Relativistic Quantum Mechanics

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Lecture - 27
Relativistic case, Particle and antiparticle Contributions, Feynman Prescription and the Propagator

Last time I introduced the concept of two point ((Refer Time: 00:20)) functions, also called the propagator which is useful in solving linear partial differential equations in presence of a source.

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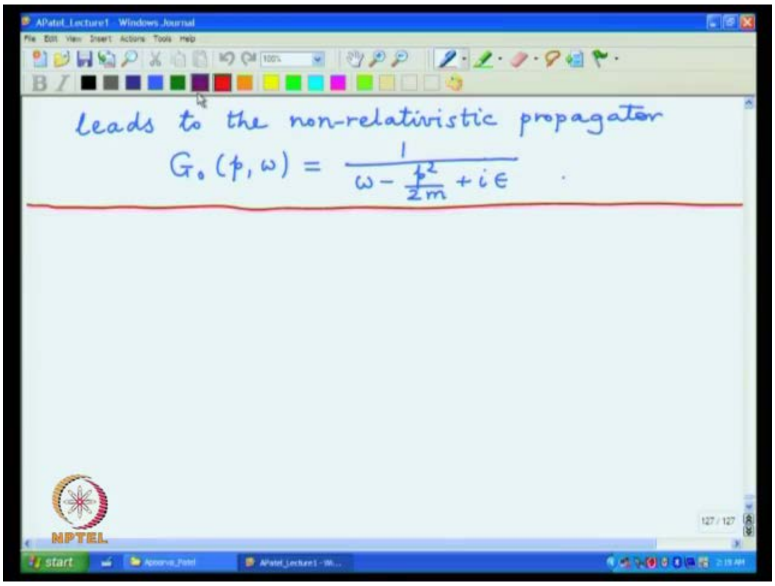

In using this for filled theories it is necessary that this propagator be calculated with specific boundary conditions. And the most important part of the boundary condition was the principle of causality. In the last lecture, I described how this principle of causality is incorporated in the calculation of the non relativistic propagator by the so called i epsilon prescription, which allows the influence of a source to propagate forward in time, but not backward in time. Now, we have to extend these ideas to the relativistic case.

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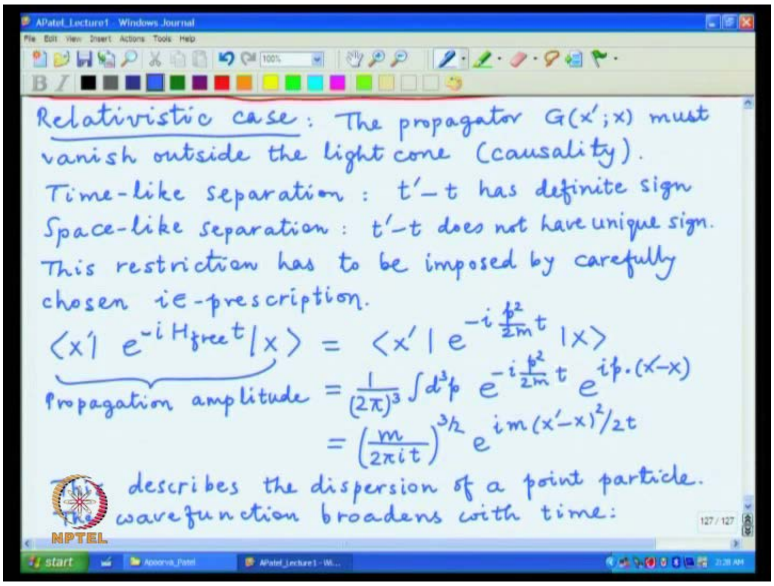

The formal definitions do not change they are the same whether the problem is relativistic or non relativistic or any other linear partial differential equations. But the principle of causality needs to be expressed little more carefully. And, so the i epsilon prescription also becomes somewhat suttled; I want to illustrate that first by a simple example. And, then we will develop the formal construction of the propagator; the causality imposes the constraint that influence can go only forward in time. But also it cannot propagate faster than the speed of light; that second part is the restriction coming from relativity.

So, the influence is not only restricted to t prime greater than t. But also restricted only in the forward light cone of the point at which the source is; it can be phrased in different ways one way is to say that the propagator G x prime to x must vanish outside the light cone. And, this becomes the principle of causality for the simple reason that for time like intervals this t prime minus t has definite sign in all inertial frames; but for space like interval t prime minus t does not have unique sign. So, one can choose Lorentz transformation for events which are space like separated to construct situation in which one event is in the future of second in one frame. But in the other frame that event becomes in the past of the second event; in such cases there is no way we can implement causality in any practical sense.

And, the only sensible construction is that in such space like regions we do not have any propagation of influence of the source at all. And, that becomes the criteria that the influence of a source is restricted inside the light cone. The light cone is the boundary separating time like and space like regions; this is a restriction that has to be imposed by carefully chosen i epsilon prescription; in the non relativistic case we saw that we just had to add a imaginary part to the energy. And, the sign of t prime minus t was always an absolute feature irrespective of the frame. And, so if it is causolin one configuration it will be causolin in any other frame as well. But here that separation of time now has to be replaced by separation along the light cone.

So, let us see the implications of this in a very simple minded example. And, that is just propagate a free particle from the position x prime to x; this is the propagation amplitude in the coordinate representation. And, one can easily evaluate it using Fourier transforms; because for the free particle the exponent has a quadratic form. And, one can do the Gaussian integrals which appear in a straight forward manner in momentum space; thus state x prime and x can be just written as plane wave factors. So, it is now the integral of e raise to i p dot x prime minus x; this is a simple integral to do the Fourier phases or just representation of the states, initial state x and final state x prime.

So, simple integral can be done by completion of the square in the exponent; that Gaussian integral gives a result which is the well known function exponential of x prime minus x whole square divided by 2 t; this describes the dispersion of a point particle where the initial state at x is a delta function. And, after sometime that delta function will spread out to be a Gaussian with increasing time the width of the Gaussian increases; and the wave function becomes broader and broader. But the important part is that; for any finite value of time the influence spreads all the way to infinity. It might be small in the sense that if you take x prime minus x to be very large one can argue that the wave function will become a Gaussian with very small tail at infinity. But the fact is that the tail is non zero.

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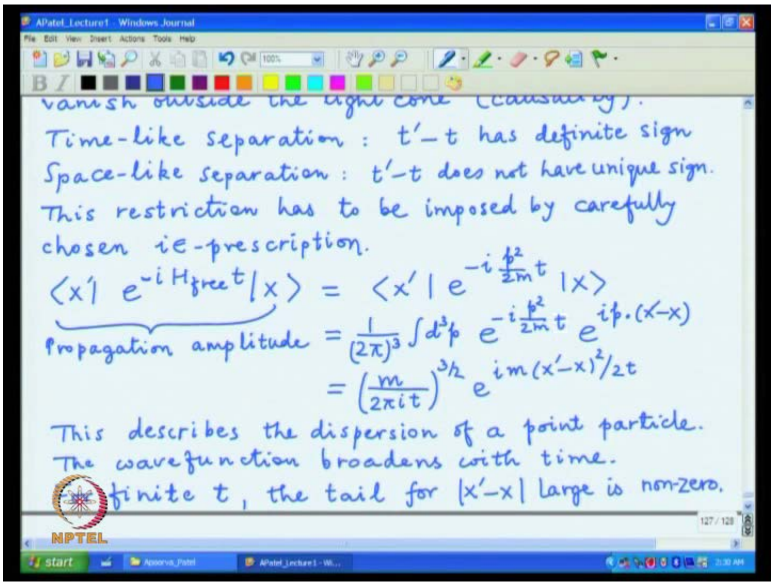

And, this is in conflict with the finite propagation speed of light; we were not expecting the relativistic feature to emerge out of this non relativistic description. But it is very clear that in the non relativistic case you can have action at a distance no matter; how large the distance is there will be essentially instantaneous possibility of interaction. Now, we can ask what are the things that need to be changed once you go from the non relativistic description where the energy was just p square by 2 m to the relativistic case where the dispersion relation is different that.

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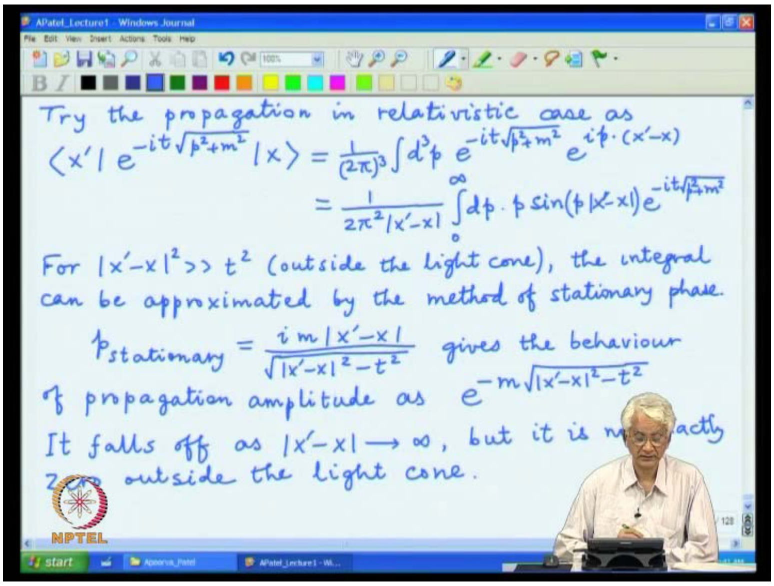

Now, can be attempted as a similar exercise; but now we will put in instead of p square by 2 m the relativistic value for the free Hamiltonian. And, see what comes out? Again the integral can be done by doing Fourier transform because the square root things are little more complicated. But the angular part of the integral is still straight forward because the energy does not depend on the angle between p and x prime minus x; that part can be done exactly those integrals were d phi and d cos theta. And, that is a simple integral which only involves the Fourier phase factor. And, the result looks like the radial integral which is left over the angular integral produces the sine function involving the product of p and x prime minus x and the energy part is unaffected; this is one step.

The next step is to do the radial integral that can also be done exactly in terms of Bessel functions. But we do not need to solve it in full generality to see the problem which crops up; you just want to get an approximate answer for this object outside the light cone where we have to obey the principle of causality. And, in that region the solution can be obtained rather easily the integral can be approximated by the method of stationary phase; in particular this sign is an oscillatory function. And, so is this exponential they both are going to produce rapidly fluctuating phases; what happens is that the main contribution to the integral comes from the stationary point; where the phase basically stops varying. And, everywhere else where the phase is varying fast the integral essentially cancels out.

You just have to find a stationary point of this integral and evaluate that particular part in the neighborhood which is again a Gaussian integral; the dominant exponent is just the value at the stationary point. And, then the next correction comes from doing the Gaussian integral around that particular stationary point. The stationary point is rather easy to evaluate by just writing e raise to i t x prime minus x in place of this sine; and making the derivative of the phase is equal to 0. And, that answer is of this particular form I have taken note of the fact that x prime minus x square is larger than t square. So, the square root is well defined.

Now, one can plug back this expression inside the integral and then the result is p is imaginary. Then the exponents become real then the product of these 2 things produces real value; that essentially is the leading behavior of this result one can evaluate it in a straight forward manner. And, the result is e raise to minus m multiplied by square root of x prime minus x square minus t square; one can see here that this integral again falls of rapidly when x prime minus x becomes much larger than t. But the point is it is not exactly 0 outside the light cone that is the troublesome part that it does fall of. But what we needed was the more stringent constraint that it has to exactly become 0 for all the points outside the light cone.

We need this exact 0 because otherwise it will imply that some small fraction of the interaction can go faster than speed of the light if not all the interaction. But that violates relativity at a basic level; that in relativity we need interactions completely restricted within the light cone or the speed of light is an upper bound not just an approximate bound. But a exact bound nothing, no matter how small can avoid that particular bound. This is a conundrum that we just tried a simplest replacement of the energy in terms of the relativistic dispersion. But it does not produce the result that we like; here we have to now invent a scheme where this small amplitude which is leaking out must be cancelled exactly by something else which is coming from the same kind of prescription; the answer we know.

And, we have seen some of this features earlier also is that we need to introduce the antiparticle degrees of freedom into the description; when you take a square root of the energy there are 2 signs it is not just positive square root of t square plus m square. But both plus t square plus m square as well as minus p square plus m square under square root sign are valid solutions of the dispersion relations; the true answer emerges if we arrange these 2 solutions in a clever way. So, that there is exact cancellation of the influence of the source outside the light cone. And that brings relativity in accord with the quantum description.

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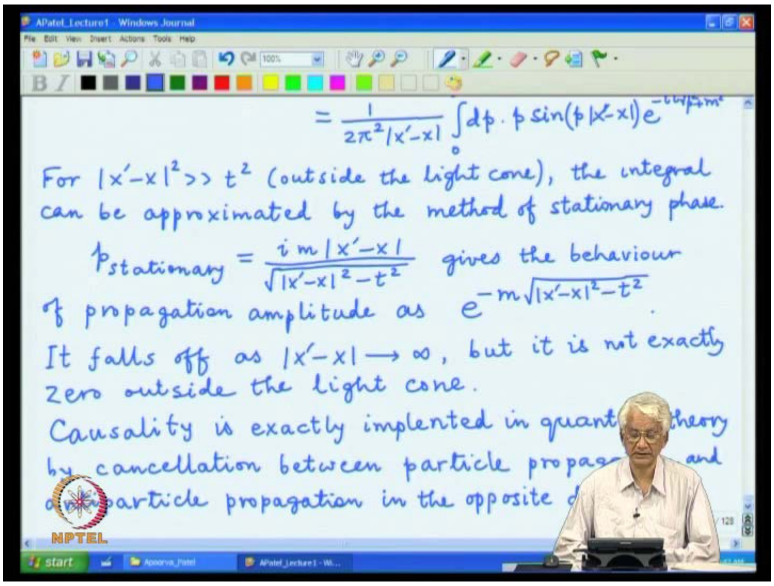

Causality is exactly implemented in quantum theory by cancellation between particle propagation and antiparticle propagation. We have seen the description of anti particle propagation while dealing with hole theory and discrete symmetries of p c and t; that antiparticle actually propagates in opposite direction to the particle in the space time description.

And, this is the resolution of the problem there are 2 contributions but they can sell out that cancellation is exact outside the light cone. That means, that once you go to a relativistic description there is no way to avoid particle and anti particle appearing together; they are always there both the contributions are needed to be consistent with the principles which we need to impose for a sensible physical theory. If you just take particle modes or just the antiparticle modes they will not respect the physical principles which we want in the theory both must appear together.

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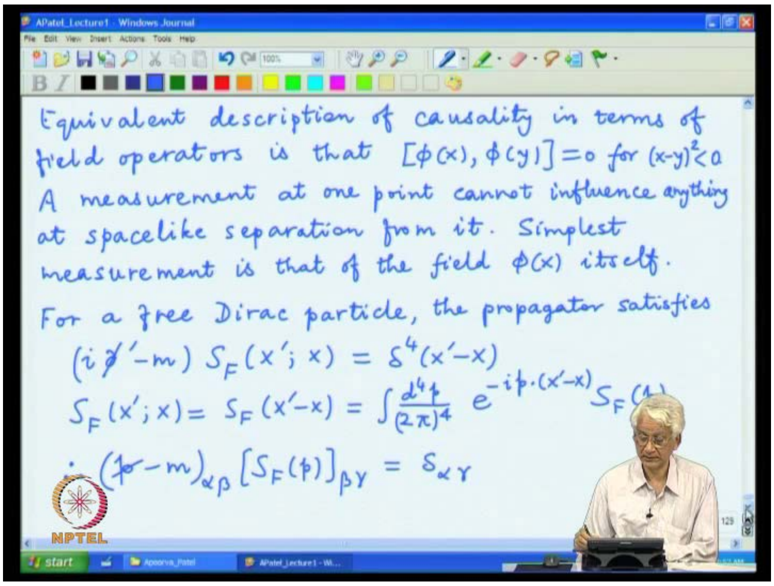

There are alternative ways to express the same idea I have described in terms of the green’s function. But sometimes it is also expressed in terms of the wave function or the field itself. The equivalent description of causality in terms of field operators is that the commutator of the 2 field operators is 0; for space like separation which are labeled by x minus y square being less than 0. This does the same job what the green’s function description tells you is that one can have propagation of a field from one point to another. And, if you measure at the second point you will get some influence coming from the first point if there is indeed a connection by the propagator if the object is outside the light cone you will get the 0 result.

So, propagator can be described in terms of the way they influence of a source give rise to certain measurements. A measurement at one point cannot influence anything at space like separation from it that is true for any operator. And, one can just say that phi is a field which is one of the operators which one can measure.

So, one can just say that the simplest measurement is that of the field phi x itself; then the commutator of the 2 operators is equivalent to saying that in one case I will measure phi by first. And, then measure phi of x later and in the other case measure phi x first. And, measure phi y later that gives a joint result for the 2 measurements; when the separation is space like one of the measurement cannot influence the other. So, the result should be the same and so the commutator vanishes.

This is in terms of the simplest operator which are the field itself. But it will be true for any arbitrary operators constructed from a basic fields; the causality will always demand that; many times this is also called the principle of causality written in a different language. But the constraint of the light cone remains the same; that in one case you are saying that the propagator cannot go outside the light cone. The other says you are saying that the commutator must vanish outside the light cone this is basically what we need.

Now, we can go to the formal description and see how this is actually realized in the case of a Dirac particle. For a free Dirac particle we have the propagator which satisfies this equation; the operator acting on the 2 point green's function I was denoting it by G earlier in the case of Dirac particle it has the standard label S with a subscript F for Feynman propagator. And, we will see why the Feynman propagator label is necessary to specify the complete description. But this has become a standard notation and the green's function equation is that this object is nothing but the delta function.

One can now solve this easily first the formal solution then we will worry about the causality. The solution is straight forwardly achieved by Fourier transforms; first point is that the system has translational invariants. So, the propagator only depend on the relative separation of the 2 point. And, we will just write the propagator in momentum space as the Fourier factors and S F of p.

Now, we can Fourier transfer the equation the Fourier transformer delta function is just 1; it is a constant. And, so the equation just becomes the momentum space description of the Dirac operator acting on the propagator is equal to 1. So, the propagator is just a reciprocal of the operator in front; you can easily write that as p slash minus m one can explicitly put some indices for the matrices if needed. And, then a simple result that this object is equal to 1 in this index space.

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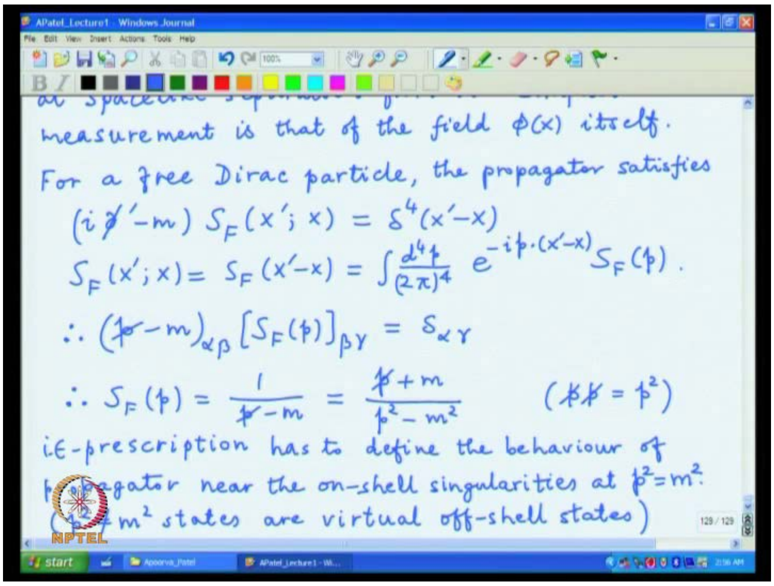

It follows that this propagator is then a reciprocal of p slash minus m; where the reciprocal has to be interpreted as the inverse of the matrix, many times this description is rationalized. So, that the denominator is not a matrix, but just a number. And, that is done by multiplying both numerator and denominator by p slash plus m. And, then the denominator become just p square minus m square. Because p slash p slash is equal to p square which follows from Clifford algebra; that product of gamma matrices in asymmetric form will just give the identity.

This is a straight forward solution of the free Dirac particle propagator. But now we have to worry about how to stick in the i epsilon description in this language to enforce the principle of causality. And, then we have to worry about 2 locations where the propagator is singular those are the zeros of the denominator. So, the so called i epsilon prescription has to define the behavior of propagator near what are called the on shell singularities at p square is equal to m square; in principle there are 2 singular points.

And, the i epsilon prescription has to handle both of them; the on shell is a terminology which is often used in dealing with quantum field theory. It describes the point where the energy momentum dispersion relation is satisfied exactly. And, those are the descriptions of the asymptotic states or exact solutions of the equation; there are the so called neighborhood of those points which also contribute to quantum behavior. Because there are fluctuations allowed in energy and momenta; quantum theory you can call them the uncertainty relations of short leaved states.

So, all the states do not have to satisfy the dispersion relation exactly. And, those states are called off shell states; they go by various different name sometimes they are also labeled as virtual states they are a necessary part of the quantum theory. But for this states actually there is no problem at all as far as a propagator is concerned; because the denominator is not going to be 0. And, then there is no singularity at all only when there are on shell singularities; we need to define this i epsilon prescription for the asymptotic behavior of the states; which should not carry any influence outside the light cone.

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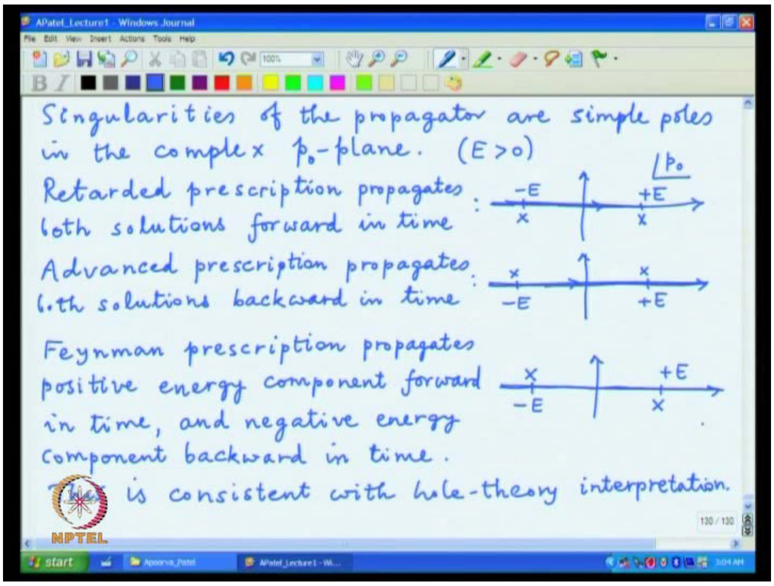

Now, let us look at the various possibilities; we have seen that this singularities in the dispersion relation happen to be simple pole in the complex p 0 plane. We saw the description of how that was implemented by putting a step function in time in the case of non relativistic problem. Then re expressing the step function as a Fourier integral and then moving that position of the pole slightly away from the real axis.

So, that you could close the count tour appropriately for one value of the time versus the other; that give rise to the causality prescription. And, we had to basically relate the time integral and a energy integral we basically have the same concept. But now instead of having 1 pole there are 2 and this 2 poles in the case of the free particle or at energy is equal to plus or minus square root of p square plus m square.

We now can give 3 prescriptions which have been defined in the literature; one of them is called the retarded prescription it propagates both solutions forward in time. As you have seen in closing of the counter in the case of the non relativistic problem; this is complex p 0 plane; there is one singularity at plus E; and matching one at minus E; where E is always assumed to be positive by definition; what the retarded prescription deed was move the poles slightly below the axis in both the situations.

So, the counter integral essentially went along the real line. And, in the forward propagation the loop closes in the negative plane; in a backward propagation it goes in the upper plane. And, so in one case both the poles are included in the other one they are not. And, so one has the description of the retarded type; that whole the thing goes only forward in time; there is a complementary result which is called advanced propagation.

And, that in the same language corresponds to shifting the pole; but now putting them little bit above the axis. So, when one does a counter integral the poles are enclosed; when one goes backward in time and closes the loop in the upper half plane. But not the other way around; the correct prescription which we need in field theory comes about in either of this 2 situation. And, that prescription goes under the name of Feynman prescription; who emphasized the description in full detail it corresponds to the feature we have seen in description with the hole theory, and interpretation of antiparticle that the positive energy solution goes forward in time.

And, negative energy component goes backward in time; in terms of the same diagram and position of the poles it corresponds to the positive energy side pole shifted to negative imaginary component. But the other one shifted to positive imaginary component; when one does the integral depending on the sign of going forward in backward in time one pole or the other one is enclosed the forward one is the particle pole and the backward one is the antiparticle pole. This is consistent with hole theory interpretation; to get this feature right we need to shift the 2 solution just in the right fashion.

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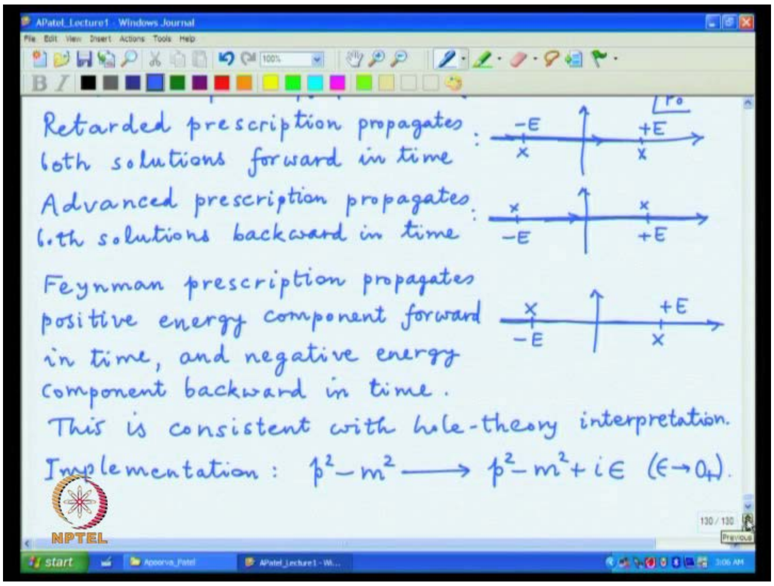

That is done by looking at the square roots. And, the sign that the 2 signs of the square root should be shifted in the opposite sense; the correction actually should go inside the plus minus sign. And, that is easily achieved by changing this p square minus m square plus i epsilon. So, the p 0 which is going to give the energy will be 3 momentum square plus m square minus i epsilon the whole thing under square root; that will give this shifts when the epsilon is going to 0.

But from the positive side it will shift this pole as indicated and implement the cosol part of the connection as we want. So, this then becomes the result of our analysis of how the i epsilon prescription should be inserted into the formal definition of the propagator. And, now one can write a fully mathematically consistent as well as physically well behaved propagator.

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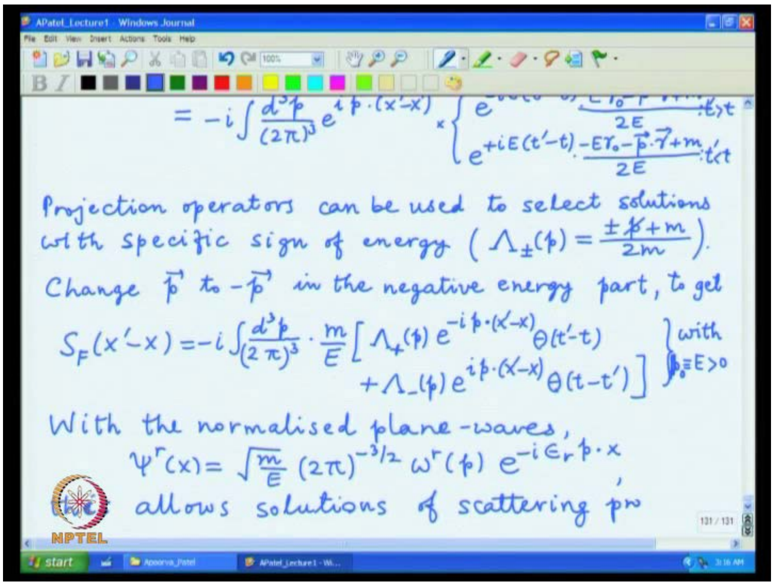

As the result integral in Fourier space momentum integral remains the standard Fourier form. But the time integral or the energy integral is now defined little more specifically with this addition of i epsilon in the denominator. This is a physically correct prescription one can do the time integral explicitly by choosing the closure of the counter in the 2 specific cases t prime greater than t or t prime less than t. And, the result I can easily write down; remember E is always defined as the positive square root. And, so there are 2 possibilities for the roots depending on t prime is greater than t or less than t.

And, then it is just a residue theorem description the numerator can be written explicitly at p slash plus m. But the denominator is now the residue at the particular pole which just happens to be 2 E. And, this is the description for t prime greater than t similar thing appears in the other case but now the p 0 value is negative. So, the factor looks almost the same but not exactly the sign is opposite. But then the sense of closing the counter is also opposite and they basically cancel each other.

And, this is the answer for t prime less than t; one now has a full structure of the propagator evaluated explicitly for the different signs of time. And, that can be rewritten in a joint form by using projection operators which will select one sign of the time versus another or equivalently one sign of the energy versus the other. Projection operator can be used to select solutions with specific sign of energy. And, in the conventional basis which we have used this projection operators were denoted lambda plus or minus p; one was the projection on the upper 2 components or the particle move at the other one was the lower 2 components are the antiparticle moved. And, there definitions were plus or minus p slash plus m divided by 2 m; this can select the specific sign of energy which is related with the particular ordering of time. So, the positive energy goes forward in time, negative one goes backward in the time.

And, the 2 now can be written as a single form; to do that there will be little further trick to get the corresponding factors to be of the same structure; the integration variable of the space component of momentum in the negative energy part which is the second half of the equation. And, then we have a form of a propagator little more compact; which is given by this momentum integral with a weight m by E; the m by E actually comes from converting the projection operators to denominator E. Then one has this lambda plus combined with e raise to minus i p dotted with x prime minus x this one goes. Obviously, forward in time the second half has the same factors but now the signs are all flipped.

So, the Fourier phases are opposite and this one goes backward in time. From this description one can easily see the interpretation which we had used in the hole theory language also emerging straight forwardly there are projection operators for particles; and holes that this Fourier phases which are identical step for the overall sign flip. And, which is one of them is going to go in one direction in space time.

And, the other one will go exactly opposite in the space time direction at the same time. So, this is a general description well one thing I should say here instead of using the E; I have put back the notation of p 0 again. But p 0 has to be interpreted here identical to E. And, it is always positive in this particular formula compared to the other formula; where p 0 was just acting as a dummy variable; this is a general form which is often expressed.

And, then one can obtain all the results from this propagator. In particular one can choose the plane wave solutions which will be used in scattering formulation; those plane waves again in the same formulation we have defined as various factors the basis function and phase factors. This allows scattering problems to be solved we have the initial wave, we have the propagator. And, now we have to add the extra potential or interaction to this propagator; to see what will happen in future I will do that next time.