Relativistic Quantum Mechanics
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Lecture - 13
Charge conjugation symmetry, Chirality, Projection operators, The Weyl equation

Last time, I introduced the definition of charge conjugation which means basically finding a solution satisfying the same equation, but with opposite value of charge.

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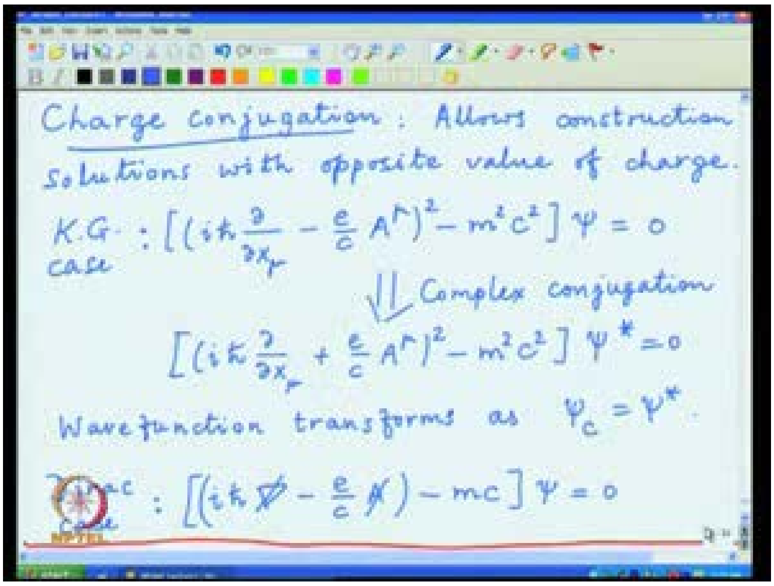

And today we will see the explicit construction of this transformation in the case of Dirac equation.

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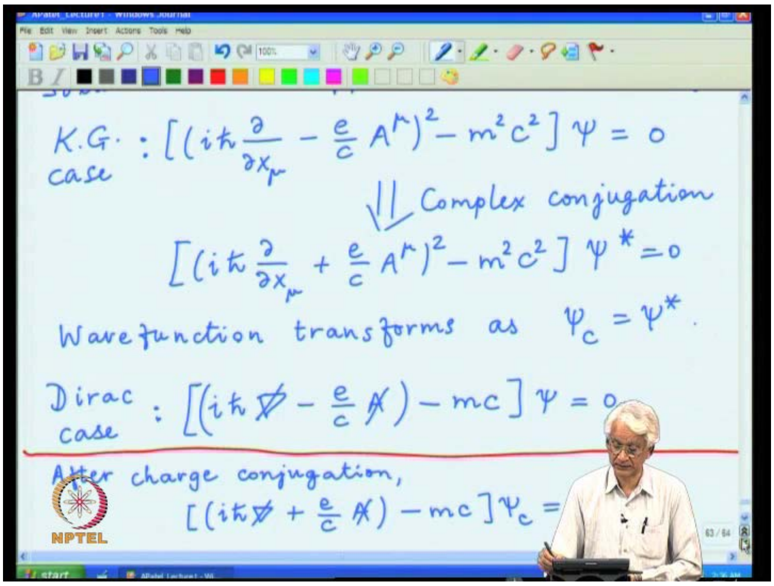

So, I wrote down the Dirac equation here, and after charge conjugation we want a transformed equation which has a plus sign in front of the a slash term, and that is the only difference. So, now we want to find a relation between psi c which is a charge conjugated wave function related to the original wave function psi. So, the first step is to just change the relative sign between del slash and s slash terms. And that is accomplished by complex conjugation exactly the same way as it is done in the case of the Klein Gordon equation. But here something else will happen, because the del slash and s slash operators involve gamma matrices, and they are not necessarily going remain invariant under complex conjugation.

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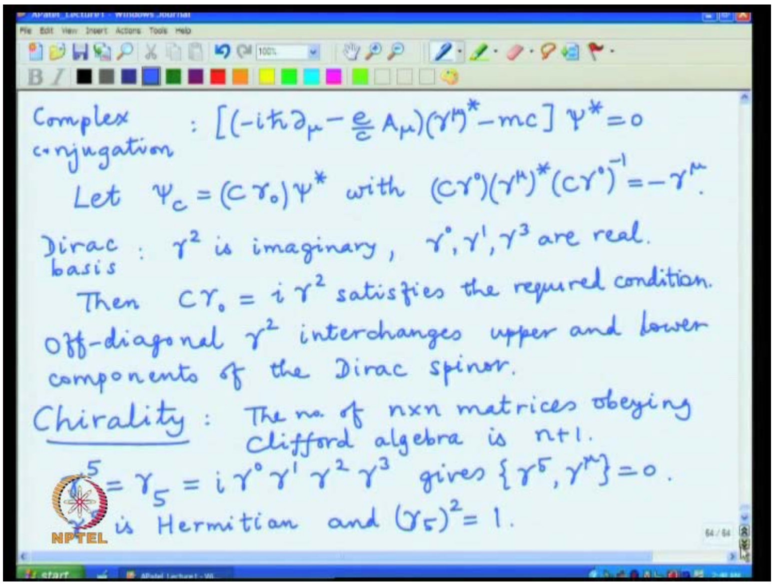

So, let us first write the complex conjugated equation which gives minus i h cross Del mu minus e by C A mu gamma mu star which I have to explicitly separate. So, the relative sign between this d mu and a mu is flipped; and to get back to the original structure. Now, we have to transform this gamma mu star to a new object which will take it back to gamma mu. But there is a negative sign to absorb from this Del slash and s slash term. So, we define a new relation which conventionally is defined as a product of C times gamma 0; where C is some matrix. And it has to satisfy the constraint that C gamma 0, gamma mu star C gamma 0 inverse produces minus gamma mu.

So, now if you operate on this whole equation by this matrix C gamma 0 it will give the equation of for a psi C. And gamma mu star will get converted to minus gamma mu which is exactly the form of the operator required when the sign of the charge is flipped. And so this becomes a definition of the charge conjugated wave function; the question is which kind of matrix will satisfy the particular transformation rule? And now that matrix can be constructed in a specific representation of the gamma matrices; in case of Dirac basis we have the particular structure that gamma 2 is imaginary and gamma 0, gamma 1, gamma 3 are real.

So, the operator should be such that when it acts on gamma 0, gamma 1 and gamma 3 the sign should change. But when it acts on gamma 2 the minus sign is already obtained by the complex conjugation. So, it should be early commute with of gamma 2; of course we know such a matrix and the convention is to so then C gamma 0 is equal to i gamma 2 satisfy the required condition. So, in Dirac basis we have an explicit construction that psi C equal to i gamma 2 psi star obeys the condition the i is used as a convention. And it is basically to ensure that the square of this matrix is essentially 1. And that is very easily seen because gamma 2 when square produces minus 1 due to the Minkowski metric signature and i square is equal to minus 1, and that has become a standard convention. But other than that choice of phase rest of the structure is dictated by the commutation rule required to satisfy the change in sign.

So, this is the construction of a charge conjugation operation in case of Dirac equation. And we can use it to construct solutions which actually correspond to positron it is kind of implicit over here that. So, off diagonal gamma 2 interchanges upper and lower components of the Dirac Spinor; this is kind of necessary in a our interpretation of anti particles which corresponded to the negative energy component of the Dirac Spinor. And in the Dirac basis they were the lower components; if you wanted to bring the anti particle degree of freedom by the charge conjugation operation as the actual solutions you must interchange the upper and lower components. And so the transformation matrix is off diagonal in Dirac basis.

So, this is the useful operation and of course we will connect it to the other symmetries; which are parity and time reversal a little bit later in the course. Now, let me introduce another construction or an operator which is extremely useful. And that is a concept of a what is known as Chirality; at the heart of this construction is a particular fact that the number of n by n matrices obeying the anti computing Clifford algebra is actually n plus 1. So, we have been dealing with the Dirac equation which had 4 matrices all of them have dimension 4 by 4. But this general fact about Clifford algebra tells us that there has to be a fifth matrix which will anti compute with all of the 4 matrices used in the Dirac equation; it is not very difficult to construct this matrix explicitly. And it has been conventionally labeled as a matrix gamma 5.

Now, this 5 is not Lorentz index it is just the fifth matrix; but it is used in the same sense as the other matrices with Lorentz index. And to avoid confusion let me stress that whether you put index up or down does not make any difference. But its definition explicitly is a product of all the other gamma matrices. And this definition ensures that gamma 5 now will anti commute with any one of the matrices involved in the produce there are totally an even number. But when you take one of them and try commutation rule it will commute with itself and not commute with the all the rest see; then the total number is even the once which were anti commute are odd in number and you will get a anti commutation rule.

So, gamma 5 with gamma mu anti commutator is equal to 0. And this works in any dimension as long as there are even number of them; you can always construct a next odd one by such a procedure in case of 2 by 2 matrices we have 3 Pauli matrices following the same rule. And the convention is again is left to fix the phase of this particular matrix and that is introduced here by a construction that gamma 5 is a Hermitian. And gamma 5 square is equal to identity and that fixes the overall phase of this particular matrix. So, this is a simple mathematical construction, but it has non trivial physical meaning. And that is the concept of Chirality which now we will explore to write down some simple relation let us go back to the Dirac equation and construct certain commutators.

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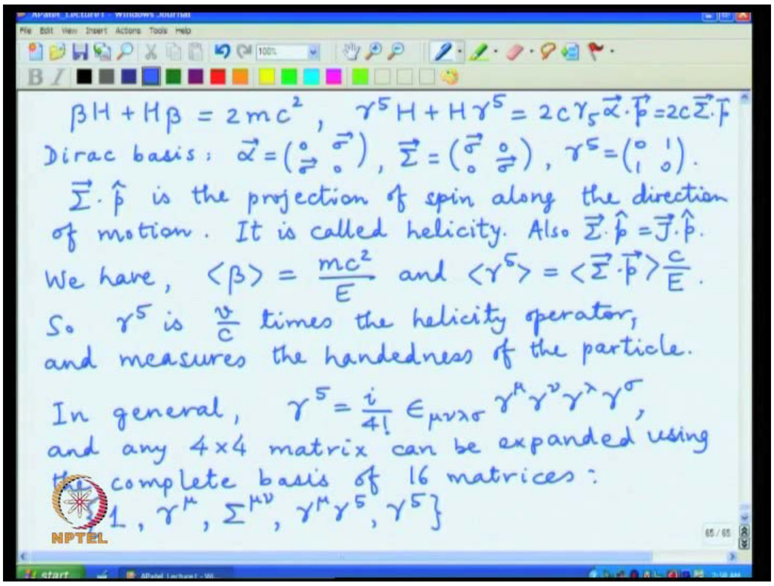

So, first we know for the Dirac equation that the anti commutator of the Hamiltonian with the matrix beta just gives the rest mass. And similarly you can write an anti commutator with gamma 5 which gives now the alpha part; because gamma 5 actually commutes with alpha; alpha is a product of gamma 0 and a spatial gamma matrix. So, there are 2 anti commutation involved in commuting gamma 5 with alpha and that makes it commutes.

So, this picks up the other term of the equation which it can be now written as 2 C gamma 5 alpha dot P. And now we define this matrix gamma 5 alpha by the symbol used earlier for the spin this construction is a identity. Because this is exactly what the spin matrices were defined to be in a Dirac basis alpha was the off diagonal; sigma was the diagonal part and gamma 5 is nothing but an off diagonal identity matrix. So, it is very easy to see that the anti commutator gamma 5 is Hamiltonian produces the structure sigma dot P. But this operator sigma dot P has a meaning and is the projection of spin along the direction of motion to be correct; it has to be a unit vector defining the direction of motion. And we have a name for it is called Helicity and since the orbital angular momentum is always perpendicular to the linear momentum sigma dot P is also equal to the component of total angular momentum along the direction motion. And this object will be conserved in situations where the total angular momentum is conserved.

So, it is useful quantity to take into account and it has a special name and it is called Helicity. So, now this gamma 5 matrix de-produces Helicity structure; and we can give it a meaning explicitly by considering expectation values of these equations written above between Eigen states of the Hamiltonian. So, we have expectation value of the form that beta will give the value m c square divided by E. And by the same logic this will give the expectation value of helicity operators scaled by the energy.

Now, if we put in the explicit relation between momentum and energy you can see that gamma 5 is v by c times the Helicity operator; it will become the helicity operator exactly when the particle is mass less. So, that v is always of the magnitude c. And many times this property of Helicity is also called the handedness; whether the particle is right handed or left handed is decided by the component of its angular momentum along the direction of motion and is gamma 5. Therefore, ends up measuring the handedness of the particle.

So, this is the interpretation of what this gamma 5 does for the particular case of mass less particle we can actually do more things; and I will soon proceed to that. But before that I also want to mention that the gamma 5 can also be used Because of all these property to construct a complete basis for the 4 by 4 Hermitian matrices which appear in the Dirac algebra; there is a general definition of gamma 5 which can be written as the totally anti symmetric tensor multiplying the product of 4 gamma matrices. It produces the same result as product of gamma 0, gamma 1, gamma 2, gamma 3 and the summation of the indices produces a 4 factorial term which is removed in the explicit normalization.

And, one can expand any 4 by 4 matrix can be written in terms of 16 components and we have convenient choice given all these matrices. And that choice is labeled as the identity matrix the 4 gamma matrices; then the 6 anti commutators of the gamma matrices I have earlier call them sigma mu nu then a product of the gamma matrices with gamma 5 and gamma 5. So, this basically gives the number 1 plus 4 plus 6 plus 4 plus 1 that adds up to 16. And it is often common to label these matrices by their transformation properties under Lorentz transformation.

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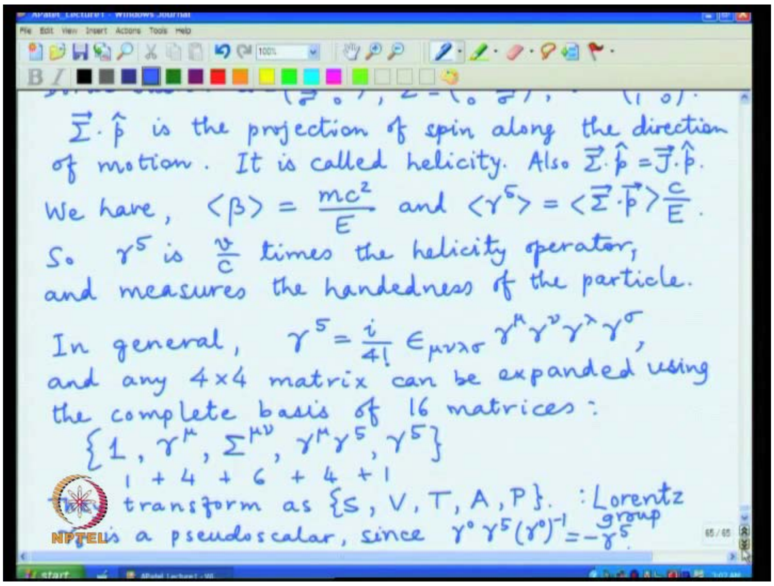

And, that is labeled as a scalar which is the identity, a vector which is gamma mu, a tensor which is sigma mu nu and axial vector which is gamma mu gamma 5; and a pseudo scalar which is gamma 5. So, this is the property under Lorentz group transformation and more generally we can look at it once we have some details about the Lorentz group. But it is at this stage quite straight forward to see that gamma 5 is a pseudo scalar; since it gets a minus sign under parity we have seen earlier that the matrix for parity transformation was gamma 0.

So, we can use the transformation rule for parity transformation and since these 2 anti commute it will produce minus gamma 5. And this flip of the sign tells us that gamma 5 is going to behave as a pseudo scalar; and that is a particular property implicit here. Now, product of a vector and a pseudo scalar will constructs an axial vector with a same vector in the x. But opposite sign of the parity and product of 2 vectors is going to construct a tensor; the details of this will appear in the Lorentz group classification when we come to it.

So, this is one particular feature and definitions gamma 5. And its use for classifying operators as well as relation to the helicity operator; there is another construction which is useful and that is to use gamma 5 as a projection operator. And that is also very a general rule; any matrix which squares to identity is a representation of reflection in some sense or the other. And if you have such a matrix then you can easily construct projection matrices which is 1 plus or minus that matrix the whole thing divided by 2. So, depending on the whether the associated reflection produces a plus sign or minus sign the 2 Eigen states can be separated by that particular matrix. And we can use it for any matrix which squares to 1.

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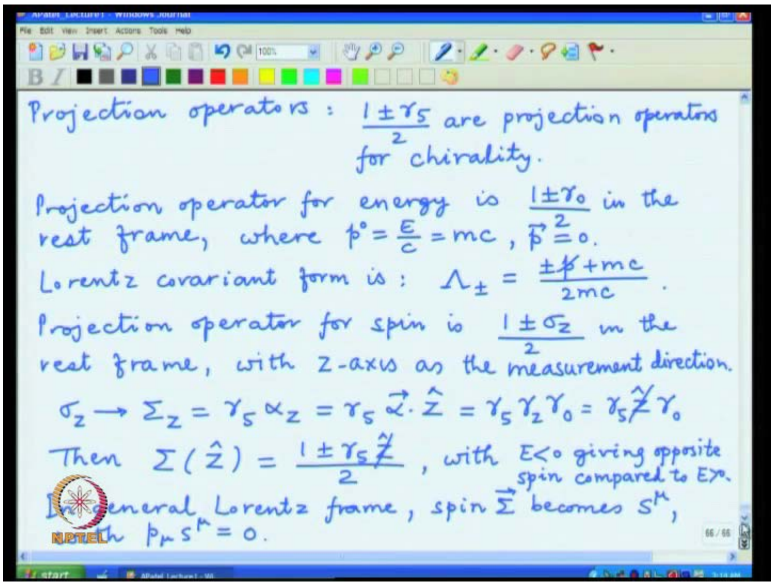

And, we can define projections operators or various cases in particular 1 plus or minus gamma 5 by 2 are projection operators for Chirality; projection operator the property the square of the projection should give the same matrix. And that is always ensured as long as the non trivial matrix appear in here gamma 5 squares to one. And then the other condition is that the sum of all the projection operator should end up being equal to identity. That means, it just covers the complete space but you can now write down many other projection operators also for the various states; which we have used to label the particles, anti particles the corresponding values of the spin etcetera. And these can be easily constructed in rest frame and then rewritten in a form which is Lorentz covariant.

So, the projection operators for energy is essentially 1 plus or minus gamma 0 by 2 in the rest frame. So, the positive energy solutions will have the upper components and 1 plus gamma 0 by 2 as a projection operator. And the negative energy components will have the lower part of the Spinor and 1 minus gamma 0 by 2 as a projection operator. And this is a structure which we explicitly put in the Dirac basis and it was visible in the case of the rest frames.

So, these kinds of structure where some particular matrix appears now can be converted in a covariant language. Because in this particular case we have the rule that the zeroth component of momentum is m c; and the 3 vector part is 0. So, this object which is 1 plus or minus gamma 0 can be rewritten in a Lorentz covariant form. You rewrite this gamma 0 as a dot product of P with a gamma matrix; in the rest frame it always has the standard form because the special parts cannot contribute. But once you have a dot product with P with gamma one can construct a complete operator p slash which has the same value in the rest frame, but it is a Lorentz covariant structure.

So, the projection operator for the 2 signs of energy is plus or minus p slash plus m c by 2 m c; the m c enters. Because I am using p 0 with the usual normalization which m c and so that everything is normalized properly. So, this is the way one can construct projection operators for various cases; one more projection operator is for the spin. And that also can be written as 1 plus or minus sigma z by 2; in the rest frame with z axis as the measurement direction for looking at the spin; it has the again same structure. And now we can convert this sigma z again in some covariant notation; which can help the first step is to take this non relativistic sigma z and rewrite capital sigma z which was the value for the spin for the Dirac equation.

So, sigma z we will convert to this capital sigma z which happens to be the same thing as gamma 5 times alpha z. And that can be written as a dot product between the alpha matrix and a direction of measurement. So, now we have a structure where this dot product can be extended to arbitrary Lorentz frame. Because that is a structure which is invariant; one more thing it is necessary to rewrite these alpha matrix as a gamma matrices which can be done by making this gamma 5 gamma z times gamma 0 which is the definition of alpha z. And then one can dot this gamma z with a unit direction to make it is a slash operator. So, which can be written as gamma 5 then z at slash times gamma 0?

Now, this is a structure where the covariant form in terms of the slash matrix is explicit and there is an extra matrix gamma 0. But we will absorb the sign which is contributed by the gamma 0 in the definition of the Spinor basis. So, an anti particle will correspond to an opposite sign of the momentum. But we will also say that anti particle will correspond to an opposite value of a spin corresponding the particle; with that particular notation we can now define a projection operator for any direction. And that direction now is denoted by this vector z which now can be written as 1 plus or minus gamma z by 2.

And, with these E less than 0 giving opposite spin compared to E greater than 0. So, this is a convention which is useful in identifying anti particles as absence of a negative energy particles. So, the absence flips everything including the spin and this is a then becomes a covariant definition of the spin operator; there is one more thing one can do to generalize the specific space direction which is useful for defining angular momentum only when the particle is addressed to an arbitrary boosted direction. And then in general Lorentz frame the spin gets extended to a 4 vector and not just a 3 vector in the rest frame which gives a standard angular momentum. So, spin sigma becomes a 4 vector S mu.

So, this was a vector in the rest frame and this was this is the 4 vector. But just there is a constraints on the spin because it had only 3 independent components in the rest frame; this also should have 3 independent components. And that can be imposed as a covariant condition which is that p mu s mu is equal to 0; in the rest frame there is p mu has only the zeroth component and so the S 0 vanishes. And that is equivalent to saying that there are only 3 spatial indices surviving to describe the spin. So, this the more general definition of the spin 4 vector and in terms of the 4 vector we can easily generalize this value of sigma as well.

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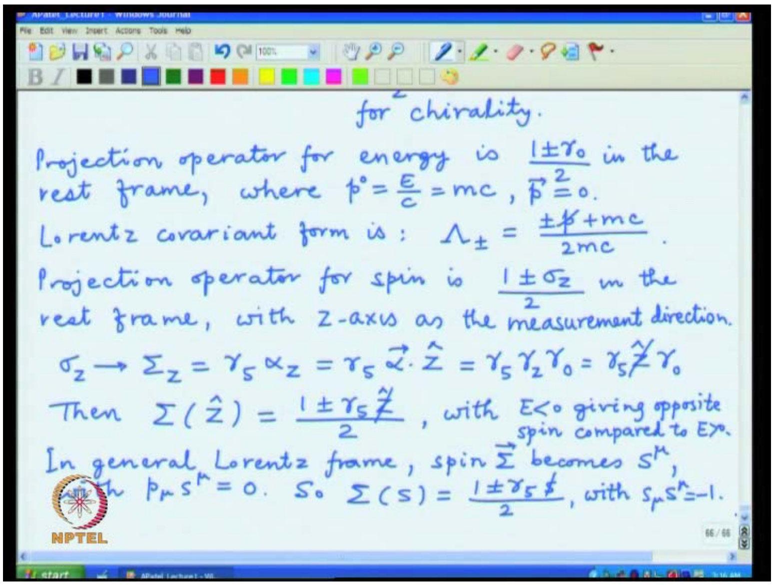

So, sigma for an explicit vector s will now have a structure 1 plus or minus gamma 5 S slash by 2 and S will be a unit vector. So, its normalization in this covariant language is s mu s mu is equal to minus 1. So, that is the general construction of various type of projection operators one can easily construct them one by one from any matrix which squares to 1. And the projection not operators for Chirality, energy and the spin are the most useful one and which I have explicitly constructed. Now, let us go back to the studying the property of this projection operator corresponding to Chirality in more detail.

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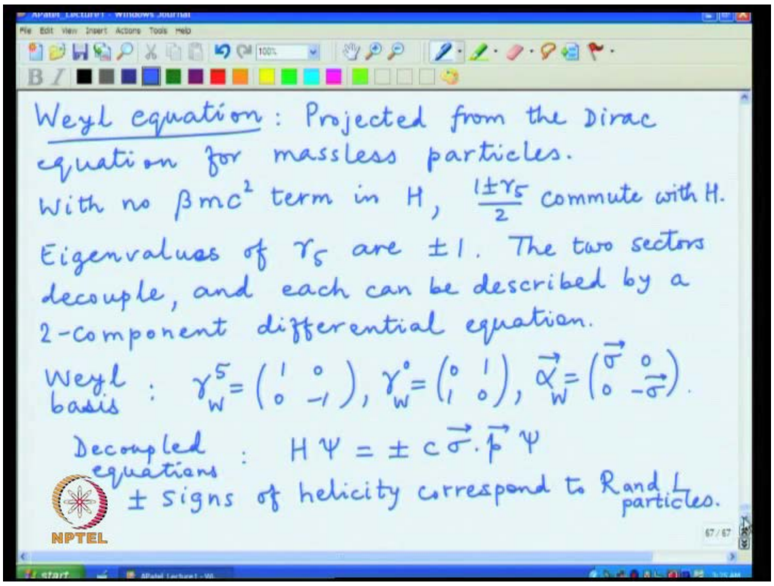

And, in that case it is useful to go to a new equation which is obtained from Dirac equation using this projection operator; and that equation is called the Weyl equation. So, this is projected from the Dirac equation for mass less particles. The mass less particle is the characterizing feature we already know that this projection operators which we constructed have a specific structure in particular gamma 5 commuted alpha but anti commutated with beta.

So, with no beta m c square term in H; what we have is that the operators which we constructed as projection operator 1 plus or minus gamma 5 by 2 commute with the Hamiltonian. The gamma 5 anti commutated with beta n that is why we had to drop this term and consider the situation for the mass less particle. But as soon we have an operator which commute with H we can give it an Eigen value which is conserved. And since these are the projection operators; the complete solution will be just decoupled in the separate sectors corresponding to different values of the Eigen values.

And, we have seen that gamma 5 measure the helicity of the particle. And for mass less particle the helicity is basically just the projection of the spin along the direction of momentum. And it becomes automatically conserved; the fact mass less plays a role in the sense that one cannot boost to a frame in which momentum will reverse the direction. And, so sigma dot p will change its value; for a mass less particle the momentum cannot be reversed because you cannot go to a frame which is faster than the speed of the light. So, this becomes a important conserved quantities and the Eigen values of gamma 5 which ends up with commuting with the Hamiltonian are plus or minus 1.

And, so then the plus 1 value will be projected by 1 plus gamma 5 by 2; and the minus 1 value will be projected by 1 minus gamma 5 by 2. And we can then the 2 sectors decouple and each can be described by a 2 component differential equation in contrast to Dirac equation; which has 4 components. And this 2 decoupled equation each having only 2 components they are called the Weyl equation. And now we can explicitly construct by them choosing a convenient basis. And that basis is called the Weyl basis; it is constructed to make this decoupling explicit from the same Clifford algebra. And what it does is compared to the Dirac basis the definition of gamma 5 and gamma 0 are inter changed.

So, gamma 0 becomes off diagonal but the beta m c square term is absent from the equation. So, it does not play any role gamma 5 is diagonal which has specific Eigen value. So, the upper components will have one Eigen value and the lower component will have the opposite Eigen value. And the other matrices which are needed to complete the equation are the alpha matrices and they have the structure that they all have to mutually anti commute. And it they can be chosen in particular basis as sigma and minus sigma on the diagonal; the commutation rules I can be very easily checked in particular gamma 0 and alpha are going to anti commute; and gamma 5 commutes with alpha.

So, now we can write down the 2 decoupled form then I can put of a subscript w explicitly avoid confusion. So, decoupled equations have the structure that the Hamiltonian operating on psi is going to be alpha dot p the beta part is absent; and alpha has 2 different values for the 2 different decoupled component. So, it is plus or minus sigma dot p times psi and this basis makes it explicit that there are 2 signs of Helicity the sigma dot p operator is very explicit which are appearing as 2 decoupled equations and we will label this plus or minus sign conventionally. So, this plus or minus signs of Helicity correspond to what are labeled as right handed and left handed particles. So, that is a easy solution for separation of the equation once you have made the objects mass less.

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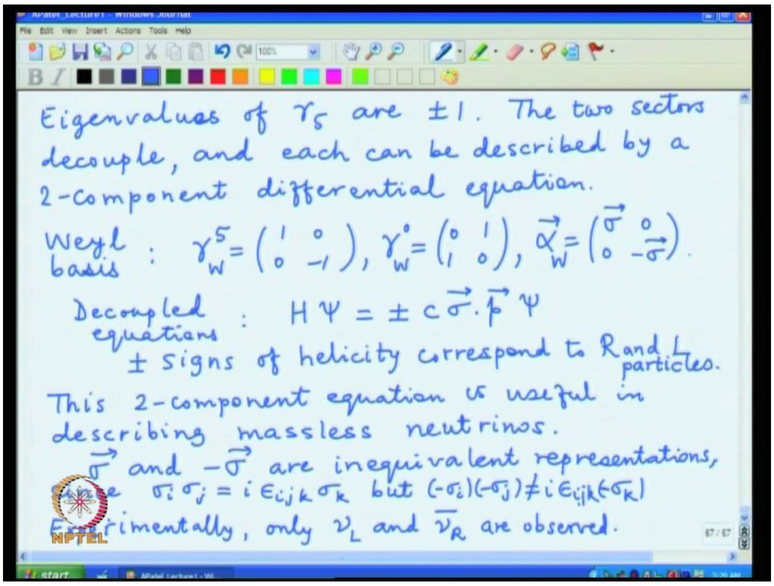

And, so this 2 component equation is useful in describing mass less neutrinos; one should note that this R, N, L corresponding to the opposite signs of Helicity are in equivalent representations. So, the sort of sigma and minus sigma are in equivalent representations seems sigma i, sigma j is i epsilon i, j, k sigma k. But if you put a minus sign the same algebra will not hold rather there will be an overall negative sign; and that is enough to say the which will be one kind of relations.

So, other one will obey the different relation and one cannot convert one of them to the other. So, these R, N, L are physically distinct objects; and they will have a different properties which can be measured by some cleaver experiment or the other. And we will use them in that particular sense; experimentally we find that both these objects do not exist. But only the left handed neutrino and the right handed anti neutrino are observed; this means that out of the 2 particular equations one corresponds to observe neutron and the other one is absent all together. Why this is so is the something which cannot be answered mathematically it is just the property of the nature. But mathematically there is no objection to having one and not the other it just corresponds to the certain degrees of freedom are not physical.

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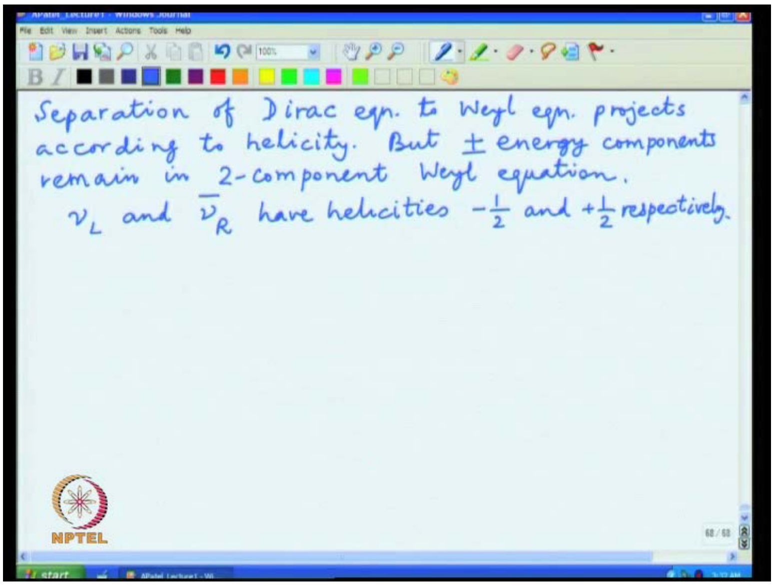

So, this separation of Dirac equation to Weyl equation projects; according to the Helicity but the plus or minus energy components remain in 2 component Weyl equation. So, out of the 4 components we have taken out the spin part essentially or more correctly the

Helicity part. But left the plus or minus energy interpretation untouched. And so we still have the interpretation of a particle and anti particle associated with the plus or minus signs of energy. And in particular this nu L and nu R bar they behave as this positive and a negative energy components one treats the absence of one has the other the Helicity also ends up flipping. And the conventional values for these things are just the same for a spin half particle minus half and plus half. So, this is what one can easily do in case of separating the Dirac equation and finding the corresponding objects labeled them by Helicity. And we have a physical quantity which also describes them.