Relativistic Quantum Mechanics
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Lecture - 28
Interactions and formal Perturbative theory, The S-matrix and Feynman Diagrams

In the previous lecture, I constructed the solution of the Dirac propagator which the boundary conditions implied by causality. And that basically give I epsilon prescription; such that the propagator vanishes outside the light cone.

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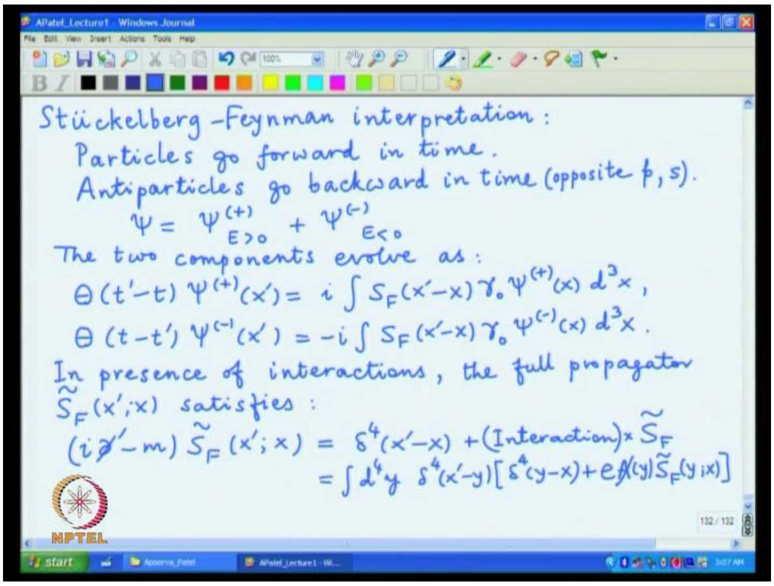

So, this formulation is known as Stuckelberg Feynman interpretation, which essentially amounts to particles go forward in time and antiparticles go backward in time. And particular we have chosen the convention that they will have opposite value of p and s compared to the particles when they travel backwards. And this interpretation and the propagator language were introduced first by Stuckelberg. But not many people paid any attention to it. It was independently reformulated by Feynman; he went further and heavily used it do many practical calculations.

So, more commonly it is basically labelled as Feynman’s method. And, he also invented a diagrammatic language which makes interpretation very easy to visualise; see the details of that particular language today; in particular introduce the interactions of electrons with the electromagnetic field. And, see how the propagator gets modified because of this particular interaction.

To revise let me just write down the breakup using a projection operators; that the total wave function was the plus component; which essentially amounted to taking the modes with positive energy. And, minus component which amounted to modes with negative energy. So, this was just the convenient breakup the 2 modes with different signs of energy propagated differently. So, it is useful to separate this thing out afterwards. Now, I can write equations for evolving both these modes; which are the same equation written together. But now projected and to different components. So that, 2 components evolved as positive energy goes forward in time.

So, 1 can use this step function. And, that is the definition of the green's function multiplied by gamma 0 psi integrated over d cube x. And, gamma 0 essentially is the operator which separates the positive and the negative energy parts in the covariant language; it appears automatically to connect between psi plus projection. And, the total wave function which is psi; it is essentially also connected to the reason that if you calculate overlap for Dirac propagator you have to use psi bar psi structure. And, not psi dagger psi and that extra gamma 0 are responsible for producing the bar in this expectation values as well. So, this is what happens to positive energy part and the negative energy part as the similar formula it now has a step function with opposite sign.

And, now one can write a minus i because I am going to reinsert this gamma 0 again. And, gamma 0 for negative energy gives a minus sign which is put in front of this equation which is minus i. And, one has the analogous evolution of the negative part. So, this is a description of a free Dirac particle evolving in accordance with the principle of causality. Now, we can bring the interaction seem; the full equation is only slightly different than the free particle equation in the sense that instead of the free Hamilton in h 0. Now, you will use the full interacting Hamiltonian which is h 0 plus h interaction.

So, one can again write down the equation in presence of interaction; the full propagator which I am going to denote by S F to the tilde on top. And, it satisfies the equation which is the free part unchanged. But now it is acting on this full propagator; Hamiltonian which interaction I am going to write it on the other side of the equation. And, this can be written as the delta function plus the extra term which is; I will just write schematically interaction, because this equation is written in the Lagrangian formulation. And, not the Hamiltonian formulation acting on this S F tilde which was on the left side.

But now you moved it to the right side and this is the generic structure. And, knowing all the detail formulation this result can be rewritten as integral d 4 y is again auxiliary variable. And, the first term I am just going to rewrite in a clever way. So, when integrated over y this product of delta function just produces delta 4 x prime minus x that is the trivial part. But now we have the full interaction term in the Hamiltonian; in this Lagrangian operator that interaction term was just obtained by changing p slash to p slash minus e times A slash. And, e times a slash I am going to rewrite on the opposite side of the equation.

So, that now becomes e A slash at location y and then S F tilde (y, x). So, this is the general which says that from x to y there is propagation. Then there is interaction at y. y could be any point in the whole space time. So, it is integrated over the whole space time. And, then there is delta function which describes the connection between y and x prime. So, it just puts in this extra interaction at this location x prime equal to y. And, that gives the modified propagator equation. If you do the trivial integral here again over y; it will become e A slash at x prime and S F tilde from x prime to x which was exactly the operator on the other side of the equation with argument x prime.

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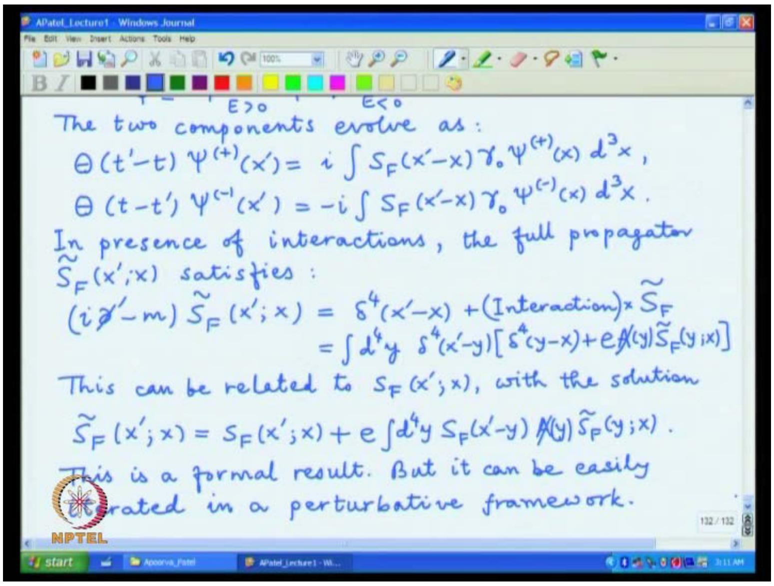

So, this is the equation, but now, we have a form which can be related easily to the free particle propagator which we calculated. S F is the solution delta 4 of x prime minus x and, the right hand side without any interaction. So, this S F tilde has an extra term which can be easily verified. It is the solution corresponding to the first term which is S F operator acting on S F gives the delta function. But we need something extra which gives the remaining piece.

And, that part can be rewritten in terms of S F as well. Because the delta function is again cleverly utilised for the purpose, you have this delta for x prime minus y converted to S F of x prime minus y; the remaining stuff is already in the correct form. And, this is the formal result. So, if you operate by this operator Del prime psi is minus m on both sides of the equation.

Every time acts on S F you get delta function it exactly amounts to the structure which we wrote above. So, if you are given an answer for S F which you already calculated in the previous lecture. Now, we can obtain an answer for S F tilde; this procedure is at this stage formal, it can be easily iterated in Perturbative framework where in the leading solution you start with ignoring interactions S F tilde is equal to S F. At the next order, S F tilde is equal to S F plus e integral S F is less another S F that is the next order. Then you take that whole expression again substitute on the right hand side in place of S F tilde. And, you get expression which is correct to second order and so on and so forth.

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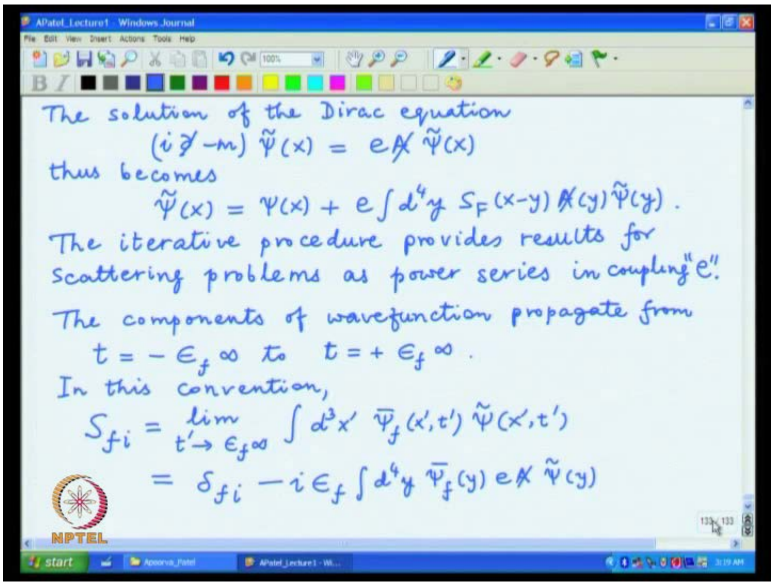

This is now general solution for propagator in presence of interaction one can also write a similar expression for the wave function, in presence of interaction the same. The solution of the Dirac equation which is now of the form again denoted with tilde; the purpose of this tiled was to take the interaction term to the opposite side of the equation. And, this structure in this Lagrangian equation is this e A slash and psi tilde. So, the free part is on one side the interaction part is the other side.

Again, in the terms of the propagator language, we just have to treat this right hand side as the source denotes the propagator is and then insert it back into the equation. So, a general solution is psi tilde x; psi x is the solution of the homogenous equation, when the interaction term on the right hand side is 0. And, the contribution now is coming from the interaction term overlapping with the corresponding green's function.

The green function corresponding to the left hand side is nothing but the propagator, but without interaction. So, this is actually S F and not S F tilde. And, the complete expression is this; where psi tilde now appears on the right hand side as well. Again, this is the formal expression if you want to use perturbation theory you can iterate it the leading order is just i tilde is equal to psi; at the next order psi tilde will be psi, e integral S F a slash.

Then, you can take that whole expression and again, put in the terms of psi tilde to get accuracy to next order and so on and, so forth. So, we have a complete procedure; the iterative procedure provides results for so called scattering problems. We are dealing with these asymptotic states which are plane waves; the solutions are now going to arises powers of what is there on the right hand side. And, the power is easily counted in terms of the electromagnetic coupling.

And, we have the value e denoting its strength. So, this is the whole formulation and it works very well. And, we will see how it works in a many different instances again one can define; so called scattering matrix problem where all these things are can be converted easily to matrix elements between certain initial state and certain final states not the components of wave function propagate from t equal to minus epsilon f infinity to t is equal to plus epsilon f infinity.

Now, this epsilon f was the symbol which we have used to denote the sign of the energy. So, this is another way of saying that the particles with positive energy are going to propagate forward in time. And, the once in negative energy is going backwards in that convention; the matrix element can be written as S f i this is schematic. But we have been using it.

And, it is defined as the overlap certain final state; with whatever, solution that has evolved from the given initial state. And, that solution we have denoted by psi tilde. So, this is just a definition one can easily evaluate the overlaps the leading term of psi x just produces the delta function. And, I can just abbreviate it as delta of some final and some initial state. But then the second part now explicitly gives the overlap. And, that can be now written as i epsilon f d 4 y psi f bar y e A slash psi tilde y.

The extra sign of epsilon f is the result of using this propagator S F x minus y together with the buried factor of gamma 0 which is part of psi f bar. This is again a formal expression if you can calculate this kind of overlaps; you will be able to calculate the scattering matrix element. The only way one can do practical calculation is in a Perturbative expansion where, psi tilde is rewritten as psi plus some corrections and then you evaluate these objects term by term.

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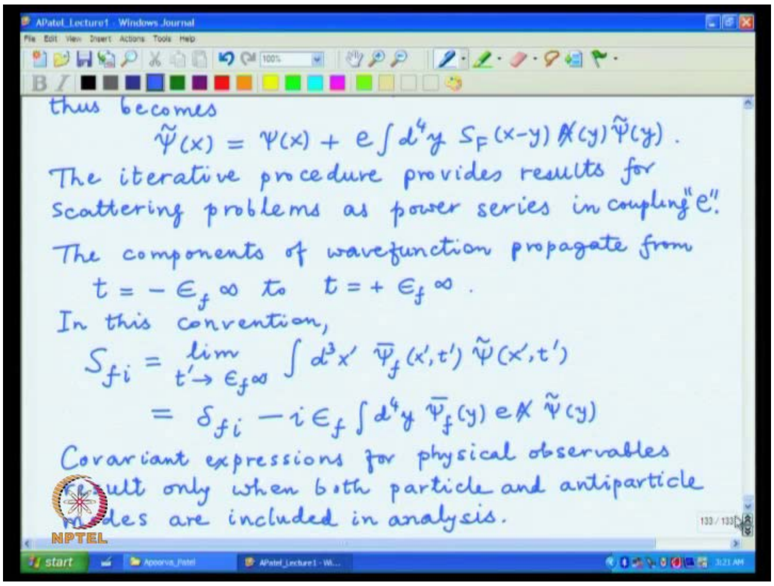

So, this is the important description and, one must remember that the covariant expressions for physical observables result only; when both particle and antiparticle modes are included in the analysis. The consequence is that, if one calculates only one part say only the particle contribution. The result will not be following the restrictions of

Lorentz invariance or if just antiparticle is included the same thing both of them are combined together; with the appropriate sign changes. And, then the total result will be Lorentz covariant; which is our requirement for implementing fully relativistic analysis together with causality.

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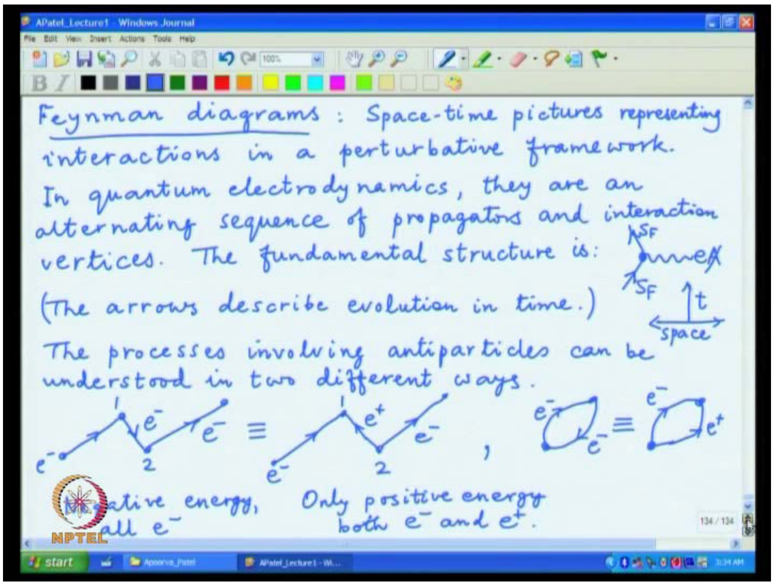

Now, let us describe some of those things in terms of what are known as Feynman diagrams? These are nothing but space time pictures representing interactions in Perturbative framework. This has gone much beyond the theory in which they were invented namely quantum electrodynamics. But in q e d has the explicit expressions. So, they are and alternating sequence of propagators and interaction vertices; one is quite to saying that a particle propagates from this location to the next location has the interaction there. Then it propagates to the third location has interaction there and so on and, so forth.

This is lose language not strictly in accordance with the restrictions of genuine quantum theory. But it expresses in an intuitive fashion very easily, what all things are happening. And, so it also a very popular language; even though, it is a lose description it heavily used in terms of doing the calculations. In the sense that to do the calculation; one actually starts out by first flowing all the Feynman diagrams. And, then for each Feynman diagram writing the corresponding algebraic expression and adding those things up. Writing the algebraic from the beginning generally turns out to be quite cumbersome and messy. And, this way you can remember the things very easily and count the various factors of propagators and, vertices and, corresponding operators in a straight forward manner.

The fundamentals structure in quantum electrodynamics is just 3 lines. There is a incoming propagator, there is a outgoing propagator and, there is vertex which represents interaction with the electromagnetic field. So, this is S F, this is S F and the vertex is represented by e times A slash. This is generalised further in full theory of quantum electrodynamics. The interaction vertex is interpreted as interaction with photon. And, wavy line which I have drawn here then we will become photon propagator. But we will come to that when we discuss the dynamical behaviour of photons interacting with fermions right. Now, the vertexes can we just the interactions of fermions with background electric field represented by e A slash in terms of operator structure.

This is a basic unit. And, as I said by repeating this one can have different descriptions in terms of how many orders calculations you want to do? Now, this description certainly has one feature which is expressed here by the arrows. So, the arrows describe evolution in time. So, in particular for every propagator; there is an arrow which says the particle going in the future in this direction. And, in this particular picture the time direction is sort of pointing upwards.

And, space is in the orthogonal direction which describes many locations. And, then one has this representation of an interaction. And, it is here that the interpretation of particle and antiparticle also re enters the language. In the sense that, an antiparticle will have the same kind of diagram which the arrow pointing in the opposite direction which amounts to flip in the sign of the momentum energy as well as the spin. And, it will be denoted by an appropriate change in the expression for S F.

So, one can have 2 different descriptions. The processes involving antiparticles can be understood in 2 different ways here it is much easier to illustrate by examples. I will just draw simple diagrams; one prescription of this interaction. Suppose has of diagram of this particular type where in electron comes in it scatters at one particular point goes backward in time. In the sense, that its energy now is negative after the scattering and, then it re scatters and, then it goes forward in time with certain positive energy. These descriptions using only electrons, but 2 times of energy and, with this Feynman

Stuckelberg interpretation; which we already built inside or formulation the same diagram can be now drawn with arrows pointing opposite way for this major segment. This will have a description, that in the initial stage there is a electron which is propagates to 0.1. At 0.2 there is production of e plus e minus pair out of nothing that is the vacuum.

And, this pair production by quantum mechanical uncertainty principles provided which is short leaved enough. So, it does not last forever; it is not going to produce asymptotic states, but one of the particles can be asymptotic and go to infinity and that is the second e minus. The e plus which is produced in this pair production. Now, can annihilate with this e minus coming from the initial state at point number one; and, there the annihilation now goes back to vacuum.

The e plus is the short lived intermediate state, but now here it is appearing as the positive energy solution. Because when you use physical e minus and e plus particles in this diagram both of them will have positive energy there are no negative energy counterparts. Here there is negative energy and, all e minus and, here there are only positive energy states both e minus and e plus. And, the way you have constructed the formalism this 2 descriptions are completely equivalent.

One just has to remember the rules of what is the electron propagator with negative energy is going to look like or, equivalently what the positive energy positron propagator is going to look like? They look identical. And, so the factors describing the various segments of this Feynman diagram produces the result. A similar description can also be given for processes which involve loops.

The most simple example of such an object; can be just 2 segments. It could be either electrons going around in loop which 2 interactions or this is certainly allowed, because the interaction can absorb the energy; to convert positive particle to negative energy particle. The alternative description of this process will look like both electron and positron moving forward in time there was pair creation at one point. And, the pair annihilation at the other point both, these results give identical answers; one can add a little bit of mathematic if one wants to see this explicitly.

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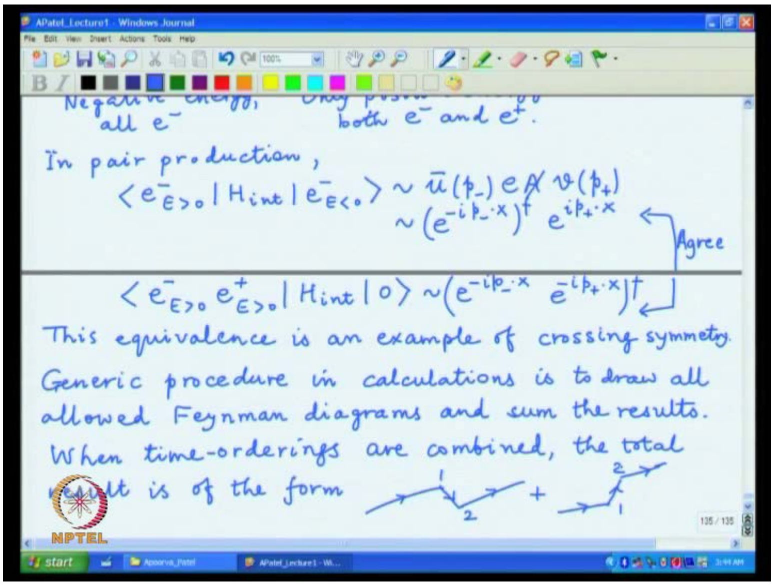

So, let me just describe the 2 languages in this simple pair production example. Then I will just describe the various plane wave phases without worrying about interactions and normalisation factors. The first language is of electron going around in closed loop. So, it can be represented as 1 electron with positive energy in the final state. It is produced by some interaction from another electron, which has negative energy in the initial state. That is the description of what happens at 1particular vertex in this diagram. Say, point number 2 in this picture; there is an initial electron with negative energy getting converted to final electron positive energy due to some interaction.

Now, one can plug in the plane wave factors. In this description, the final state electron will have wave function u bar with some momentum which is p minus, minus here refers to an electron the interaction will be something of the nature A slash. And, then the initial state is the negative energy electron. And, we saw it is the fundamental variable was this wave function denoted by v. It has energy momentum which I am going to denote by p plus, because it is describing the antiparticle mode. And, p plus is just the label for this particular language. Then by plugging in factors this object is going to behave like e raise to minus i p minus dotted with x the whole thing complex conjugated. And, then e raise to i p plus dotted with x, because that is the behaviour of this wave functions u and we have opposite phases.

This is one particular language, the other language is similar that corresponds to now, writing that there was nothing and then pair got produced out of it. So, now, we have an electron and the final state with the positive energy. As well as, positron in the final state with the positive energy with interaction and produced from the vacuum. The behaviour of this is now, both particles are positive energy are there in the final states. So, its e rise to minus i, e minus dotted with x then e raise to minus i p plus dotted with x p plus is momentum of the positron, but now, it is in the final state. So, it comes with the opposite sign of the phase and this whole thing is complex conjugated.

So, this is the description, what one can easily see all the phases and the signs of momenta between particle and antiparticle are arranged in our convention. So, that both these descriptions produced the same mathematical expression. Even though, in one case you are writing some object in the initial state. In other case you are writing the antiparticle in the final state.

This illustrates symmetry which mentioned this equivalence is an example of crossing symmetry. It is something which, we had seen in the discussion of the whole theory that you can change the particle from an initial state to an antiparticle in the final state flipping the signs of all momenta. And, we can easily see that illustrated here, both in terms of the matrix element as well as, in terms of the Feynman diagrams. This is very useful constructions and terminology which is heavily used in calculations in perturbation theory.

The generic procedure in calculations is to draw all allowed Feynman diagrams and sum the results. This procedure include summation over both particle and antiparticle modes. As well as, interactions accruing at different location in space time; one has to remember the rules of statistics while summing over this particular processes. In the sense that, some diagrams might be related with some other ones by interchange of particles. And, when you interchange the particles in this particular diagrammatic language, it corresponds to interchange lines that will incorporate the Fermi level or Bohr statistics as the case may be when we add the diagrams you might have an extra sign coming from statistics in the particular time part.

The illustration is that, the total result is of the form where, all possible geometries of these propagators is included in particular in this intermediate points 1 and 2 are going to get integrated over the whole space time. There locations are not restricted they are the interaction positions. But the interaction positions could be anywhere. So, they can be ordered either in situation where1 is the feature of 2 or it could be the 2 is in the feature of 1. When 1 sums over all possibilities, 1 has to include all this different orderings simultaneously. And, the total object will be the sum over all possible interactions or in other words, it will include all this propagator lines irrespective of whether they are going forward or backward in time when we draw all combinations.

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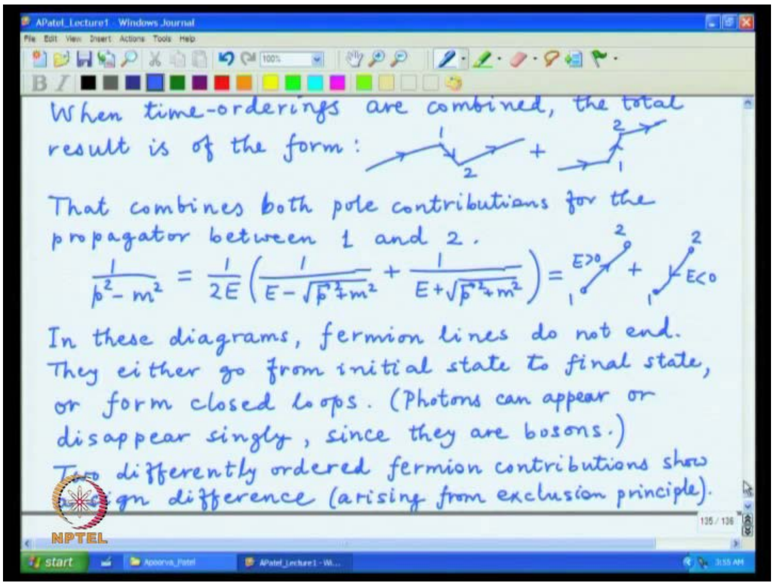

And, then adds them together in this particular case, these 2 ordering combine both pole contributions for the propagator between 1 and 2. In practice, this sort of happens automatically; because of the algebra which we have constructed with the prescription of I epsilon. When 1 has 1 over p square minus m square regulated in terms of 2 different poles. It can be written as, positive energy pole plus a negative energy pole. And, as long as 1 uses this total combination, this object is equivalent to the sum of 2 contributions. Where, the first is a positive energy poles, it corresponds to a particle going from 1 to 2 evolving forward in time. And, the other 1 is the same thing.

But its negative energy part equivalently, the antiparticle corresponding to evolution backward in time. When 1 uses this total Lorentz invariant combination, p square minus m square; both these things are automatically summed. And, 1 does not have to worry about these 2 different diagrammatic orders. When 1 uses this covariant expressions that is useful and, keeping track of, what all things are calculated? And, what have to be combined when doing detailed calculations with, what other contributions? It is generally, always useful to combine various contributions to get quantity which I Lorentz invariant. We will see that this thing will generalise to other possibilities, where the interactions with photons will come extra symmetric or gauge symmetric. And, in those cases again it is very useful to combine all contributions which are gauge invariant together.

Those calculations then provide important of symmetry. And, whether the calculations has any inconsistency or error inside them or not? This is all the machinery I just want to add 1 more comment in this diagrams fermions lines do not end. They always need to be continuous. So, they either go from the initial state to final state or from closed loops. This is an example of the super selection rule which I mentioned while describing Lorentz symmetry that the bosons and fermions form different sectors. The vacuum belongs to bosonic sector. And, the single fermions belong with the fermionic sector. There is no way one can superimpose one with the other.

So, if the fermion has to be there it will go all the way from the initial state to final state remaining in the same fermion sector or otherwise the closed lops will involve particle antiparticle creation and annihilation; where one can mix fermion with boson sector. But only in pairs that is the topological feature which is buried in this diagram. On the other hand photons which are represented by a wavy line can appear or disappear out of basically nothing. That is the allowed because they are bosons. So, in the same sector as the vacuum this distinction is also a buried inside this particle diagram.

The feature that is useful thing to keep in mind is that 2 differently order fermion contributions show sign difference which is in some sense connected to the poly exclusion principle. It is part of this particular algebra, because we are viewing a single fermion going both backward and forward in time. If you are look at fermion line which intersects a particular time slice at 2 points. One can see that one of them is the positive energy electron and another is a negative energy electron.

Since thus the same fermion line, it should obey the principle of Fermi statistics. In particularly there interchange, there will be negative sign. One can also look at the diagrammatic representation where 2 points are interchanged in that particular case corresponding to 2 different ordering; the algebra we have constructed is such that the negative sign already is built inside; the various definitions of propagators between positive energy solutions and the negative energy solution. But this will become very explicit in detailed calculations later.