Relativistic Quantum Mechanics
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Lecture - 20
The spinor representation of SL (2, C), The spin-statistics theorem

In the previous lecture, I defined the generators for the homogenous Lorentz group, the three rotations and the three boosts and also showed explicitly how they produce a representation that transforms four coordinates; that representation is by 4 by 4 matrices, and it is called the defining or the adjoint representation of the group. This group is called SO 3, 1. It is a orthogonal group involving four coordinates, but three coordinates have a plus sign in the metric, and the one remaining has a negative sign or vice versa. It turns out that there is another way to write down the four coordinates and define a representations corresponding to that, and that representation is different than the adjoint representation which acts on a four component vector by 4 by 4 matrices. This alternative way of writing the four coordinates is to rearrange them as a 2 by 2 complex matrix which is Hermitian by constructions.

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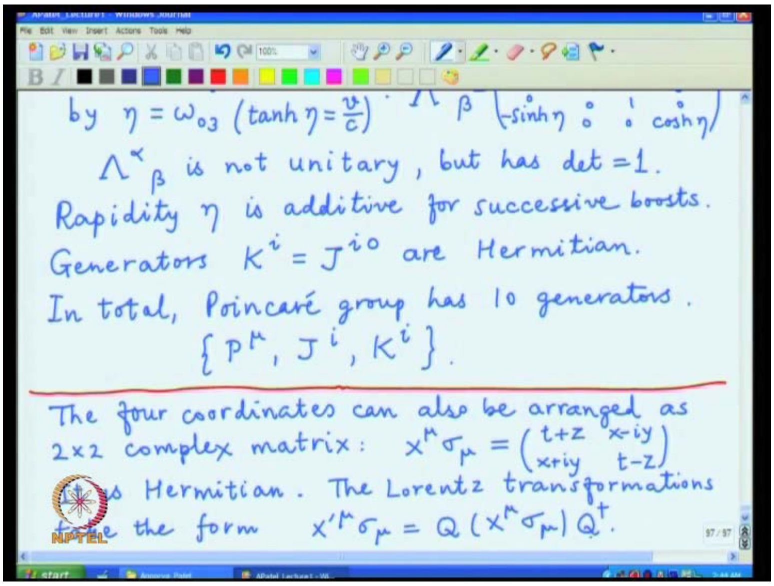

So, it again has four different components and that matrix is written as the combination of the four component with the four Pauli matrices, the zero th Pauli matrices corresponding to the identity, and if one explicitly writes the components it becomes t plus z and t minus z on the diagonal and x plus i y and x minus i y on the off diagonal. Now this matrix is Hermitian by construction, and one can ask what kind of transformations the Lorentz group corresponds to in this particular notation. Again the individual components we already know how t transforms, how x transforms, how y transforms, how z transforms, but now we want to rewrite the transformation as this matrices, and that turns out to be the Lorentz transformation take the form where x prime in the same basis is a 2 by 2 matrix Q multiplying x mu sigma mu with Q dagger.

On the other side clearly this transformation takes Hermitian matrices two Hermitian matrices, and so the new matrix will have the same structure, and it can be written again by the four components t x y and z. The point is to identify which particular matrices q give the same kind of transformations given by angle theta and rapidity eta which we wrote in a 4 by 4 matrix notation earlier now they will appear as 2 by 2 matrices, and since this matrix is a complex matrix q and q dagger are also going to be complex matrices in general. I will give the answer first, and then explain how it does the job of transforming according to the Lorentz transformation.

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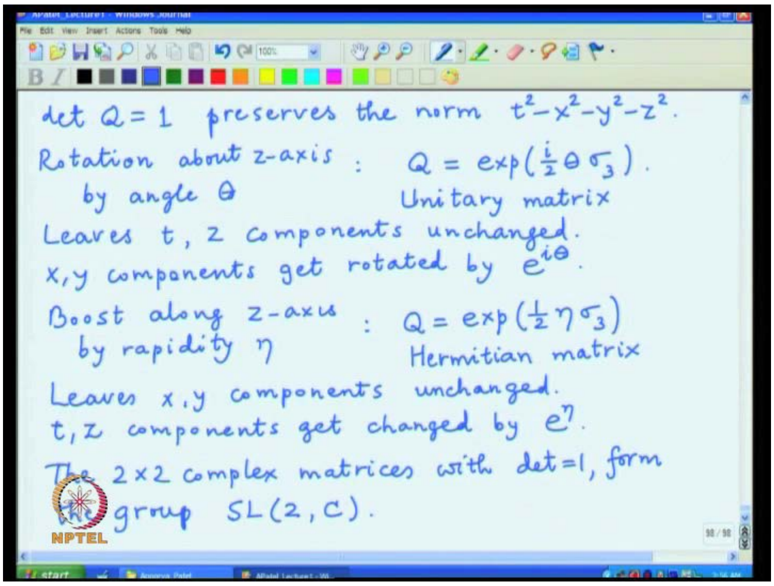

It is convenient that determinant of Q is chosen to be 1. It preserves the norm which is t square minus x square minus y square minus z square, and we would like to use that condition, because it is consistent with Lorentz symmetry. And now we are left with the job of finding out 2 by 2 matrices with determinant one which does the required job, and this can be classified into two separate parts. They are the three rotations and the three boosts, a 2 by 2 matrix actually as eight components, but this particular condition determinant Q equal to 1; it is a complex equation. It takes out two components from the eight and the sixth ones remaining are made up of three rotations and three boosts, and one can write those things as again in the structure of Pauli matrices.

So, the rotation about z axis which we wrote down earlier; now it has a form that Q is exponential i by 2 theta times sigma 3. So, this is unitary matrix, because a Pauli matrix is a Hermitian, and one can take all the three rotation axis direction by just choosing sigma 1, sigma 2 and sigma 3 inside this exponential; in the particular case when this is sigma 3 now one can ask what happens in the combination Q acting on one side and q dagger acting on the other side of x mu sigma mu. Clearly this matrix is sigma 3; it will commute with the sigma 0 and sigma three components of x mu sigma mu. So, then Q and Q dagger will cancel out for those two particular components Q dagger becomes Q inverse since this matrix is unitary and so the time and the z components are left unchanged under this particular operation.

On the other hand if one talks about x and y components then they correspond to the structure of sigma 1 and sigma 2, and Q is going to anticommute with them. So, one can take Q dagger through this x mu sigma mu on to the opposite side, and the anticommutation changes the sign of the exponent, and then one has the transformation which is essentially q square acting on x 1 sigma 1 and x 2 sigma 2, and q square removes this factor of half in the exponent, and one just has a multiplying factor of e raise to i theta. So, x and y components get rotated by e raise to i theta, and that is exactly what we need for rotation about z axis by angle theta. Everything here appears in the complex notation, so the actual component is the x plus i y which is equivalent to the complex combination of the two coordinates, and that has a multiplication by e raise to i theta which is a phase in the complex notation.

So, this is exactly the job which we wanted, and that explains this factor of half which is crucial in understanding this particular representation of the Lorentz group. One can similarly find the corresponding factors needed for boost by rapidity eta along the z axis, and we know that all we have to do is now instead of getting these trigonometric functions we have to get the hyperbolic function. So, the answer is easy to guess and also very easy to verify, and that is exponential of half eta sigma 3, the only part is i is now gone from the exponent. So, this will again act little differently when q and q dagger are involved in the operation of transforming the coordinates.

This matrix is not unitary rather this is a Hermitian matrix. It is an exponential of a Hermitian matrix. So, it is also Hermitian matrix, and so now one can ask what happens when with x by Q and Q dagger; Q dagger is equal to Q in this particular case, and if one has the components x and y they are going to anticommute with sigma 3, and so one takes Q dagger onto the other side. It will become the exponent with the opposite sign, and since Q dagger and Q are the same thing the two exponents with the opposite sign will just exactly cancel out, and so leaves x, y components unchanged. On the other hand the t and z components get altered, and in that particular case the q actually commutes with sigma 0 and sigma 3. So, Q dagger goes through Q, and you will have Q square; Q square will remove this factor of half, and it is just overall exponent of eta times sigma 3, and the components were again t plus z and t minus z.

So, the overall factor will be e raise to eta for t plus z and e raise to minus eta for t minus z, and that is the transformation rule for a Lorentz boost in this particular combination of time and space coordinates. It can be written as a single exponential. So, this does exactly the same job as the 4 by four matrices did in the adjoint representation, but now we are doing the job with 2 by 2 complex matrices. So, there are six degrees of freedom explicitly constructed here. So, the 2 by 2 complex matrices with determinant equal to 1, they form a group which is denoted as SL 2, C; two represents the dimension of the matrix, c means that the matrices are complex, L is a linear operation which is what matrix algebra does, and s stands for again determinant equal to 1.

So, this is a notion and so now, we have a different label for the Lorentz group that it is something which can describe by this group SL 2, C, and this is also a very useful notation, and one can ask what are the properties or what are the various types of things one can define? And in this particular group one is actually allowed to define a new kind of representation or transformations which act on certain objects not x mu sigma mu, but we can define other kind of objects and those objects are called spinors.

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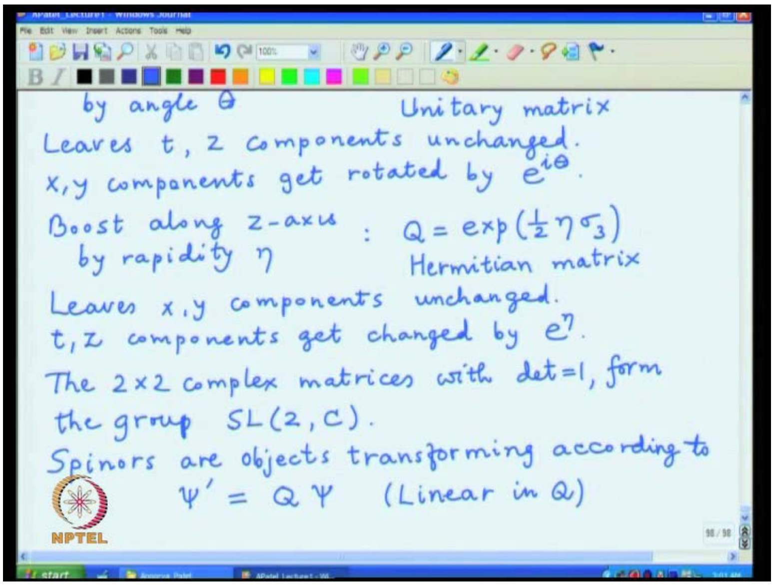

So, the definition is that spinors are objects transforming according to the rule that psi prime are equal to Q times psi where Q is exactly the same structure defined above. So, this is a transformation which is linear in Q, and that is referred to as the spinor representation; the adjoint representation which we saw earlier it is bilinear in Q acting by Q on one side and Q dagger on the other side, and that is a different representation compared to a spinor one. In particular if you now look at the matrices corresponding to Q once again we will see that the same rotations and boost will occur, but now the exponents have only half the value compared to what happens for the adjoint representation. So, these objects actually behave differently than those in the adjoint representation.

And the spins as we know already from non-relativistic quantum mechanics have the smallest representation which belongs to spin equal to half, and this spinor actually go back to that spin half structure, the adjoint representation the lowest one is actually a vector representation, and it refers to the non-relativistic analog of L equal to 1 for the angular momentum. So, this is the representation where the angular momentum has half the value compared to the angular momentum for the representation of coordinates. So, it is a new representation. It is just a cleaver rearrangement of the four components which allows us to define these particular objects, but it plays a very important role, because spin half particles are ubiquitous in our physical theories as well as in nature; rather most of the particles which we consider the fundamental particles they are all fermions and they belong to this spinor representation, and we have to study it in more detail.

These defines the state, and by construction these states have to be complex states, because we constructed Q to be a complex matrix; even the four components were all treated as complex numbers, and that is a different parameterization compared to the usual x y z and t where everything could be done with real numbers. So, there is a history behind it, and this particular aspect of being able to define rotations with half the transformation angle goes back all the way to the study of rigid bodies rotations and classical mechanics. And this in that sense is a phenomena one can see in classical study of rigid bodies, and one does not have to come all the way down to quantum mechanics to understand this particular topic, but anyway we often find the spin half being introduced only in quantum mechanics, and then this is the representation which shows up over there.

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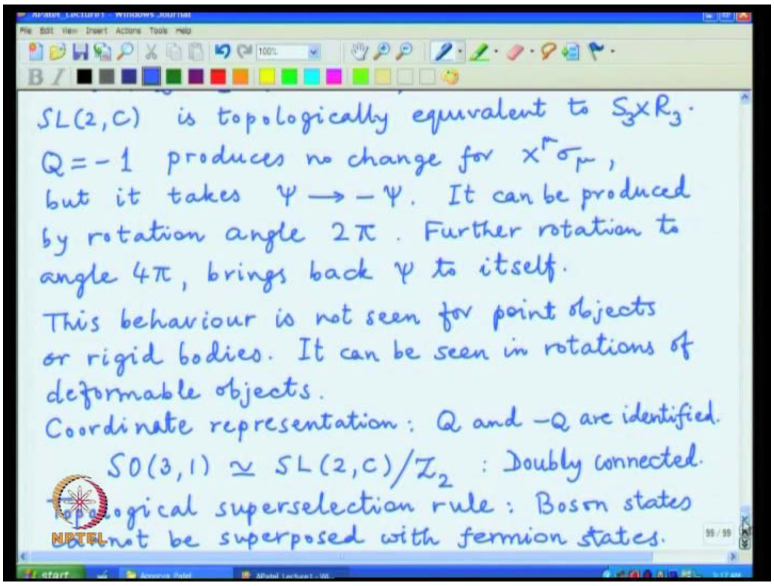

Any arbitrary Q can be broken up into rotations and boosts by what is known as a polar decomposition, and this is very general result of linear algebra that any matrix can be written as a product of a unitary matrix and a Hermitian matrix, and Hermitian matrix I can write as an exponent. So, if one factorizes in this particular way the unitary part decides the rotation, and the exponential of the Hermitian matrix defines the boosts, and we have the usual conditions that determinant Q is equal to 1 which implies that determinant of u is also equal to 1 as well as trace of the Hermitian matrix is 0. So, there are four degrees of freedom both in the unitary and the Hermitian part, but one each has been taken out by this condition on the determinant, and so one is now left with the remaining degrees of freedom, and that allows us to consider this breakup.

So, SL 2, C is topologically of the structure described by these two factors, and the unitary matrix with three degrees of freedom is S 3, and the Hermitian matrices with three degrees of freedom is R 3. So, S 3 is a sphere with three degrees of freedom, and R 3 is a three dimensional real space again three coordinate degrees of freedom. So, this is a little bit of mathematical structure, but what is now peculiar is the relation between this particular structure of SL 2, C and the first structure we described in terms of SO 3, 1, and that can be seen by looking at the spinor representation which we introduced in the case of adjoint representation both the notations as we explicitly constructed produce the same Lorentz transformation, and the peculiarity for spinor representation shows up for the particular case where the matrix Q is equal to minus 1.

Now because the adjoint representation is bilinear this minus sign does not do anything to it and Q equal to minus 1 produces no change for x mu sigma mu, but it takes psi to minus psi. So, this is the transformation which does not change the coordinates, but it does change the sign of the spinor, and one can now go back and ask how one can generate this particular structure Q equal to minus 1 we had explicitly constructed it in terms of the angle of rotation. So, it can be easily produced by rotation angle 2 pi since q was e raise to i by 2 theta sigma. If you put theta equal to 2 pi it will become exponent of e raise to i pi times sigma since sigma square is equal to 1 that becomes just equal to minus 1.

So, this now gives a peculiar behavior of this spinor representation that rotation by angle 2 pi actually changes its sign if you continue further rotation to angle 4 pi brings back psi to itself. So, this corresponds to states where if you rotate by angle 2 pi they actually change sign; it is a different state not the same as the original one, but if you continue rotating all the way to 4 pi you come back to the original state, and this is something which is not easily seen. So, this behavior is not seen for point objects or rigid bodies where we know that if you rotate by 2 pi we basically get back to the same confirmation which we started with. But it can be seen in rotations of what are known as deformable or flexible objects where the 2 pi rotation does not bring the state back to where it was, but 4 pi rotation does.

And this is something well known from the study of classical mechanics without any reference to quantum states, and there is a demonstration which I can show you with my hands. So, here it goes. I take the object to be my hand; the palm is oriented in a particular way, and I am going to rotate my palm about vertical axis which is the z axis. The whole arm is a deformable object; I will rotate my palm, but the shoulder will remain unrotated. Here is a rotation by 2 pi which has brought the palm back to roughly where it started. Its orientation is the same, but my whole arm has got on twisted in the process, and now I can continue rotation by 2 pi again and bring back my palm to the original position. I will repeat; this is the first rotation, my arm is not that flexible.

So, it is not a perfect rotation about z axis, but you can get the idea, and I continue once more, and it comes back to the original position. So, 4 pi rotation brings back this classical structure to its starting configuration, but if I only rotate by 2 pi it is a different confirmation and so it is not the original state. So, this in essence is an illustration so called theoretician's experiment to explain what a spinor is, and it is an object which after rotation by 2 pi does not come back to its starting point but a 4 pi rotation does. So, this essentially gives a mapping, but there is a factor of two in describing various states in terms of So 3, 1 notation and SL 2, C notation, and this extra factor of two corresponds to this ambiguity of Q equal to plus or minus signs, and one can state the same result little differently in the coordinate representation.

Q and minus Q are identified with each other in the sense that both the transformations corresponding to Q and minus Q are considered to be the same while in the spinor representation these two objects are clearly distinct, and so the group SO 3, 1 is the same. The language is it is homomorphic to SL 2, C divided by this factor z 2 which is the sign of Q, and this is a kind of relation which explains this 1 is to 2 mapping, and one can also refer to this saying that the Lorentz group is actually doubly connected, and the peculiarity of this half angle spinor representation arises from this doubly connected nature of the Lorentz group, And it is important feature of space time transformations in the three plus one dimensions that we are working with. I should mention that this kind of peculiarities are dimension independent, and if one works in different number of dimensions then the behavior of the Lorentz group can be different in those particular cases.

One also has a consequence from this doubly connected nature to the states of the Lorentz group, and the fact that the spinor will flip sign while the adjoint ones do not. It gives a kind of topological super selection rule which essentially can be stated that the bosons states cannot be superposed with fermion states. The fermion states will belong to this half integer angular momentum representations the spinors; bosons will belong to the integer spin number representation the adjoint one, and so one cannot have a superposition where one state transforms according to one rule, and the other component transforms according to other rule; the necessity is that all the components of a superposition must transform the same way, and so it follows that one cannot consider physical state where some part is bosonic and some part fermionic.

Either all the components are bosonic or all are the components are fermionic, and this feature arises from this doubly connected nature of the Lorentz group, and it is generally referred to as a topological super selection rule, but it is an important consequence of the group theory. Since I have mentioned bosons and fermions there is also another connection which I would like to point out, and that is the connection to statistics. The one way to classify boson and fermions is in terms the value of the spin whether its integer or half integer. The other one is that if there are two particles whether the exchange will produce plus sign or a minus sign in their wave function; now that is the description of statistics, and statistics require that you at least have two objects. The Lorentz group part you can deal with by describing only one object, but still there is a connection between the properties of Lorentz group and the part of the statistics.

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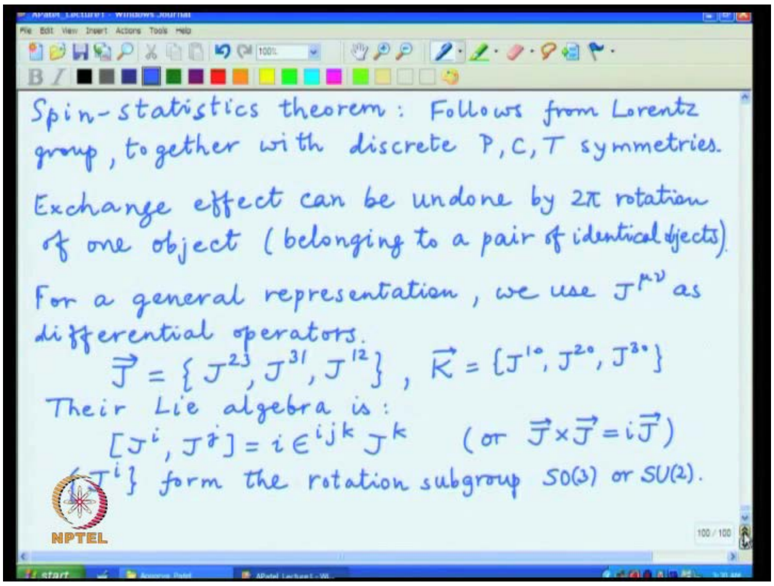

And that goes under the name of the spin-statistics theorem, and it follows from the Lorentz group together with the discrete symmetries of P, C and T, and one puts all those things together, and one can actually mathematically prove that under these restrictions of symmetries all the objects which have integer spin must commute with each other, and all the objects which have half integer spins they have to anticommute with each other. I am not going to give a mathematical proof of this statement though it follows from this same theory little bit extended, but I will give a simple again illustration the theoretician's experiment about how this theorem is routed together with the Lorentz group representations we just discussed, and to do that again I will show it with my hands.

I have this is a strip of paper with two ends, and I am going to interpret these two ends as two different particles. Both of them are identical, and I want to now see what is the relation when I exchange them compared to those configuration were they were unexchanged. So, this is again a flexible body. So, when I exchange things are going to get twisted, and you observe how the things get twisted. So, I take these two ends around onto the other side. The two ends can be made to look the same, but the intermediate part of the strip has got untwisted. I can remove that twist by holding one end fixed by rotating the other end, and that is how it goes. This is half a turn and this is now complete turn, and so with a rotation of 2 pi the strip has become untwisted, and I am back to the original configuration of two ends with untwisted stripe connecting them.

I repeat; I take one of them exchanged with the other. The strip gets twisted; once again undo the rotation by making one turn of one end, and I get to the starting point, and what this illustration shows that exchange produces a certain effect which can be undone by rotation of 2 pi of a single particle. So, that connects the exchange part which is a statistics to the spin part which is a rotation by 2 pi. So, exchange effect can be undone by 2 pi rotation of one object. This one object can be any one of a pair of identical objects, and now we already have seen what a 2 pi rotation does. If the particle is a boson the spin is integer, and 2 pi rotation will bring it back to where it was, and so the exchange effect basically does not do anything the whole exchange is commutative.

On the other hand for a fermion the 2 pi rotation will produce a negative sign as in the spinor representation. And so the exchange effect essentially changes the sign of the state which means the two fermions are anticommuting with each other, and this again is a powerful properties of spinors which one can understand in this way, and it is fundamental principle which occurs over and over again in discussions of quantum field theory, how do fermions behave and how do bosons behave when dealing with multiparticle states. So, this is some amount of discussion of the spinor representation and the consequences it produces, and now we can back go to classifying all possible representations of the Lorentz group; you have only seen two examples so far the coordinate representation or the adjoint representation and the spinor representation.

But one can have many other representation as well, and to do that we have to now go and solve the algebra of the Lorentz group in a general form without making any explicit matrix representation like 4 by 4 matrices or 2 by 2 matrices. And so we will do this general analysis by treating the generators as operators, and we have a explicit form for these operators in terms of differential structures and various coordinates, and we have listed all ten of them. Actually we are discussing only the six corresponding to rotation and boosts in the homogenous Lorentz group. For a general representation we use this J mu nu as differential operators, and the assignment which I had done before was that the three components of rotations where J 23, J 31 and J 1 2 essentially dictated by the epsilon I J K symbol and the three operators for K are J 10 20 and 30.

The convention is to take both k and J Hermitian, and so when you exponentiate them to produce different rotations in one case you get a unitary matrix and in other case you get a Hermitian matrix, and that comes out, because of the simple sign of the Minkowski metric whether you put i together with the exponent, or you do not put the i essentially that is the structure J and K are defined to be Hermitian. So, this now follow the algebra which we wrote down in case of J mu nu, but now you can break it up into simpler parts, and these parts are well known, and they can be easily obtained by restricting the indices of the commutators of J mu, and the Lie algebra is given by these commutators where J i J j is i epsilon i J k.

J k many time this thing is written in a index free notation as J cross J is equal to i J, and it probably is a little less writing instead of writing these whole set of commutators, but one can now write all the other commutators. These are the commutators for the angular momentum, and they basically close within themselves. So, they form a subgroup of the Lorentz group; that is this J i form the rotation subgroup which in case of the coordinate representation is SO 3, or if you want to use the 2 by 2 matrix notation it is referred to as SU 2. Again the relation between SO 3 and SU 2 is this. Q going to minus Q counting, so the SO 3 is essentially doubly connected version of SU 2, and one can also write down a super selection rule which comes down that the spin half integer values do not mix with spin integer, and that is all derivable only by looking at this rotation part alone without worrying about what is happening in the full Lorentz group, because it is a subgroup within itself.

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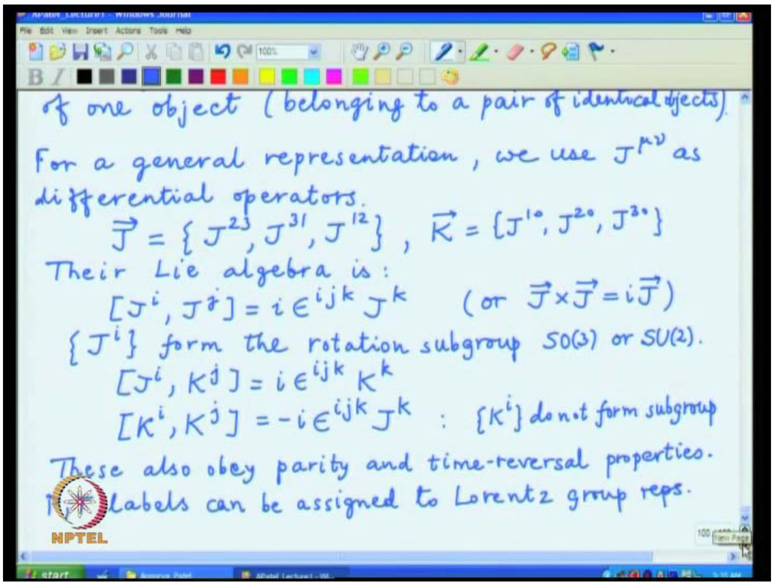

But the other degrees of freedom can now be included by writing down the other commutators, and they now take the form what is the commutator of J i and k J? One can easily work it out, and it turns out to be proportional to k, and so J and k basically mix with with each other, and one can also ask what happens in commutators of K i with K j, and that produces j which is different than the other relations and that J actually comes with a negative sign in front as well. So, this k's by themselves the commutators do not close. So, the rotations form a subgroup, but the boost does not form a subgroup. So, k I do not form, but this describes the six generators, and it clearly shows that the algebra of the six generators closes under commutation, and so that is the criterion for constructing a group. So, the whole group is closed.

But these combinations are simple enough that one can actually try to rewrite them, and somehow factorize this j and k generators in a different combinations, and once that is done one gets a complete classification of all representations of the Lorentz group. Before doing that let me just point out that these commutators also obey the properties of parity and time reversal, and these are the features which are useful in combining with Lorentz group. They are discrete symmetries not included in the Lorentz group sector one of the four sectors which I started out discussing; they are outside, but these generators do obey those particular rules under parity, nothing happens to J and k gets flipped in sign. So, all the rules respect that under time reversal j switches sign but k does not.

Note that time reversal includes complex conjugation which changes i to minus i in the definition of the generators. Then the commutation rules remain satisfied because i changes to minus i in their right hand sides as well. These properties are useful again in classifying the representations of the Lorentz group, because the parity and the time reversal labels can also be assigned. P and T labels can be assigned to the representation whatever we construct for the Lorentz group, and this is a common procedure to give this label as well. It is possible because in quantum mechanical science these two properties are simultaneously obeyed by these equations, and in operator language these objects commute. Now let us try to simplify this particular algebra by making claver combinations.

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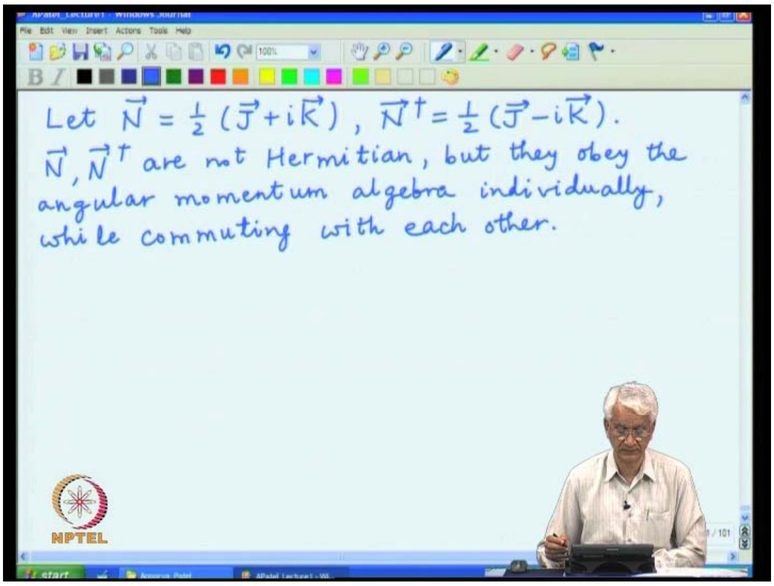

This combination turns out to be non-Hermitian. Let us define N as the combination J plus i K and its Hermitian conjugate as J minus i k, and there is reason for taking this combination and the result is that N N dagger are not Hermitian, but they obey the angular momentum algebra individually while commuting with each other, and that leads to factorization of the Lorentz generators into two sets of three components each, and that in turn gives a complete classification of all the finite dimension representations of the Lorentz group. We will go through that analysis in the next lecture.