Relativistic Quantum Mechanics
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Lecture - 25
Lorentz Group Classification of Dirac Operators, Orthogonality and Completeness of Dirac Spinors, Projection operators

We constructed the transformation matrix for the Dirac spinor by demanding that the equation remains covariant when a Lorentz transformation is applied. And that lead us to this explicit form for the matrix S and the corresponding generators sigma mu nu. Now, we are in position to construct various operators and more complicated quantities, which will have specific transformation rules under Lorentz transformations. And so we can now just work out some simple examples one by one.

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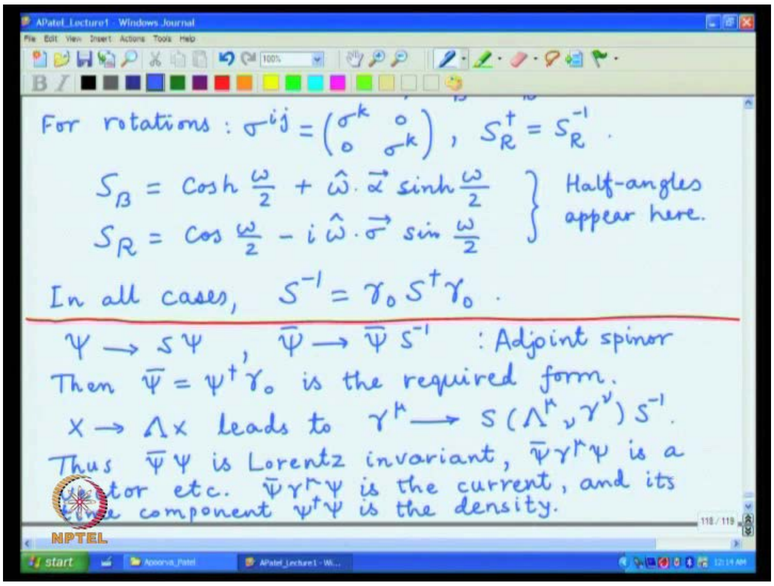

The first thing is to construct the adjoint spinor; just like psi goes to S times psi, we want to find a quantity, which transforms as psi bar goes to psi bar S inverse. And this defines the so-called adjoint spinor, and one can construct invariance by combining psi bar and psi, so that the factors S cancel out. Now, the Hermitian conjugate of psi actually does not obey this property, because the matrix as is not always unitary. But, we saw how to find an inverse of S by taking the Hermitian conjugate of S, but adding extra factors of gamma 0. And that allows us to see the solution of this particular result that then we can choose psi bar is equal to psi dagger times gamma 0. When one takes Hermitian conjugate equation of psi, one get S dagger. But, then the extra factor of gamma 0 converts the S dagger to S inverse. Various constructions involving Dirac Spinors heavily use this particular form that, everything is written in terms of psi and psi bar; psi dagger actually is very rarely used. So, this is one particular structure.

What about the gamma matrix itself? We saw the relation for the gamma matrix, which involved both the transformation rules for the space coordinates as well as the transformation rules for the spinor coordinates. And that is because the gamma matrices of both the vector space-time index as well the spinor internal space index. And so when x goes to lambda x, that leads to a double transformation rule for the gamma matrix, which we already seen before. Let me show it again, which is lambda gamma is equal to S inverse gamma S.

So, we can rewrite this equation in an equivalent form that, gamma mu will go to S times lambda gamma times S inverse. The lambda gamma basically performs the transformation on the vector index. And S and S inverse on either side performs the transformation on the two spinor indices that each gamma matrices has. So, now, this all machinery for psi as well as gamma allows us to create lots of different invariant quantities. Some simple examples are that, if I construct psi bar psi is Lorentz invariant. One can combine some of these gamma matrices together with it.

Let us look at the simplest case, which is psi bar gamma mu psi. In this combination, under the Lorentz transformation, the factors of S completely cancel out; only the lambda remains and the lambda just transforms space-time coordinates. So, this is a vector and one can go further. In this kind of construction, the particular structure psi bar gamma mu psi is important, because it is also the expression for the current. So, psi bar gamma mu psi is the current; and its time component, which happens to be psi dagger psi, is the density. We have seen both these structures before psi dagger psi as the density and psi bar gamma mu psi as the special component of the current. And they now naturally appear in this form and we explicitly see that, the Lorentz transformation of properties indeed that of 4-vector, which is necessary for describing a current.

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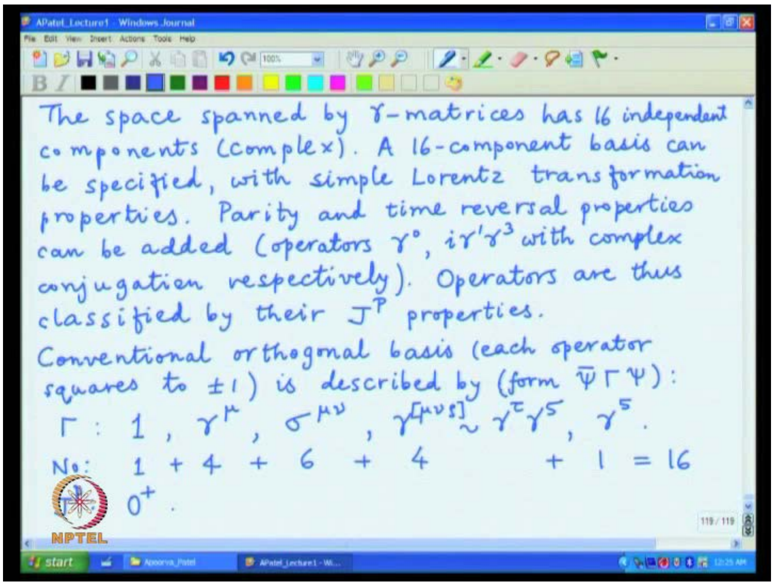

One can now go further and define all possible structures, which can be constructed by products of spinors and various kind of gamma matrices. This internal space spanned by gamma matrices has sixteen independent components. These are just the complex entries of the 4 by 4 matrices. And one can now construct a basis, which is useful in dealing with Lorentz symmetry. As far as just the matrix is concerned, one can just look at each entry as the independent orthogonal component. But, the Lorentz symmetry requires specific transformation rules. And it turns out that, little bit of rearrangement helps. 16-component basis can be specified with simple Lorentz transformation properties. We have so far only dealt with the matrix as per continuous Lorentz transformation. To this, we can add the usual discrete symmetry parity and time reversal.

In the case of Dirac equation, we have already seen what these discrete operations of parity and time reversal are and those operators, where gamma 0 and i gamma 1 gamma 3 with complex conjugation respectively. Operator for parity was gamma 0 and operator for time reversal was i gamma 1 gamma 3, and then the complex conjugation of the spinor. This time reversal operator was specific to the Dirac basis, while the gamma 0 is independent of the choice of the basis. We will use basically the Lorentz transformation together with parity to specify generic properties. So, that allows classifications of operators by their J P properties, where J comes from the continuous part of the Lorentz group and specifies the angular momentum or equivalently the spin value for the operator. And P is the discrete part. It is the value coming from what happens to the operator under parity. So, now, let us construct the orthogonal basis.

This is a conventional orthogonal basis and it has a normalization that, each operator squares to plus or minus 1. The minus 1 is necessary, because some of the operators are anti-Hermitian. And then you have to have either a factor of i or let it be minus 1. That cannot be avoided, because the Clifford algebra itself says that, the anticommutator of the gamma matrix produces the Minkowski metrics. And Minkowski metric has both plus or minus 1 sitting inside them. These are the 16 operators; I am just going to write down. First is identity; the second are the 4 gamma matrices themselves; the third one is objects described by 2 gamma matrices. And the symmetric part just leads back to the Minkowski metric. So, we have to look at the antisymmetric part. And that antisymmetric part we have already selected as the generators of the Lorentz transformation. And I might as well choose these objects to be sigma mu nu.

Then, comes the stage; one can have three Lorentz indices; but using again the Clifford algebra, the three are giving a nontrivial contribution only when they are completely antisymmetrized. Antisymmetric component reduces to Minkowski metric. So, we have three antisymmetrized products. And that can be written as say mu nu rho – anti symmetrized. But, one can also now rewrite it as the fourth component multiplied by gamma 5, because gamma 5 has all the 4 components antisymmetrized inside it. And the last one will be the product of 4 gamma matrices completely antisymmetrized and that might as well be written as gamma 5. The list ends here in a sense, because if you count the number of operators, there is 1; there are 4 here; there are 6 here; there are another 4 here; and there is 1 here and there add up to the 16 components.

Beyond this, any product of more than 4 gamma matrices can always be reduced to this form either using Clifford algebra or using the definition of gamma 5. If you have more indices, you can always either symmetrize it or antisymmetrize it. And then it all boils down to these 16 components. And since we have chosen them as a specific constructions, there are all orthogonal to each other. And orthogonality can be specified in many different ways. One way is to take the product of any two components. And these are matrices and take a trace. And if two components are not equal, the trace is 0. That again follows rather trivially, because the matrices obey Clifford algebra as well as the matrices are trace less. So, if you pick two operators from a different set, you will end up with a traceless matrix; hence, that implies orthogonality.

Now, we can work out the value for the angular momentum together with parity for this particular case. This is a list of matrices gamma. And this whole object has the form psi bar gamma psi. So, the gamma matrix is inserted between psi bar and psi. And we saw that, if you pick identity, psi bar psi is Lorentz invariant. And so the corresponding value of J P in this particular case happens to be 0. The parity is plus, because identity matrix does not transform under parity operator, which is gamma 0.

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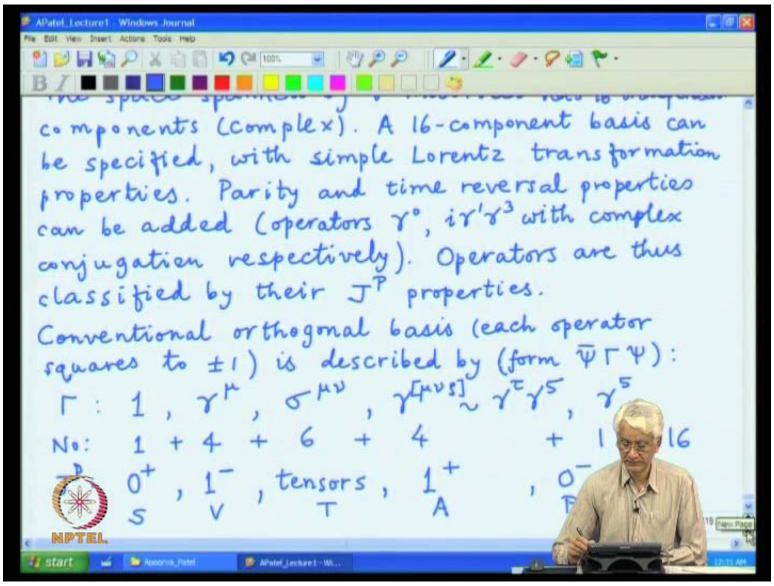

What about the next one? This is the vector. The single Lorentz index indicates that, its transformation property is like a spin 1 object. One can go to the rest frame; and in that frame gamma i defines the same object as the three components of the space coordinates or equivalently the momentum coordinates. And its property under parity is given by P gamma mu P inverse, where P is essentially gamma 0. The space components anticommute with gamma 0. And so this object transforms as 1 minus, which is the property of a genuine vector under transformation of rotation and parity. What about sigma mu nu? It has two indices.

And so it is going to transform as a 2-index Lorentz object. And one typically calls these things as tensors. And you can construct objects of various spins and parity from these constructions by projecting on to various different components. I will not write them down explicitly. But, it is an asymmetric tensor similar to the electromagnetic tensor f mu nu. So, one can in principle break it up into the electric part and the magnetic part with specific transformation property under rotation as well as parity.

The next object gamma tau gamma 5 has the same property as far as the spin is concerned, because gamma 5 is a scalar under Lorentz transformations. The gamma 5 basically commutes with the matrix S for the simple reason that, the S involves the generator sigma mu nu, which has two gamma matrices; gamma 5 anti commutes with each of them. So, gamma 5 ends up commuting with sigma mu nu. And so it left time invariant under S. So, it is a scalar. So, the spin part of this gamma tau gamma 5 is still 1 just like as in case of vector. But, gamma 5 does change the parity of the operator, because under gamma 0, gamma 0 gamma 5 gamma 0 inverse produces a negative sign times gamma 5. So, it is an object with opposite parity compared to the vector. And it is sometimes called as a pseudo vector, sometimes it is called axial vector.

And, gamma 5 – we already dealt with it; it does not transform under S, but its parity is negative. So, it behaves like a pseudo scalar. And these abbreviations are often used. So, it is S, V, T, A and P. This is now a very useful organization of the 16 independent components specifying the internal space spanned by gamma matrices. And we have broken up those components according to the simple properties of Lorentz transformations. These are now helpful in dealing with many detail calculations of interactions of electrons with photons and many other kind of fields, where spinors appear interacting with other fields, which are consistent with Lorentz symmetry.

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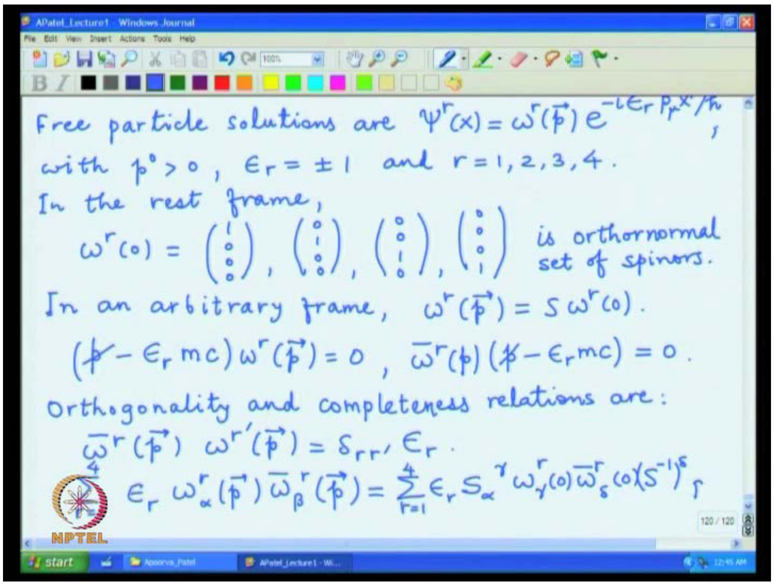

We have a complete list of operators and basis. Let us now specify the spinors themselves explicitly by going to the little group. And that is convenient to give a complete description of the states in any basis or any frame. General form of the so-called spinor wave functions; we have already seen that, the generic solutions are kind of plane wave. And what we want to do is reorganize these plane waves in a form consistent with Lorentz symmetry. So, we will first specify the wave functions in the rest frame of the particle in which case the momentum is basically zeroth component, is a mass; and special component all vanish. And then you can go to a general state by a boost operation to whatever momentum you desire. Or, in the rest frame, the spin coordinate can also be specified by various directions and you can apply the rotation operator to go to a particular spin direction as well.

Free particle solutions will denote them as psi with a superscript r and r will take 4 values corresponding to the 4 spinor components. And this will be organized in terms of the spinor description and an overall phase, which describes the plane wave structure. It is p mu x mu by h cross. And we have explicitly put a label epsilon r, which is actually the sign of the energy, because it is conventional to describe all the solutions with the component p 0 always positive. And so whenever the energy is negative, we will take into account that fact by putting epsilon r equal to minus 1. This is a choice for certain conveniences; it is not always necessary. But, it has become common place. So, we will just use this. These are the four so-called plane wave structure solutions, where omega r p is now completely in the spinor space; and we want to find a convenient basis to describe these 4 components.

In the rest frame, you can just take the 4 values rather in a simple-minded fashion as just the 4 components: 1, 0, 0, 0; second one will be 0, 1, 0, 0; third one will be 0, 0, 1, 0; and fourth one will be 0, 0, 0, 1. And this basis is orthornormal. So, it is quite convenient; just 1's and 0's; nothing more complicated. And then in an arbitrary frame, we can obtain omega r with some p. Because of the conventions, we have chosen; and the description of the wave P 0 is always positive square root of p square plus m square. So, it does not matter whether you specify only the 3-momentum or a 4-momentum; they are equivalent. And all that happens is one has to apply the boost or rotation operation, which we discussed just before; which is the matrix demoted by S. So, this gives a complete specification.

First, the rest frame is the frame of the little group in which we specified a complete basis. And then we can go to an arbitrary frame just as we discussed for generic one particle states in the classification of Lorentz group solution. In this particular case, it is just the operator S, which we need do that. This can be now constructed case by case, because we exactly know what the operator S is, we know what the generator sigma mu nu are; and the parameters omega mu nu will describe what is the rapidity for the boost or what is the angle for the rotation. The equations satisfied by these objects can also be written and they correspond to p slash minus epsilon r m c acting on omega r p equal to 0. So, this is the Dirac equation after removing all the space-time derivatives in favor of the eigenvalues p mu. That is the effect of this phase factor in the solution.

And, the epsilon r, which comes together with p mu – I have just multiplied it through and put it in front of the mass term. So, this is the Dirac equation. But, now we do not worry about the space-time; p's are basically just some numbers; they are eigenvalues corresponding to the plane wave. The corresponding adjoint equation can now be easily obtained. But, now, we will use this notation omega bar; and that is useful for taking the adjoint, because gamma 0 is Hermitian, but gamma i are anti-Hermitian. And that extra sign is canceled by the gamma 0 buried inside this omega bar. So, the equation does take essentially similar looking form as the original equation; just the multiplication order is opposite. The equation for omega dagger would not have such a simple form. And that is why it is much more convenient to deal with these objects omega bar. This can be also directly solved to obtain omega and omega bar if desired, but it is generally more convenient to start with a rest frame basis and apply the necessary boost or rotation transformation.

The point of doing all these exercises is that, one can now construct the orthogonal relations not in just the rest frame, where it is rather trivial, but in an arbitrary frame, which specifies basically just the fact that, you have a complete basis. That relations are orthogonality and completeness relations that, omega r bar with some momentum p and omega r prime with the same momentum p, is delta r r prime; which is fine, because omega transforms by S; the omega bar transform with S inverse and they cancel. And so the product will be the same as what it will be in the rest frame. But, in the rest frame, the product involves an extra gamma 0, which is buried in the definition of omega bar. And so you will get plus 1 or minus 1 depending on whether you are using upper component or the lower component. And that sign is precisely what is specified by epsilon r. And so this is an orthogonality relation of these general Lorentz spinors.

And, one can now write down a completeness relation as well, which sums all the 4 components and guarantees that you will get back the original objects. That can be now written as the same objects, but multiplied in an opposite order. And I will again include the factor of epsilon r, which is necessary to get rid of the extra contribution coming from gamma 0 part of omega bar. So, this relation is now given in terms of these indices not contracted. So, there is an overall matrix leftover. And the completeness says that, the matrix should be the identity. We can now construct this product and evaluate it explicitly. It is the contraction of S with some indices alpha gamma omega gamma with an index r of 0. Then omega bar has an opposite transformation rule S; now, it is inverse together with the indices delta and beta.

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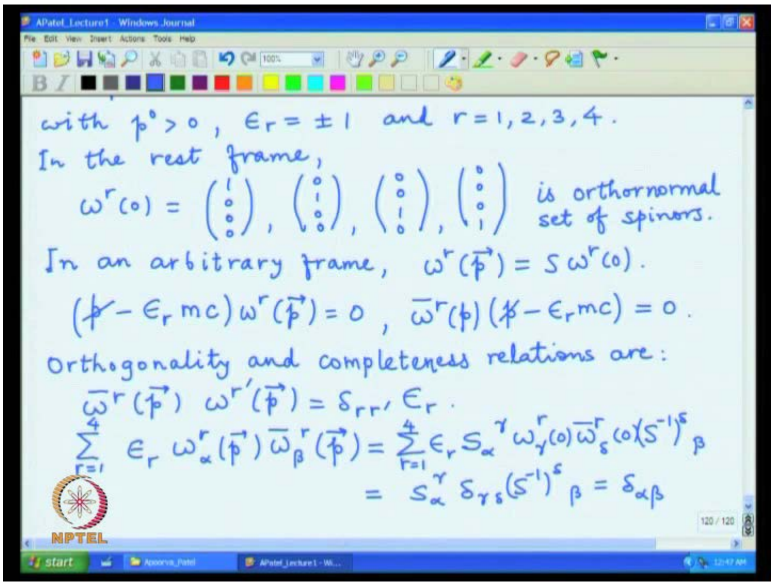

And, we have all the necessary machinery to evaluate this thing in a rather simple-minded form. The completeness for the zeroth part already holds in a very simple fashion; it is identity, but this epsilon r is needed to cancel the gamma 0 buried inside it. Using that completeness in the rest frame, we will end up with S alpha gamma, then delta gamma delta, and S inverse delta beta, which is nothing but the Kronecker delta of alpha beta, which is what we need for the completeness relation to hold. So, this is a rather simple form; just because of this convention of putting epsilon r, we have to keep track of it. And every time we use omega bar, the corresponding factors of epsilon r will appear to cancel the sign buried inside gamma 0. So, this is a useful property. We have now a complete orthogonal basis; it is normalized and it also satisfies the completeness relations. Any arbitrary spinor can always be decomposed in terms of various components in this basis.

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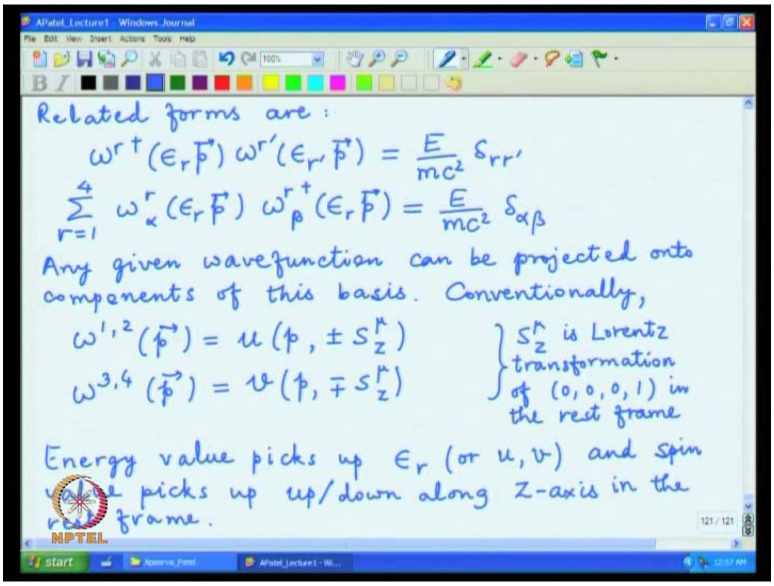

I also want to point out related forms, which do not involve omega bar, but involve only omega dagger. And one can actually derive them in a rather straightforward fashion with little algebra. They can be written now as the contraction of omega dagger with omega. And here it is necessary to put in these factors of epsilon together with p as they appear in the plane wave expansion directly. So, this specifies the actual momentum. And when you contract omega dagger with omega, there are no extra gamma 0 floating around anywhere. This then behaves not as a scalar, but this is the zeroth component of a 4–vector.

And so it will give a result, which is the transformation property of the zeroth component of 4-vector. And so it will have the value. Whatever value it had in the rest frame, multiply by this Lorentz contraction factor; and that is the orthogonality relation. It is still delta of r and r prime. But, a Lorentz contraction factor explicitly appears delta r and r prime just follows from the result in the rest frame. And this Lorentz contraction factor comes from the boost operators S.

A similar exercise gives a relation for completeness. But, it again looks little different when only omega dagger is involved and not omega bar. Epsilon r always goes with p and will use here the same momentum to define the completeness. So, this object will have the transformation rule again with all the S and S dagger involved, but it is not a place. Whereas, S dagger will completely cancel out; it will still leave behind Lorentz contraction factor. In the rest frame, the answer is still same; then there is no Lorentz contraction factor. And in an arbitrary frame, this ((Refer Time: 42:42)) produces delta alpha beta, but multiplied by E by m c square. These relations are sometimes useful. And so it is worth remembering, where the deviation between the omega dagger and omega bar comes in. So, this is now a complete basis.

Any given wave function can be projected on to components of this basis. And the reason for doing that is we can now assign specific meanings to what this basis means. The rest frame we have actually chosen very cleverly with all the 1's and 0's. So, the meaning in that rest frame was very clear. When the upper components were nonzero, they were the positive energy solution; and the lower components were nonzero, they were the negative energy solutions.

On top of that, there are two upper components and the two lower components, but they can be identified by the eigenstates of sigma z or equivalently the third component of the spin; it is either up or down. So, we have the notations 1 and 2 of this frame, are denoted by spinors u. They correspond to some momentum. And the spin can be the Lorentz transformation of whatever is S z in the rest frame. And similarly, the lower components are denoted with the notation v, which has the same thing, but the convention is to flip the sign of this S mu z.

One can interpret that flip of the sign as coming from factors of gamma 0. It is also related to the fact of how we are identifying the antiparticle and the particle modes. Antiparticle modes are going to be treated as absence of the particles modes that was the essence of the CPT theorem. And that absence label comes back here as flipping the sign of the spin. So, the two modes will have opposite signs of the spin. Here the definition of whatever is S mu z is nothing but the Lorentz transformation of the 4-vector in the rest frame, which denotes the positive spin.

And, that is the vector we have chosen as z component equal to plus 1 in the rest frame. So, this then becomes the 4-basis vector 2 u 2 v; and each of the two components are labeled by plus or minus for the corresponding spin labels. Energy value picks up epsilon r or equivalently the choice of u or v; and spin value picks up up-down along z-axis in the rest frame. That is a convenient basis.

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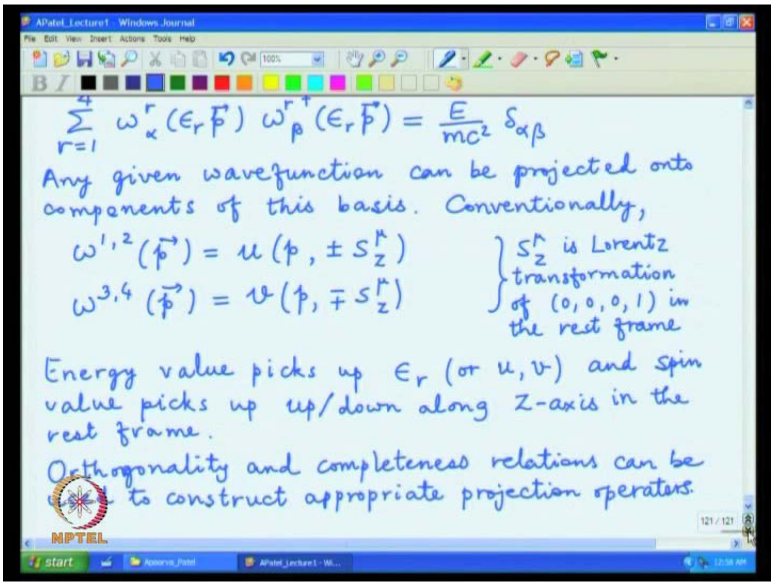

One can use this completeness and orthogonality relations to construct appropriate projection operators for these 4 choices. And we can do that now very easily.

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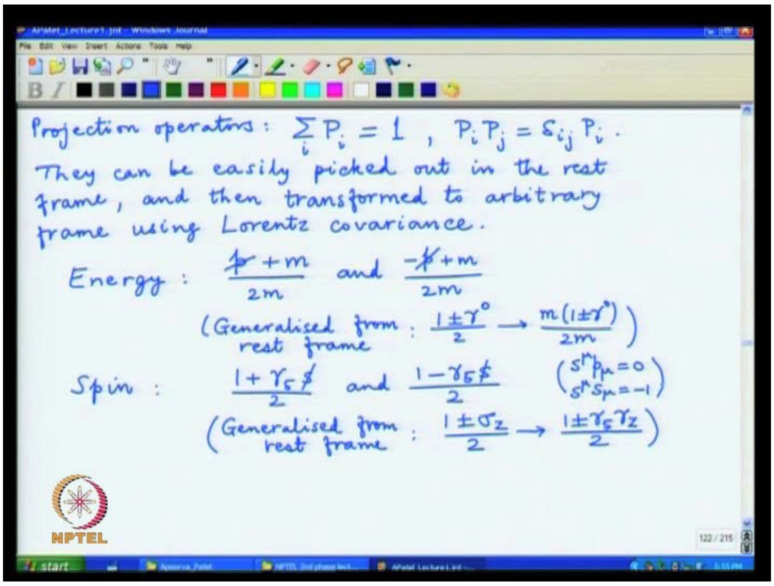

Remember the properties of projection operators. Let me denote them by index i on some basis P i. And a complete set means that, all the projection operators should add up to 1. And the orthogonality statement says that, if I have two different projections, it should give me 0. But, if I project along the same direction, once again I should get back whatever was there at the time of the first projection. The second projections does not do anything. And that relation is expressed as P i P j is equal to delta i j times P i. We need operators, which specify this particular property.

And, they can be more or less rid off from the completeness relation, because we have this set P I, which add up to 1. Completeness relation also adds 4 objects, which were adding up to 1. And we just have to convince ourselves about what were the correct objects. Again, they can be easily picked out in the rest frame, where we had nothing but 1's and 0's. And then we can transform them to arbitrary Lorentz frame by using Lorentz covariance.

So, what are the projection operators for energy? The structure which we had in the rest frame, the energy is just specified by the matrix gamma 0. Now, we have to write the matrix gamma 0 in a Lorentz covariant form. We can take advantage that, one can contract gamma 0 with the momentum. And in the rest frame, the momentum has only the time component. So, P 0 gamma 0 divided by m will give gamma 0 in the rest frame. And then to use that as a projector, that P 0 gamma 0 in arbitrary frame can be generalized to p slash. It has now 2 signs. The projection operator constructed for gamma 0 is either 1 plus gamma 0, which corresponds to the upper two components or 1 minus gamma 0, which corresponds to the lower 2 components.

What has become common place that, 1 plus gamma 0 and 1 minus gamma 0; one can now generalize to p plus m and minus p plus m. These are generalized from 1 plus or minus gamma 0 by 2 to m times 1 plus or minus gamma 0 by 2 m; and using the fact that, m is also value of p 0 in the rest frame. So, these are the projection operators for the energy. They have been made into the Lorentz covariant form by using this p slash. So, they will now work in any arbitrary frame. And similarly, for the spin, one has to start with an operator, which has the correct structure in the rest frame and then generalize it to an arbitrary frame. The rest frame structure is nothing but 1 plus or minus sigma z by 2. But, now, one has to convert sigma z into a Lorentz covariant structure. Again one can insert the particular vector corresponding to the direction of measurement. And we have seen what that vector is little earlier.

Let me write down the answer first and then explain. In the 4 by 4 matrix form, sigma has Pauli matrices on the diagonal. So, sigma z can be replaced by gamma 5 gamma z adopting the convention that, particle and antiparticle have opposite spins. Now, when the spin direction is Lorentz transformed from z-axis to general S mu, gamma z changes to S slash equal to gamma mu S mu. The spin remains orthogonal to the momentum vector S mu p mu equal to 0 and also space like S mu S mu equal to minus 1. That essentially, now gives a complete prescription of taking an arbitrary component in any frame and working out what the various parts are; which degree of freedom has which sign of energy; which degree of freedom has which sign of spin.