Relativistic Quantum Mechanics
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Lecture - 14
Weyl and Majorana representations of the Dirac equation, Unitary and antiunitary symmetries

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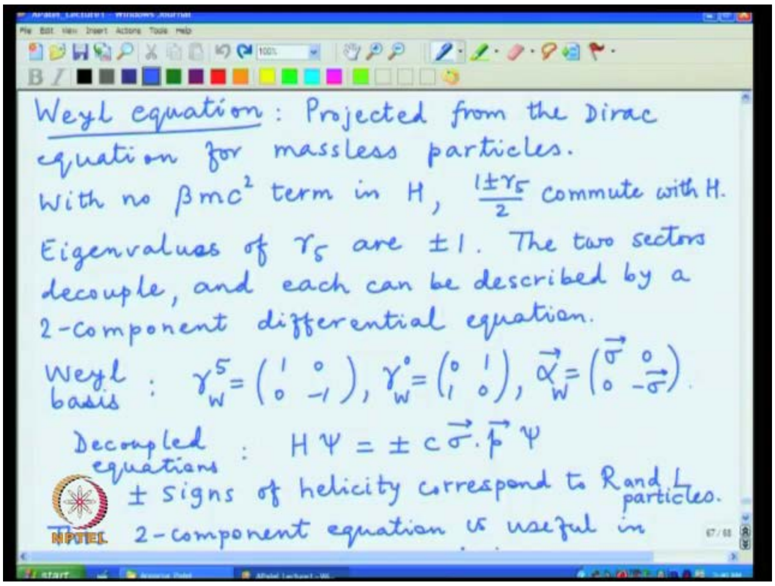

So, in the last lecture I was talking about the Weyl equation. It is a special case for mass less fermions, and because the mass equal to 0 allows us to write down equation with one less number of anti commuting set of matrices. And then you can choose the matrices cleverly enough. So, that the 4 components can decouple into 2 sets of 2 components, and that produces the Weyl equation with 2 different Helicity.

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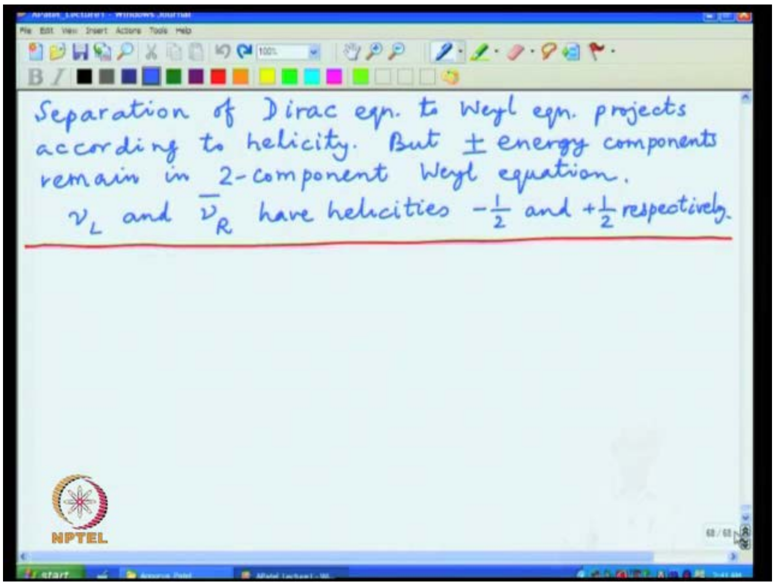

Now, before going further I would like to explain a few more properties of these Weyl fermions. Last time I said that this equation is a good description of mass less neutrinos which have 2 Helicities possible. But in nature we observe only the left handed neutrino and the right handed antineutrino.

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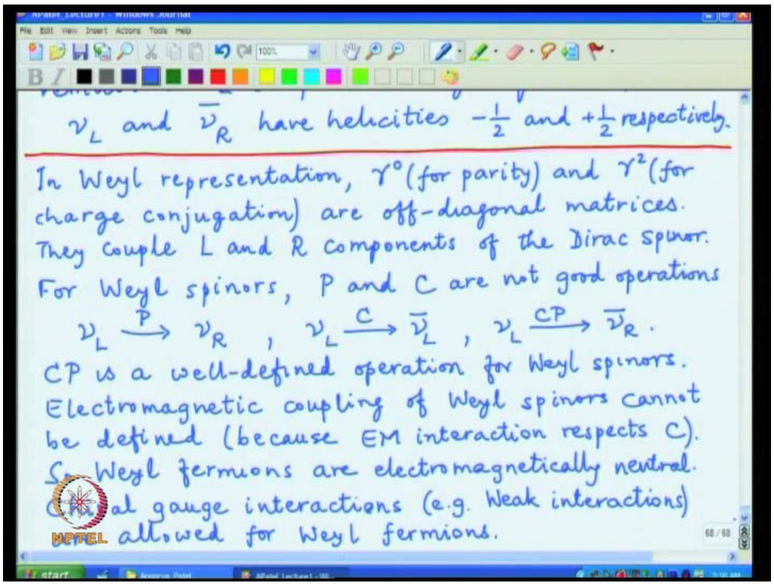

Now, one can see several other properties of the equation as well from the structure of the anti commuting matrices which we have used in particular in Weyl representation, both the matrices which is gamma 0 which is a transformation for parity; and gamma 2 which is the matrix needed for charge conjugation transformation are off diagonal matrices. The charge conjugation retains the same structure for the transformation; because just as in the Dirac representation only gamma 2 is a matrix which is imaginary. And since the charge conjugation transforms involved a complex conjugate only that degree of freedom corresponding to gamma 2 transforming to minus gamma 2 star is important for charge conjugation. And so that transformation matrix remains gamma 2; now both these matrices are off diagonal matrices that means they couple left and R components of the Dirac Spinor or in other words the parity and charge conjugation cannot act completely on particles of a single Helicity.

So, for Weyl Spinors, P and C are not good operations. In other words you cannot define if for a objects of a single Helicity to be more explicit if I take a left handed neutrino. And apply a parity transformation it produces a right handed neutrino or equivalently I take a left handed neutrino and apply the charge conjugation transformation; it produces left handed antineutrino neither of them actually exist in a pure description of a single Helicity Weyl equation. If you want to get a object which is completely defined for a single Weyl equation one can apply both these operations simultaneously. And since gamma 0 and gamma 2 are off diagonal the product of the 2 of them turns out to be a diagonal matrix.

And, so one can have a combined operation which is commonly referred to as C P and that produces a right handed antineutrino; and that indeed is a solution of the Weyl equation with an opposite values of energy. So, it turns out that even though C and P cannot be individually defined for Weyl Spinor; C P is a well defined operation for Weyl Spinor which indeed couples the particle and the antiparticle degrees of freedom. There are other consequences also of the charge conjugation symmetry not being defined or one can also say that these symmetries are violated; that one cannot couple Weyl fermions to a interacting field which obeys charge conjugation symmetry. And the important example for such a interaction is the electromagnetic interaction.

And, the electromagnetic coupling for that reason is something which cannot be properly defined, because this electromagnetic interaction respects C or the charge conjugation operation. In other words the Weyl fermions are electromagnetically or under any other interaction which respects charge conjugation it have to be neutral; they cannot carry that particular charge one can on the other hand have some interaction which violates charge conjugation. And then Weyl fermions are allowed to have that interaction and that is indeed realised in the standard model where the weak interactions are not charge conjugation symmetric; and they indeed couple to the neutrinos. So, these are called Chiral gauge interactions e g the weak; they are allowed for Weyl fermions. So, this much about the various kinds of interactions Weyl fermions can have depending on their properties of parity and the charge conjugation. The 2 component description actually also turns out to be very helpful in discussing various properties of the Dirac fermions as well.

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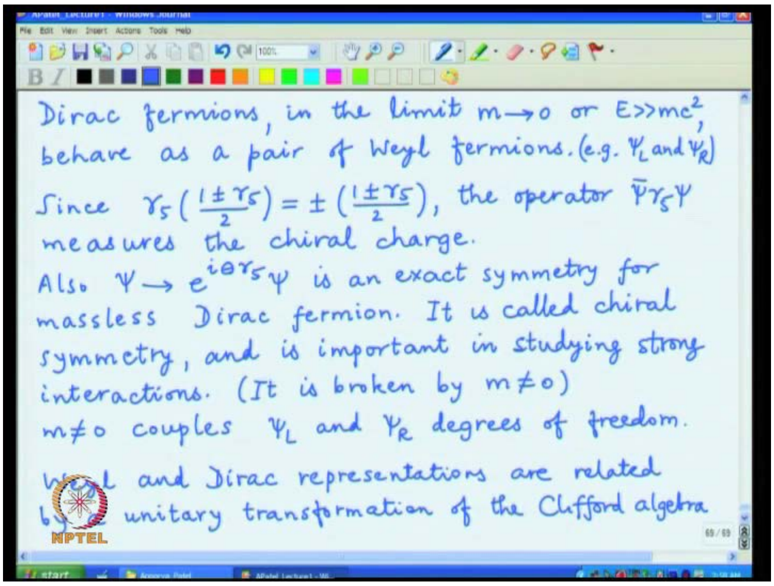

And, that is possible either where the mass is very small or equivalently the fermions are highly relativistic. So, the energy is much bigger than m c square; and in that case we can neglect the contribution of the mass in the Dirac fermions behave as a pair of Weyl fermions. And that description is also very helpful in describing how the left handed and the right handed components behave? And one can straightaway mention a few properties one of them is something which is referred to as a chiral charge leading to a concept of chiral symmetry; it follows from the fact that the operator gamma 5 acting on this projection operators for the Helicities it just gives the plus or minus sign of the helicity operator.

And, so if one wants to determine the plus or minus one can measure it just by measuring the expectation value of gamma 5 inside the state. And that operator then psi bar gamma

5 psi measures the so called chiral charge of the Dirac fermion. And that will essentially tell us whether the particular component we are looking at it contributes as a left handed part or the right hand part. And one can then extend this use of the operator to a sort of symmetry which mixes the 2 components.

And, that also can be done again in the case of Dirac fermions where one can define a transformation which is exponential of i theta gamma 5; where theta is some rotation angle this is a an exact symmetry for mass less Dirac fermion; for the simple reason that gamma 5 commutes with the matrices alpha they anticommute with the matrix beta. But beta is a matrix which drops out in the limit of the mass going to 0. So, under this transformation the equation remain unchanged. And that is called chiral symmetry; it plays a very important role in studying the so called strong interactions; where it is actually broken by a spontaneously generated mass for the Dirac Fermion. But that is getting too far ahead into the property of a field theory.

But it is important that this is a certain symmetry and if there a mass term it will break this particular symmetry. And that is again a useful concept related to the transformations of left handed and right handed components. If one write downs the contribution of the mass it rather explicitly appears as a interaction it couples left handed and right handed degrees of freedom. And this can be physically understood as saying that if I have a massive particle I can boost to a frame which is moving faster than the particle in that frame the momentum will get reversed. But its spin will remain unchanged and the so helicity will change sign and it will transform a left handed particle into a right handed particle; such a transformation is not possible if the mass is exactly 0. So, a non zero mass indeed ends up coupling the left and right components.

And, that is a way to understand the breaking of chiral symmetry which is produced by a mass term. The mass term may be explicitly present in the Dirac equation or it can be dynamically generated as in the case of strong interaction. So, these are the various properties of Weyl equation; and one can look at the clever choice made by choosing a particular set of anti-commuting matrices as a transformation on the basis of the gamma matrices satisfying the Clifford algebra. So, the Weyl and Dirac representations are related by a unitary transformation of the Clifford algebra and if one requires it can be easily constructed. So, then one can use the transformation to relate not just the basis matrices. But any arbitrary components of the speed as how do you go from one bases to another.

So, this is 1 aspect of the convenience in choosing the bases of Clifford algebra. And it allows you to basically how the degrees of freedom of the Dirac equation; when the particle has no mass. There is another transformation again corresponding to a clever choice of the bases of the Clifford algebra that is possible. And which also ends up having the degrees of freedom of the Dirac Spinor and that is what I want to describe.

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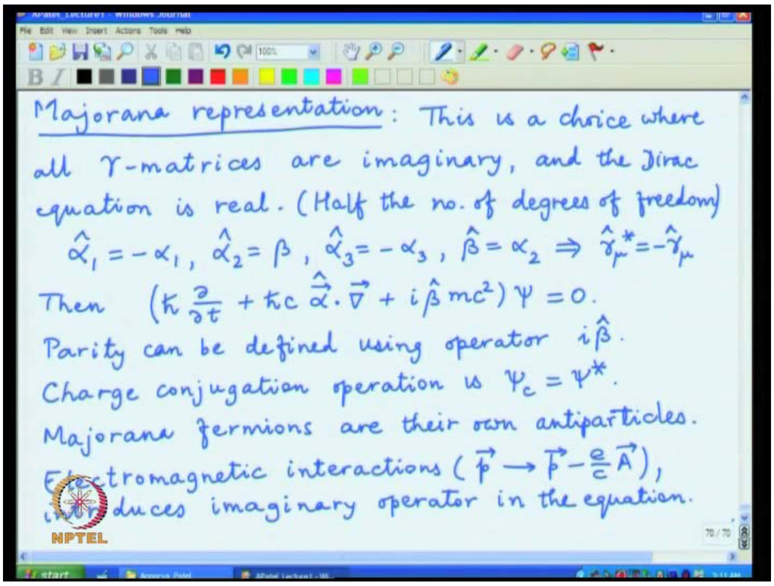

Next and that representation is called Majorana representation. And objects which follow this particular structure they are referred to as Majorana fermions; this Majorana is an Italian name. And the later j which is appearing in that here, in Latin it is actually pronounced as and that is how I am referring to the name. And what Majorana cleverly observed is that one can rewrite or rather choose a form of gamma matrices. So, that the Dirac equation instead of having a complex numbers becomes completely real. And this is a clever choice which is possible for the relativistic equations; such a choice actually is not possible for the Schrodinger equation there you are forced to introduce complex numbers. But in case of Dirac equation this is a clever trick. So, this is a choice where all gamma matrices are imaginary and the Dirac equation becomes real; which means half the number of degrees of freedom.

And, it is very easy to see by just again a simple permutation on the various matrices which we have already used. And I will write down the representation which was defined by Majorana; and the Majorana matrices I am going to denote by a symbol hat on top of the object. So, alpha 1 hat is defined as minus alpha 1, alpha 2 hat is equal to beta alpha 3 hat is equal to minus alpha 3 and beta hat is equal to alpha 2. And this implies that gamma mu hat star is equal to minus gamma mu hat; because beta is already one of the gamma matrices it is imaginary. And other ones which are products of beta and alpha all the alphas are real. And so alpha turns beta which are the gamma matrices also are imaginary.

And, so one can now use this definition to rewrite the Dirac equation without any complex number. So, at the unit i and that structure looks like h cross del by del t plus h cross c alpha hat dotted with gradient plus i times beta hat m c square psi is equal to 0. And i has been multiplied throughout and i times beta hat is real and so are all the coefficients in front. And so this will have solution psi which also can be completely real; and then it will have only half the number of degrees of freedom compared to a full complex solution.

So, one can have a this has a mathematical possibility; the question is what are the physical properties of a such a fermion? And that now one can try to infer from a this quantity; the parity transformation can still be defined its will be still defined as a matrix gamma 0 or in this particular case it is it will be defined by the matrix beat hat. And there is a choice available which is just i times beta hat; there is always a complete freedom in choosing the overall factor of the matrix; if you want to make it real choose a phase cleverly now for otherwise leave arbitrary it does not matter. There is no problem in defining such an operator for the case of Majorana fermion. But for a charge conjugation operation the relation was a transformation which converted gamma mu star to minus gamma mu.

But that operation is already two for the gamma matrices in the Majorana bases. So, the transformation is essentially nothing but identity in the Spinor space and charge conjugation operation. Then is that psi c is equal to psi star and if we choose the solution psi to be real; that means, the charge conjugated solution is equal to the original solution that is a unusual case. But it means that the Majorana fermions are their own antiparticles; under charge conjugation they come back to themselves. And if you are going to interpret the charge conjugation as a object which goes to the antiparticle; which will have an opposite value of the charge. That indeed is not possible in case of Majorana fermions; you get the same solution back after applying the charge conjugation operation and this a unusual feature.

So, the structure now is quite obvious what happened in the Weyl bases and what happened in the Majorana bases. In case of Weyl bases you reduce the 4 components to 2 components by separating the 2 different values of the helicity or the spin degree of freedom. In case of Majorana representation we again separated the 4 Dirac components into 2 parts of 2 components each by separating the particle and antiparticle degrees of freedom. And so the spin part will remain inside the 2 component description of a Majorana fermion but there is no separate label for a particle and a antiparticle.

Another way to look at it is to introduce the electromagnetic interaction into this Majorana structure of the Dirac equations. So, electromagnetic interactions which are introduced by taking the operator p and writing p minus A into the equation the p is i times the gradient. And so only tries to redefine the gradient by this particular transformation; it happens that one introduces back an imaginary operator which was carefully avoided in the other terms of the equation. So, if one wants to stick with the structure that I will only work with a real equation and a real operators it is not possible to introduce electromagnetic interaction.

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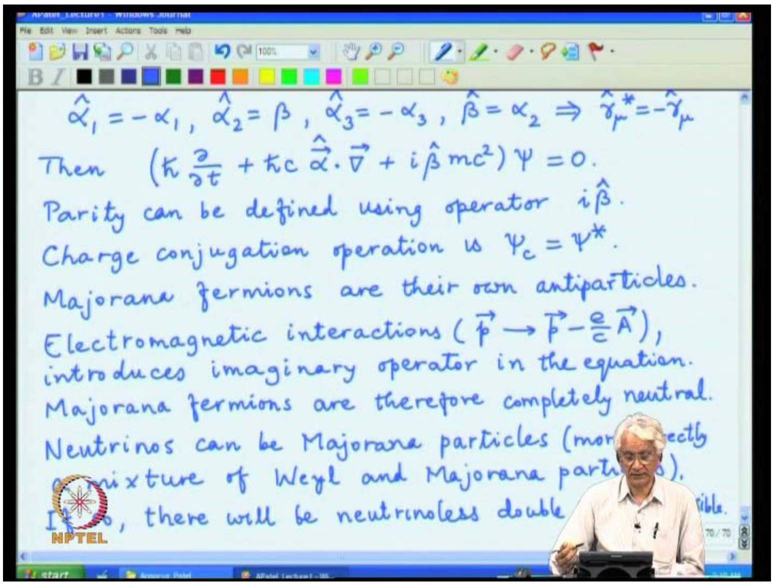

So, Majorana fermions are therefore completely neutral; they cannot have any interaction of this p going to p minus A type. And that includes both electromagnetic type interactions and also the chiral type of interactions which are possible in the case of weak interactions none of them exist, because all of them will convert this gradient operator to an extra term which is imaginary.

And, if you want to stick with the real bases one has to stick to a object which is completely neutral it does not carry charge under any type of interaction what so ever. And this is a rather strict constraint and it also makes Majorana fermions the very hard to detect. Because if they are not going to carry any charge; then how are you going observe them; they can be only observed through their space time effect. Because the Majorana fermions will carry spin even though they do not carry charge. And that is again a leads to a possibility that may be the neutrinos are Majorana particles.

So, neutrinos can be Majorana not strictly Majorana particles; because they do have a weak interactions. So, more correctly a mixture of Weyl and Majorana particles that is again possible by choosing a bases; which is slightly rotated away from the Weyl bases in the direction which points towards Majorana fermions. And this remains an intriguing possibility that may be there is some property of neutrino; which has a Majorana characters. And the only property which will display this Majorana character is this identity between a particle and antiparticle if so there will be so called neutrino less double beta decay possible.

And, this is the only clinching test which one can make to demonstrate a Majorana character of the neutrinos; that normally double beta decay will be one beta decay followed by another; in which case first one will emit a neutrino, the second will again emit a neutrino. But if the neutrinos have Majorana character it will allow a possibility where the 2 neutrinos can mutually annihilate and disappear into vacuum. And in that case there will not be any neutrino seen at all; and that is what is referred to as a neutrinos less double beta decay. And this is a unique property which will give a unambiguous signature that neutrino has a Majorana character or not. At present there have been lot of investment and efforts to study this particular property but the question is not yet answered.

People have observed double beta decays but always corresponding to emission of 2 neutrinos; the search for neutrino less double beta decay is on. But barring that reaction one can allow for a theoretically possibility that one has a Majorana component into the neutrinos. And that can be included in the Dirac equation by this mass term; which is quite a different possibility compared to the Dirac mass term. The Dirac mass term couples the left handed and the right handed neutrino modes while the so called Majorana mass term which appears in the Majorana equation; it will associate itself with a coupling a the particle and the antiparticle degrees of freedom. So, this is a different character and in theoretical models both these kinds of mass terms are investigated; whether the neutrinos has a Dirac mass or it has a Majorana mass and that question is important.

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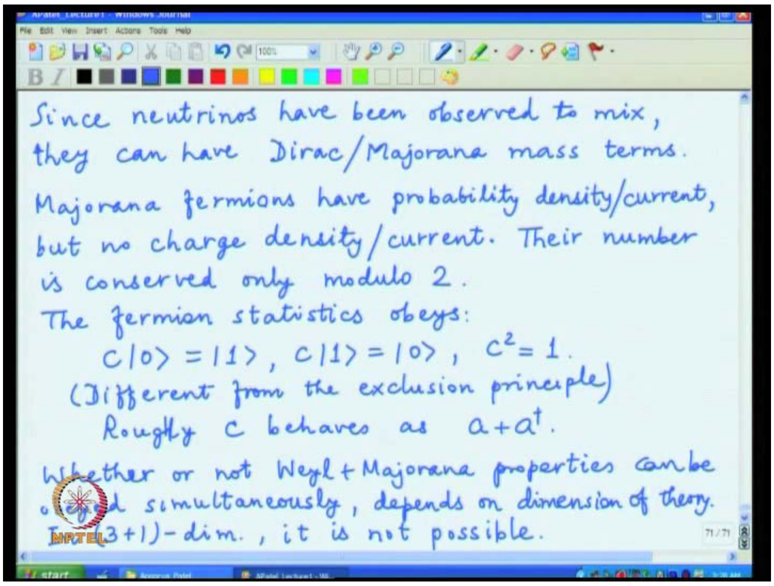

Because of the observation of neutrino mixing says that one must have coupling between the different components of the neutrino wave function; they can have Dirac or Majorana or both mass terms. And the models having these particular properties have been constructed and they are under investigation. And trying to figure out what the prediction are and what kind of a parameters will obey the experimentally observed consequences?

So, this is a particular feature that Majorana fermions are their own antiparticles and that leads to it peculiar effects. It also leads to a feature that one can have a rather peculiar looking fermion statistics compared to the usual exclusion principle in case of a

Majorana fermions. But let me first just say that Majorana fermions have this probability, density or current that is perfectly find. But no charge density or current on top of that the numbers of the Majorana fermion is conserved only module 2. Because 2 Majorana fermions can annihilate into the vacuum or there can be pair produced from the vacuum; just like a particle, antiparticle pair.

Here the particle and the antiparticle are identified with each other. And the statistic which follows from this kind of rule is rather peculiar. The fermion statistics then obeys these are peculiar rule that if I define a C as a operator; which creates a particle acting on the state 0. In the second application of the same operator can take you back to the original state or equivalently C square is the identity operator; this is different. And the exclusion principle where fermion operator acting on the vacuum state; it will create a one fermion state. But if I apply a second fermion operator it will produce the number 0; it will annihilate the state completely not produce the 0 fermion state. So, this is different from the exclusion principle.

And, that is a effect that the numbered in this statistics is only conserved modular 2; it does not have the more standard rule of Pauli exclusion principle where the number can be either 0 or 1but no other values. So, this is a different sort of story and roughly this operator behaves as a plus a dagger of the operators of Fermi Dirac statistics. And this particular super position that we have both a and a dagger combined together is again manifestation; that there is no distinction between a particle and a antiparticle.

So, this is the feature of Majorana fermion which appears as a statistics in a many body theory; it is again little bit unusual. And one can ask whether these 2 different properties of reducing degrees of freedom of Dirac equation are they mutually compatible or not. And it turns out that in some situations which depend on the number of dimensions of the space; in which one is working the 2 properties can be compatible. And in some cases they are different. So, depending on dimensionality one might even have a situation where the fermions can be defined. So, that they have both Majorana and Weyl properties simultaneously. So, whether or not Weyl plus Majorana properties can be obeyed simultaneously that depends on the dimension of the theory.

But in the or usual case 3 plus 1 dimensions, it is not possible; it is actually possible in a little bit of a simplistic scenario of 1 plus 1 dimension; there the matrices are the Pauli matrices. And one can play around with a trying to impose both these condition simultaneously and it works out. But I would leave that as an a exercise to be explored without going into further details. So, this is a consequence of playing around with the bases of a Clifford algebra and one can indeed find new features as we have seen. Now, I want to move on to discussion of another kind of symmetry and that is important together with a parity and charge conjugation. And it actually forms a triad of the discrete symmetries important for classification of particles and that is a concept of time reversal symmetry.

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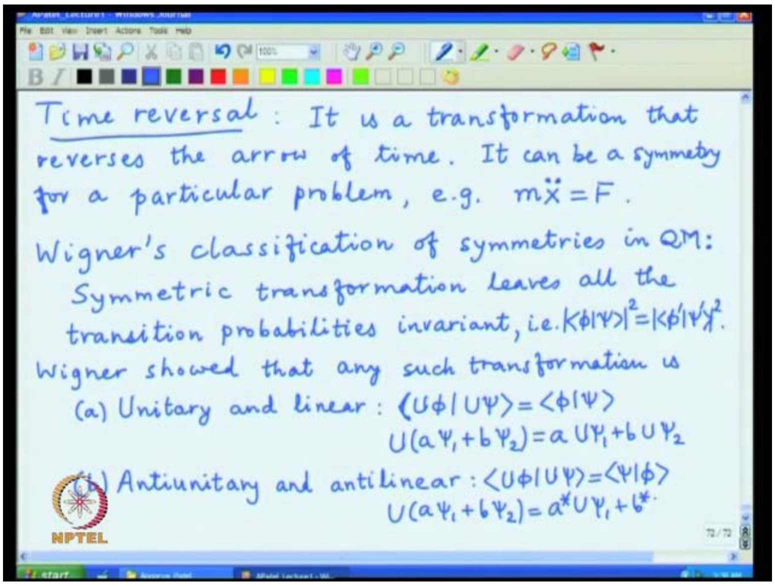

Now, this is a symmetry which is difficult to consider in practice because we know that indeed future and past are very different in our experience. So, there cannot be any symmetry which will equate future with past in any sense. But in mathematical formulation it turns out that many of the equations which we consider for many of the different kind of interactions; they do have this kind of symmetries. And through that equations we will not distinguish between past and future in a straight forward fashion. And then one it can indeed define a transformation which flips the arrow of time a simple example is the Newton’s second law; where the second derivative with respect to time which is the acceleration is related to the force. And if you change the arrow of time the second derivative with two flips of sign actually does not change its value.

So, the Newton's second law is symmetric with respect to this time reversal symmetry. And such considerations also exist in a relativistic theories and we want to study that particular situation a little bit in detail. So, this is a transformation it reverses the arrow of time and it can be a symmetry for some particular problem. And as I said e g m x double dot is equal to force in case of Newton's equation of motion. But now let us define this particular transformation more accurately in the case of quantum mechanics.

And, there we will define a general rule of what is a transformation which can be associated with the symmetry and quantum mechanics? And that is a analysis which is associated with the name of Wigner. And he gave a very general theorem which listed explicitly; what are the possibilities of various kind of symmetries in quantum mechanics? And the condition which one requires for the quantum theory to be symmetric is all the observables do not change under the symmetry transformations. And in quantum mechanics all the observables are the absolute squares of all the transition amplitudes one can construct. And that condition is very simply formulated as a condition between overlap of various kind of states which exist in the theory.

And, that can be written as overlap of phi and psi mod square in one particular bases will be the same; after you have change the bases according to the symmetry transformation. And Wigner's classification basically listed all the things that can be allow such a invariant of the transition probabilities. So, in general Wigner showed that any such transformation falls into 2 distinct possibility; one of them is called unitary and linear which means that phi prime is a unitary matrix acting on phi.

And, it can be summarized as U phi U psi is equal to phi psi this is a condition of a unitary transformation. And the linearity means is that a acting on psi 1 plus psi 2 is the same as a times U psi 1 plus U times U psi 2. And most of the transformations which we have considered so far fall into this particular category; the translation, rotation etcetera they all correspond to changing a bases by certain unitarity transformation.

But Wigner showed that there is a second possibility also available and that is called anti unitary and anti linear. And that is a little bit unusual situation where you have the change which not only applied as unitary transformation but it also flips the amplitude to its complex conjugate. And this is the transformation which Wigner labelled as anti unitary. And anti linearity is a property which goes along with this transformation; where the coefficients become complex conjugate. Based on this general classification we will discuss time reversal next time.