Relativistic Quantum Mechanics

Prof. Apoorva D. Patel
Department of Physics
Indian Institute of Science, Bangalore

Lecture - 24
P and T Transformations, Lorentz Covariance of Spinors

Now, we will discuss how the eigenstates of the Poincare group transform under the discrete symmetries of parity and time reversal. For that, it is sufficient to know the action of these discrete P and T operators on the Poincare group generators. The ones which are most relevant for specifying the eigenstates are the generators, which produce the Casimir invariants for mass and spin. And in particular, what we need are the simple transformation rules for the angular momentum operators, which can easily be extended to the necessary quantum numbers.

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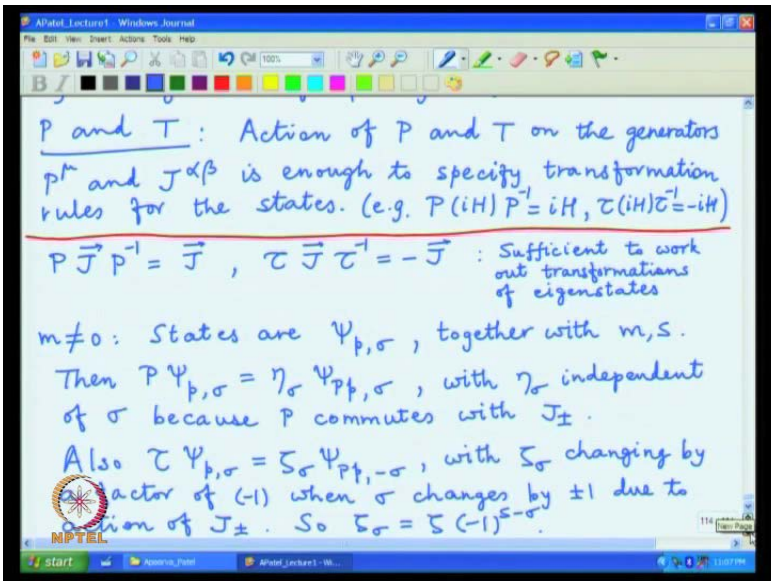

So, these rules are that under parity, the angular momentum does not change. On the other hand, under time reversal, angular momentum does change sign. So, this basically dictate, how that states are going to transform. And so let us illustrate this concept for the two important classes of states, which we have seen: one is massive particles and the other is massless particles. So, first, for the massive particles, we have the states denoted by the quantum numbers of the 4-momentum and the spin. Out of the 4-momentum, the time part can be always decided if you know the value of the mass. So, it is literally the

3-momentum, which has to be specified. And those two quantum numbers m and s – they are not going to change under parity or time reversal; they have been defined in a specific way and they remain invariant; they are the quadratic operators in particular.

So, now let us consider what happens under a parity. What we expect is for the eigenstate, you might get a phase; and then the transformed values – the 3-momentum transforms to p times p; where, p is the parity operator. And the spin remains invariant under the parity. This is just a generic rule, because we are dealing with a unitary transformation and an eigenstate. And then the question is – what is this particular phase? And that turns out to be independent of the spin, because one can apply the raising and lowering parts of the angular momentum operator on to this particular state, which change sigma to sigma plus or minus 1. And since the operator commutes with parity, the phase for eta sigma has to be the same as eta of sigma plus or minus 1. And so the whole multiplate of spin states in case of massive particles transforms according to the same phase, which can be picked by a certain convention.

We can apply the same logic for the time reversal operators in this particular case. This will change to some other phase, which I am denoting as zeta of sigma. And that now action the state time-reversal transformation of 3-momentum is the as the parity transform at the 3-momentum. It just changes sign. So, instead of using a different notation, I can just call this thing p acting on the 3-momentum. On the other hand, the spin changes sign and the time reversal as well. So, there will be a state with a minus sigma appearing.

Now, we need to determine what is zeta sigma. Again the same logic can be used together with the action of J plus or minus. It changes sigma by plus or minus 1 unit. But, now, the fact that the time reversal anticommutes with the angular momentum operator. There is a minus sign in the transformation rule; means that every time sigma changes by one unit, the phase also ends up changing sign. So, zeta sigma changes by a sign when sigma changes by plus or minus 1 due to the action of J plus minus. One way to absorb this sign and write a overall phase as some constant value for the whole multiplate of spin states; we can define this zeta of sigma as some overall phase times minus 1 raised to sigma. But, it has become convention that, this phase is expressed as minus 1 raised to S minus sigma; S is constant for the whole multiplate; and sigma is projection along the third direction, which changes by one unit under the action of J plus or minus. So, we have a complete specification now for how the states are going to transform including the overall phases in the presence of parity and time reversal transformation for massive states.

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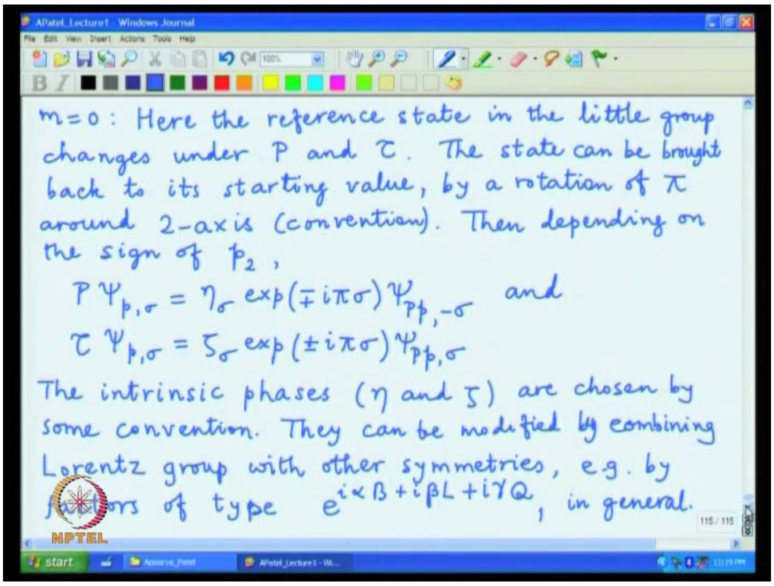

Now, let us go to the massless states and see again what we can obtain. Now, in the case of massive states, transformations were easy, because the state in the little group was left invariant under both parity and time reversal. It was a state at rest. On the other hand, for massless states, the little groups states is not invariant under either parity or time reversal. So, we have to now modify the definition of the transformation used for parity and time reversal to bring the little group state back to where it was.

Here the reference state in the little group changes under P and tau. The one way to do it is add an extra operation, which is part of the Lorentz group, so that the state comes back to its starting reference position. And that can be done by an operation, which again flips the sign of the third component of the momentum, which is the one undergoing change under either parity of time reversal for massless particles. And that can be done by applying rotation operation, which takes the third axis towards the negative of itself. Again, it is a convention, which axis is chosen; and the convention is to take the rotation to be around 2-axis.

So, now, we define the eigenoperation; not just as parity and time reversal, but parity and time reversal combined with this extra rotation by pi. Then the reference state is the same as what we started in the little group. This problem is absent for the massive particles. So, one has to treat the two cases: massive and massless particles rather distinctly. And working out all these separate eigenvalues, what happens actually depends on the component of momentum along the two directions, because that is something, which get involves by our convention that, we are applying a rotation around 2-axis. So, we can again define the transformation rules.

What happens with this modified parity operator? It gets a phase; then it gets a extra phase, which comes from this rotation by pi. And that rotation produces the e raised to i pi times sigma. And then momentum changes by the parity operation, but sigma also changes its sign, because of this rotation by pi around the 2-axis. So, it is a little bit complicated looking transformation, but various phases do appear. The other difference for massless state is that, now, these various spin states denoted by sigma are not components of a multiplate. So, you can define the transformation rule for each individual label of sigma and you do not have to have a complete multiplate of a 2s plus 1 states. In particular, at the most, you will a doublet corresponding to sigma and minus sigma when parity and time reversal are good symmetries.

Similarly, the transformation rule for time reversal is psi parity acting on the 3-momentum and sigma. There are two flips of sign of sigma: one by the time reversal and another by rotation of pi. And so you came back, where you started within this case. But, then there is an overall phase coming from the rotation part and that still contributes this phase e raise to plus or minus i pi sigma. Whether it is a plus sign or a minus sign depends on the sign of the second component of momentum. In particular, there is a discontinuity when the second component of the momentum is 0, but one has to live with this peculiarity in case of massless particle. This is now the specification for each value of sigma. There is a separate phase; one cannot simplify this expression further. And these various objects eta, zeta, etcetera, are many times chosen by picking up some convention. It is not something, which is absolute unless there is a relation, which connects different states of particles.

The intrinsic phases – eta and zeta are chosen by some convention. They can be actually modified if you go to a larger group beyond Lorentz group by adding some other symmetries like gate symmetries or even some discrete vales of charges, which are not part of any gauge group. For example, one can add baryon number or lepton number or a charge, etcetera and get extra phases. B is say baryon number; L is lepton number; Q is that electromagnetic charge, etcetera; alpha, beta, gamma are some parameters.

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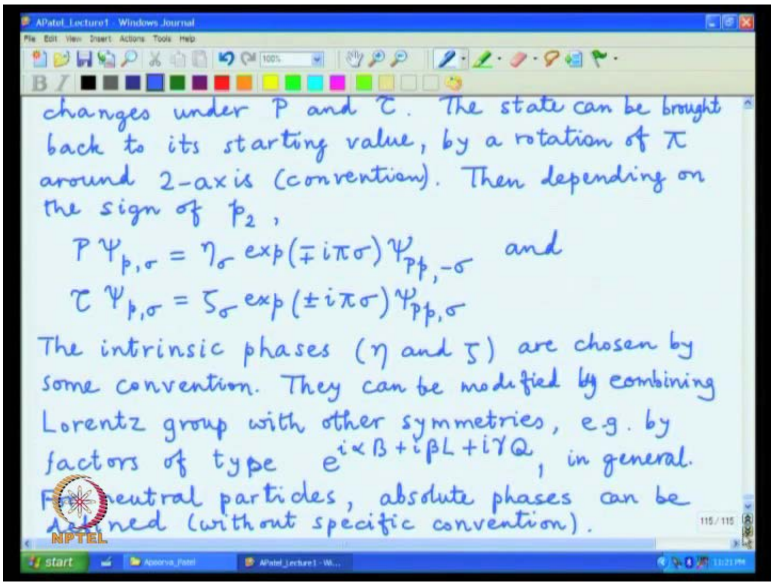

All these things are possible and one can have modifications of these phases. The only thing is that, such phases do not exist in case of neutral particles. So, such convention is not necessary, because they will not have these kind of charges – baryon number, lepton number or electromagnetic charge. And in that particular case, the convention is not really necessary; one can assign absolute values of parity or time-reversal phases and they cannot be modified by changing some convention. For example, it is possible to define specific properties of parity and time reversal for a photon. And then it cannot be changed by picking up some new convention. It is a neutral particle and there are absolute values of what phases apply in those particular cases. So, this is essentially the results for these discrete symmetries in case of both massive and massless particles.

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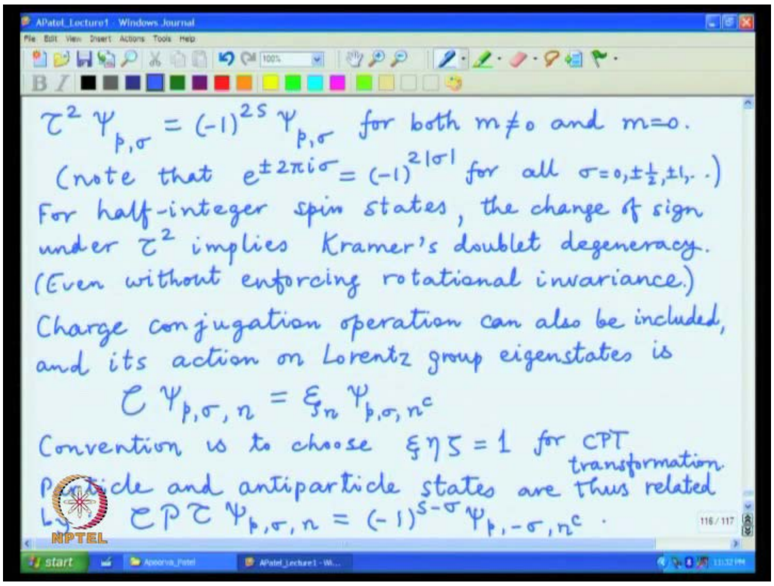

There are a couple of extra comments which I can make; which are some things which go beyond the Lorentz group, but which still are helpful in understanding the nature of particles. One is that, if you apply the time reversal operation twice on the state, then the phases kind of multiply. There is the minus 1 raised to S minus sigma. But, since sigma changes sign under time reversal, that factor appears twice. And so we will end up getting minus 1 raised to 2S. The parity operation also occurs twice on the 3-momentum, but that cancels out. And so ultimately, we just left with 2S. There are phrases as well, but time reversal involves a complex conjugation.

So, first time, the phase will appear and the second time this complex conjugate will appear and the two will again cancel out. So, there is a particular simplification in case of time reversal. We are just left with a state coming back to itself up to a sign. And this is true for both massive and massless particles. And in particular, you can use the simple fact that, e raise to 2 pi i sigma is equal to minus 1 raise to 2 sigma for all the integer as well as the half integer values of the spin.

The absolute value of sigma ends up being S for massless particles. So, this is a rule. But, it now tells you something that, for half-integer spin states, this operator basically takes the state to its negative. Clearly, the negative of the state is not the same as itself. And to find a solution out of the dilemma, we must have not 1 state, but actually 2 states with all the identical quantum number of mass and spin. And this implies what is known as

Kramer's doublet degeneracy; that means there is not one state with that quantum number, but there is a set of 2 degenerate states; does not have to be 2, it can be 2, 4, 6, etcetera; but it has to be an even number of state. This statement is true even when rotational invariance does not exist. And that is helpful because time reversal is a symmetry that holds in a much larger class of interactions than the set of interactions that obey rotational invariance. Explicitly, if psi is one eigenstate, then the other degenerate eigenstate is given by the time reversal operator tau acting on psi. This is a peculiarity, which comes out of this analysis, but it is also necessitated by the application of the Lorentz group. So, this is one caveat.

The other caveat is to include the operation of charge conjugation. And one can ask – how the Lorentz group can combine with this? This is strictly not a space-time transformation; but because of the CPT theorem, its properties are tied with what happens under parity and time reversal, which are space-time transformation. And so it is useful to give an appropriate definition, which is consistent. With everything else, we have constructed for Lorentz group. To do that, we now have to include some extra quantum numbers. We had this p and sigma as usual, but I will denote all the other charges by these numbers n. And then one can have another phase. Nothing happens to p and sigma under charge conjugation, but the phase can depend on the value of n. And what happens is the state goes to p, sigma and n conjugate; which means all the charges will have to be flipped in sign.

Also, one gets a new phase, which is denoted here by psi. The convention again is that, this is a unitary operation. And so this modulus of psi is 1. And convention is to choose the combination of all the three of them – psi, eta and zeta is equal to 1. And this is what one gets for the complete action of CPT transformation. And that produces a certain simplification in specification of the states that, now one can put all the things together and generate the relation between particle and antiparticle states. The two are related by this combined operation, which is CP tau acting on p, sigma and n; and it produces a phase minus 1 raise to S minus sigma.

And then the momentum comes back to its value spin; flips its sign; and all the charges also flip their sign. And this is a general rule, which is extremely useful in terms of interpreting the space-time picture of what is the particle state and what is an antiparticle state. And it is heavily used in constructions of various kind of Green's functions in quantum field theory; in particular, the strategy of Feynman to relate propagations of particle and antiparticle in the space time and denoting them by simple diagrams. So, this is as much as I would like to say about general properties of Lorentz group and its combination to various discrete symmetries.

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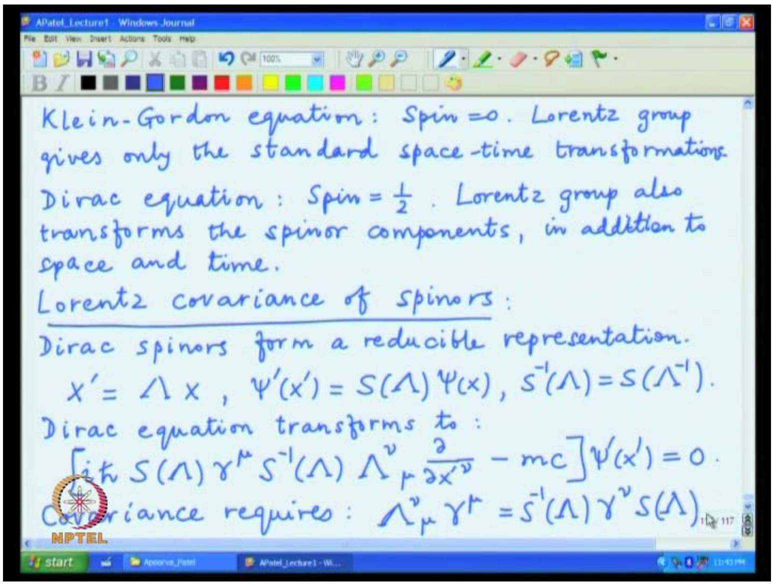

Now, I would like to go back to specific applications of this whole formulism to the problems or the fields, which we have discussed. There is hardly anything to add in case of a Klein-Gordon equation, because the spin happens to be equal to 0. And then Lorentz group gives only the standard space-time transformation. But, the case of Dirac equation is different, because in that case, you have spin is equal to half and the Lorentz group also transforms the spinor components in addition to space and time. And it is worthwhile to analyze this transformation rather explicitly, because Dirac particles are very important ingredients in our model of physical world. I will now discuss the special part of Lorentz covariance of spinors in more detail. It is also little more interesting, because Dirac spinors are actually not irreducible representations of the Lorentz group. The reducible representations are the Weyl Spinors. So, we have a joint action on both left and right-handed particles by the Lorentz group. And that combination is also useful to understand.

We now start with just basic definitions; and here I will stick to the homogeneous Lorentz group first. For the case of simplicity, the translation part can be added rather easily little later with a homogeneous transformation space-time transforms as x prime is some 4 by 4 matrix lambda acting on x. But, in addition, the spinor components are going to transform. And how they transform will depend on the matrix lambda. And it is another 4 by 4 matrix, which I am denoting by S acting on psi of x. And because this whole structure is a group, we must have an inverse transformation as well. And that gives the condition that, the S inverse in this particular case is nothing but the S of lambda inverse. If one applies inverse transformation, one goes back from x prime to x.

Now, our job is to derive the explicit form of this matrix S and see how the spinors actually transform under this operation; and also, find out the corresponding eigenstates starting from the whole description under the little group and then applying various boost operation. We start by applying this rule to the Dirac equation and derive the condition for a consistent property for this particular group. The Dirac equation transforms to the structure; where, I can write the action of this S of lambda on the whole equation. We do not know how this thing is going to act on the internal space.

So, we cannot let it go through the gamma matrices. But, one can write it as S gamma mu S inverse. And then the S inverse S is an identity operator, which is inserted in this structure. And that can now go through the remaining part, where there are no gamma mu's. And then the remaining action of S will just produce psi prime. So, that is essentially the result in the transforms coordinate. d by d x mu can be written by chain rule of differentiation as this d by d x prime. The mass term does not do anything at all. And then we have S acting on psi which produces psi prime of x prime. So, just take the original Dirac equation, where all this lambda was absent. Apply S of lambda and get this modified equation.

And, now, our job is that, we want to find the structure of this matrix S, so that this transformation is a covariant transformation. The purpose of writing everything in this form was that, the mass term actually is in the original form already. And so what we need is basically rule that, the first term also comes back to itself. And so covariance requires that, this whole combination in front of this derivative must be the same as the gamma matrix. In the x prime coordinates, this lambda can be combined with any matrix; they are essentially the numbers – lambda mu nu. The object it is contracted with is the matrix gamma. And then this whole S and S inverse are applied on either side of this result. S – this structure and S inverse should become equal to gamma nu, which will contract with these partial derivatives. And flipping the operation on the other side, we can rewrite this whole thing as S inverse gamma nu S lambda. So, this is essentially what the covariance condition requires.

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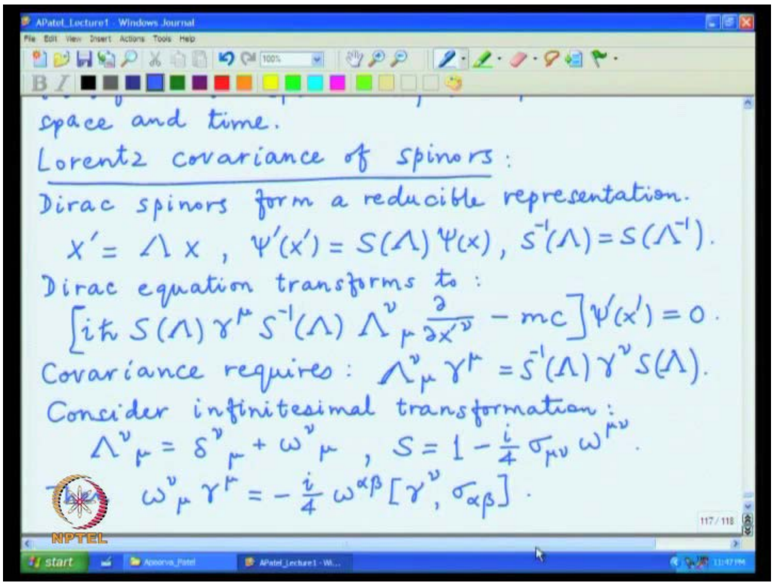

And now, one can derive the structure of this matrix by demanding that, this particular condition be satisfied. For any arbitrary Lorentz transformation, what we need is actually to only construct the infinitesimal version of this condition; and the rest of the stuff can be then obtained by exponentiation. So, let us choose that infinitesimal transformation, where this matrix is identity plus some little parameters. We have seen these parameters both in the case of Lorentz transformations of the rotation type and the boost type. And similarly, we will now parameterize this matrix S as identity minus some coefficients, which are chosen conventionally; some structure sigma mu nu, which we have to determine and the same parameter, so that this transformation is also a linear transformation.

Now, we can find out what these structures sigma mu nu is by substituting both these form in the equation and see what happens. The identity part essentially just cancels out on both sides. It will produce gamma mu is equal to gamma nu on both sides. And so we have to keep the next order term on both sides of the equation and see what it implies for sigma mu nu. And that condition now looks like omega nu mu gamma mu, which is what is there on the left-hand side. On the right-hand side, this sigma mu nu term will come twice: one from expanding S inverse and the other from expanding S. But, they are on opposite side of this matrix gamma nu; and that produces the well-known commutator structure. So, there are these parameters, which are left unchanged in this whole commutation game. But, we have gamma nu commuting with sigma alpha beta. This is a condition, which now has to be solved to obtain what is sigma alpha beta. And that can be figured out with a little bit of algebra. It is very easy to verify the answer once we have obtained the structure. And that is what we can see.

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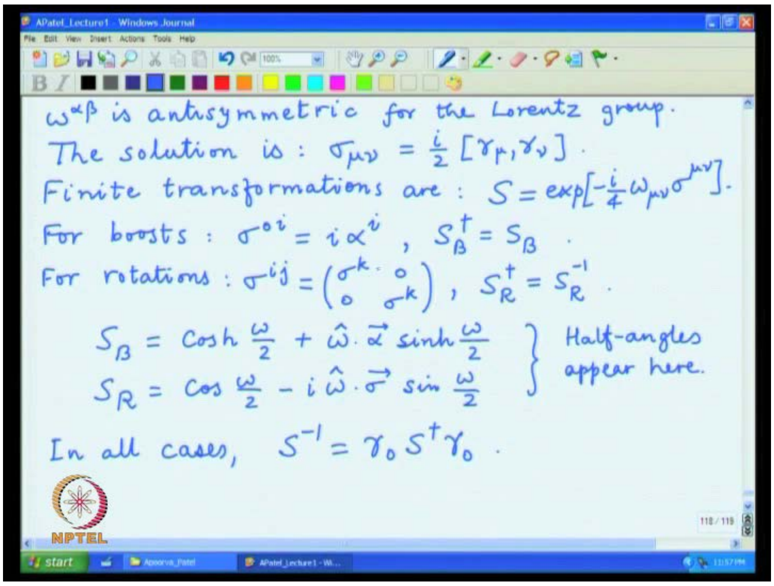

One thing which we know is that, this parameter is antisymmetric for the Lorentz group. And so sigma alpha beta, which is contracted with it, will also turn out to be antisymmetric in its structure. And now, there is only one matrix on left side and there is a commutator of gamma and sigma on the other side. Given the algebra of the gamma matrices, one can see that, a set of two matrices appropriately arranged can get contracted with a single gamma matrix to reduce to one gamma matrix. And that solution is unique. One can work out the details of the Clifford algebra. And that result is sigma mu nu is i by 2 commutator of the two gamma matrices. That is the only antisymmetric structure one can construct. And having done this, now, one can directly substitute inside here and verify that it is indeed the result, which works.

We have in a sense solved the problem of covariance. In case of infinitesimal transformation, we got this particular form. Now, this object behaves in different ways when the operation is a rotation versus its boost; its properties under Hermitian conjugate are different under the two cases. And one can easily evaluate from this commutator rule what they are going to be. One can now work them out in a very straight forward fashion. Finite transformations are just simple exponents of these objects. That is a standard way. The structures build up in group theory. This sigma mu nu are essentially the generators represented in this particular representation. And the overall transformations are the exponents of this particular generator. These objects are exponential of omega times sigma. For the specific cases of rotations and boosts, one can easily work them out. For boosts, we have the matrix sigma 0 i, which happens to be the same as i times the alpha matrix and S. Let me put a suffix B for boosts, is Hermitian, because alpha matrices are Hermitian.

On the other hand, for rotations, one has the structure sigma i j. In the Dirac bases, these are nothing but the matrices, which can be written using the antisymmetric symbol as a structure of sigma k acting on the diagonal. I had used the notation capital sigma in the earlier part of this course to denote matrices of this particular type. And in this particular case, sigmas are Hermitian object and the extra i means that S in case of rotation is unitary. So, this is a generic structure, which emerges with explicit solution of the covariant transformation of the Dirac equation. One gets these matrices sigma mu nu as the generators describing the Lorentz group.

In this particular representation, one can actually now write down a general form corresponding to transformation about arbitrary access. So, then one can say the boost transformation along an arbitrary access corresponds to hyperbolic cosine of omega by 2; where, omega is the rapidity. And the second term is sine hyperbolic, which will now depend on the direction of the boost. The particular feature, which is different than the same structure from the Lorentz transformation, is the appearance of the half angles. And these half angles are clearly tied up with the spin half structure, which is what Dirac equation represents.

One now can write down both these half angle rules. So, this is the explicit transformation. And one can now work out various consequences in as much detail as necessary. Only one more trick is useful that, how do you deal with these matrices for boosts, which are Hermitian and not unitary. And that is a standard structure that, one can exploit. In all cases, one can rewrite the inverse of this matrix S inverse as gamma 0

S dagger gamma 0. And that is a useful identity, which is true for both boosts as well as for rotation. And we will exploit this. In case of boosts, the gamma 0 is going to anticommute with alpha; and that is how this identity holds. In case of rotation, gamma 0 commutes with these diagonal sigma operators; and that is how the identity holds. But, this turns out to be a useful structure, which will use henceforth in defining transformations for Dirac spinors in Lorentz group.