interaction are contracted with the corresponding photon wave function A mu. And, A mu is simplified to this factor of epsilon mu.

So, we only have the photon part appearing as epsilon slash. So, as the rule goes, we start with the end point of the electron line mu bar f and then work backwards all the way. So, the first diagram, the first interaction produces is epsilon f slash. Then there is an electron propagator. And, we have to use energy momentum conservation to find out what its momentum is. And, that momentum is p i slash plus k I, slash and then minus m plus i epsilon. So, that is the propagator. And then there is a second vertex again contracted with the A mu. So, it gives epsilon i slash. And, that is the first diagram.

And now, the similar thing for the second diagram; but now the vertices are interchanged. The first we have epsilon i slash. Then there is an i times the propagator for the photon. But, the momentum conservation now says that, this intermediate momentum is p i slash minus k f slash. That proton got emitted first, and then minus m plus i epsilon. And then now, we have epsilon f slash. So, that now gives the sum of two diagrams; and then the last wave function – the electron, which u i. So, this is the structure, which emerges after using all the Feynman rules and it gives a matrix element. We of course, have to use all the normalisations for the various wave functions and fluxes when we convert this to the cross-section. But, that will be done in the next step.

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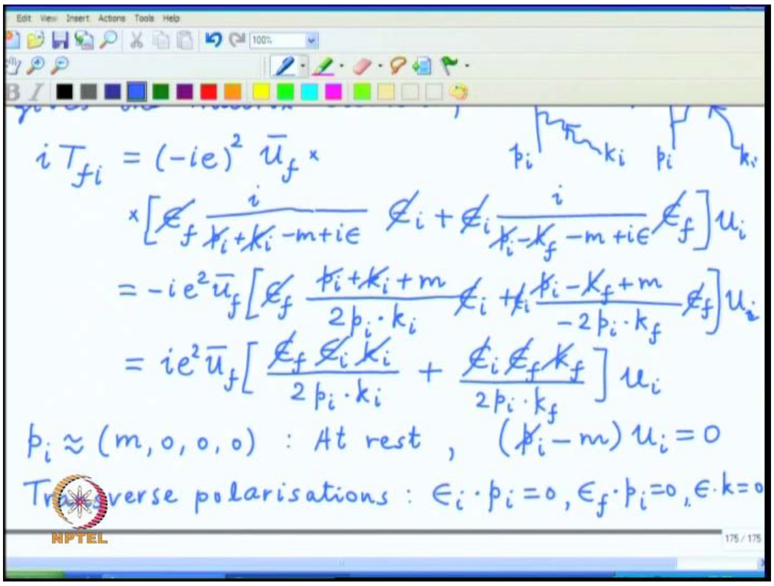

We can still simplify this calculation a little bit more. So, first is this all the factors of i can be combined. And, that give minus i e square u f bar. And, the denominators can be rationalised. So, all the gamma matrices now come in the numerator and the calculations will become simple for taking all the traces of the gamma matrices. So, this object will produce p slash plus k slash plus m. And, the denominator – now, we will have the square of the total momentum minus m square.

And, that can be simplified as well, because the initial and final momenta belong to particles, which are on ((Refer Time: 45:31)). So, we exactly know what is p square and k square corresponds to within those particular cases. So, here for example, p i slash square produces just m square, which cancels with the minus m square, which is there in the rationalised denominator; k i slash whole thing square – this is just k square, which happens to be 0, because physical photon. And so the p i slash plus k i slash whole thing square just produces the dot product of the two, which is 2 p i dotted with k i. Everything else is gone from the whole gamma matrix algebra. And, then the epsilon i slash remains.

Same thing can be done on the other side or the other diagram as well, where the result is p i slash minus k f slash plus m. There is an epsilon i slash in the front, epsilon f slash at the back. And, the denominator follows the same rule; p square cancels with m square; k f slash square is 0. And, we only have the dot product surviving, which is 2 p i dot k f. And so one can now close the bracket and put the electron wave function outside it. So, this is the simplification. We just rationalised the propagator denominators. Can still do little bit of simplification, which comes from the various properties of this polarisation tensors. In particular, we can commute this polarisation epsilon slashes to be just next to each other by taking them through these momentum factors. Of course, as far as the mass is concerned, there is no problem; it just goes through. But, as far as the other objects are concerned, we just have to be little careful about commuting the matrices.

So, let us see what that produces when epsilon i slash is commuted with k i slash. We will get the object in the reverse order with a negative sign plus a dot product of epsilon i and k i, which happens to be 0, because of photon is transverse. So, we just get a negative sign. That negative sign can be combined with the negative sign outside. And, we have the object, which I am going to write down from commutation with k i slash. And, the denominator is the same thing 2 p i dot with k i. What about p i slash? Again you can anti-commute with what is there on the other side. That is fine. And, the dot product will now has epsilon dotted with p i. But, here we can use another property that, the initial electron is essentially at rest; and so its structure is m, 0, 0, 0. And, the polarisation is transverse, which has some component only in the space directions. So, again, p dotted with epsilon gives 0. And so we have only the negative term remaining. So, it will be now minus p slash plus m.

And, that whole object, which will act on the u i; so we have also this relation that p i slash minus m acting on u i is equal to 0. So, the commuted part of p i slash just cancels with m, because of this wave equation satisfied by the initial electron wave function. And, the only part, which survives is then the k part; and, which produces this particular object. Similar analysis can be done for the other side. Again p slash can be commuted with epsilon f slash. For the same reason, epsilon f is also transverse. So, its dot product with p is 0. Again, we will have minus p i slash plus m acting on this wave function u i, which gives 0 from the Dirac equation. So, the term which survives is essentially k f slash.

But, since again k f slash and epsilon f slash can be commuted together with a change in sign, the reason being that, k dot epsilon is again 0. So, we have now getting plus epsilon f slash k f slash. This is a negative sign in the denominator; but that can be eaten up by the overall negative sign outside. And so the final result looks like a product divided by 2 p i dot k f whole thing acting on u i. So, transverse polarisations, which means epsilon i dotted with p i is 0, epsilon f dotted with p i is also 0; and, epsilon dotted with its k is also 0. So, these are all the rules, which we have used. And, that simplified the expression to this particular form. At this stage, there is not much else one can do; but the structure is in a form, where one can see the various symmetries.

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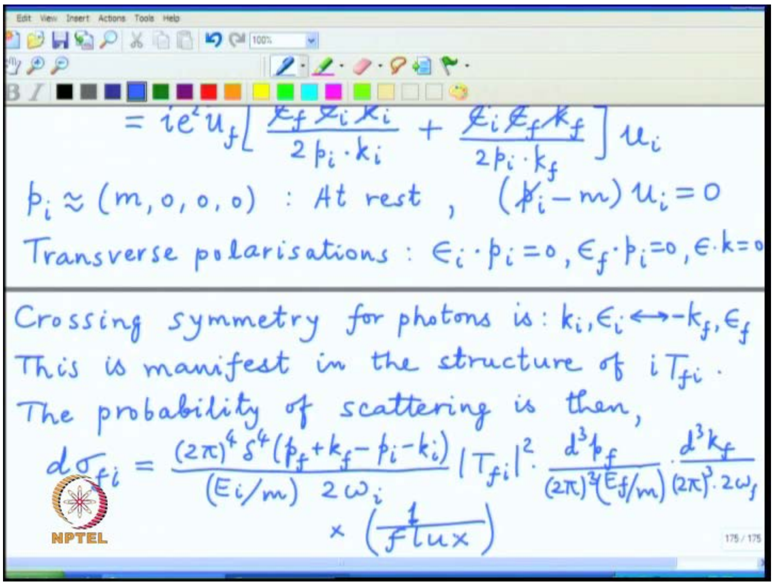

Explicitly, in particular, there is this symmetry between the two contributions, which come from the two diagrams. And, that is very obvious in this particular structure. It is something, which I have earlier called a crossing symmetry – interchange of the two photons, where one of them is in the final state and the other one is in the initial state. And, in particular, for photons, it has its own antiparticle; there is no distinction between particle and antiparticles. So, one can move them just from one side to the other. And, that will interchange k i and epsilon i with k f and epsilon f with the only change is that, the momentum of the photon is reversed.

And, if one applies this crossing symmetry to the two diagrams I drew above, one immediately sees that, one diagram goes into the other. In the algebra, we have to include this negative sign, which comes from converting a particle in the initial state to an antiparticle in the final state. And, in this process, the 4-momentum is completely reversed. And now, one can see that, the two terms, which are written in the expression above, are indeed related by this particular interchange. If i and f are interchanged, all the indices of epsilon and k appear in the correct order; and, the negative sign of k f does not play any role, because there is a factor of k f both in the numerator and the denominator. So, the sign essentially cancels out. So, in this particular form, even this crossing symmetry is a manifest in the structure of i times T f i. So, that is as much as one can do in terms of calculating this matrix element.

And now, to calculate the probability of this particular scattering, is obtained by taking the absolute square of this T f i's object and including all the normalisation factors coming from the volume parts. And, that object is – first, there is this factor from the momentum conservation, which I include only once; the second factor of 2 pi to the 4 delta 4 0 is scattering per unit volume and the unit volume part is divided out.

So, that is the scattering part. And, the wave function normalisations of the initial states for the electron gives E by m. And, for the photon, it gives twice omega. Then there is factor of T f i square. And then one has to now include the final state differential volume elements for the electrons and for the photons. And, the last factor to include is 1 over the incident flux of the particles involved in this particular process. And, I will continue with calculation of this flux and simplification of this scattering probability in the next lecture.