Applied Optics

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Lecture 07

Matrix Method in Paraxial Optics - ll

Hello everyone, welcome again to my class, this is lecture number 7. And in this lecture, we will learn about matrix method which we have started already in the last class, we will dive deep into this method and see how to model a fraction in matrix or how to write refraction in a 2 by 2 matrix. As you know this is module 2 and we are learning matrix method in paraxial optics. In the last class, we learn how to write the matrix for translation, in translation if we have two points, then we replace this translation with a 2 by 2 matrix. Now, we will try to formulate a matrix for refraction, this matrix is called refraction matrix.

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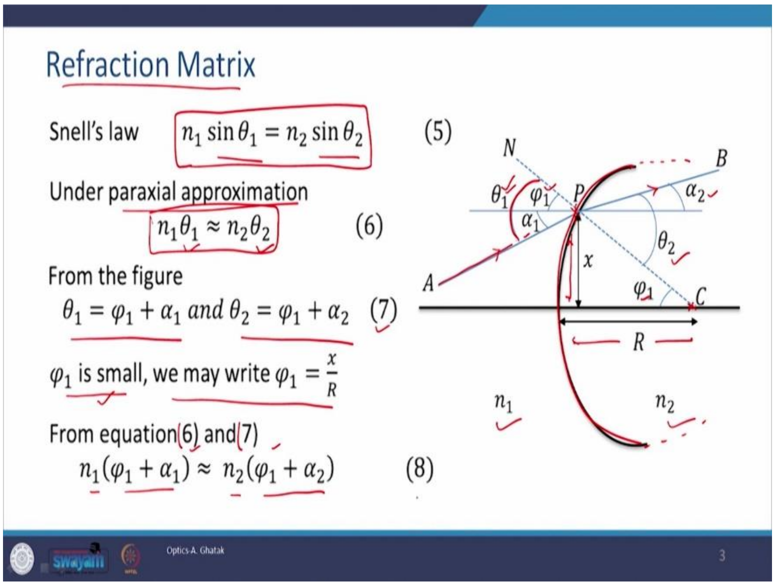

Now, suppose we have a spherical boundary or spherical refracting surface which is separating two media of refractive index n_1 and refractive index n_2 and suppose, an object is placed at point A and the light which is emanating or which is coming from point A it falls at point P on the refracting surface and after getting refracted, it goes to point B this is the after refraction the direction of propagation of the ray is PB. Now, since the refracting surface is a spherical, we can extend it into a circle and this circle has a center C. Now, if we want to draw a perpendicular at point P, we will have to join point P with C and therefore, NC will represent perpendicular at point P.

Once perpendicular is known, you can easily calculate the inclination of the incident ray with a perpendicular which is represented by θ_1 in this diagram and the angle of refraction as is shown in the figure is θ_2. Now, we will apply Snell's law here and Snell's law says that $n_1 sin\theta_1 = n_2 sin\theta_2$. Now, since we are in the paraxial regime, we will use paraxial approximation and therefore, we can write $\sin\theta_1 = \theta_1$ and $sin\theta_2 = \theta_2$. Therefore, under paraxial approximation, the Snell's law reduces to $n_1\theta_1 \approx n_2\theta_2$.

Now, from the figure we can see that $\theta_1 = \varphi_1 + \alpha_1$ where φ_1 and $\theta\alpha_1$ are angle measured from the horizontal. φ_1 represents the inclination of the perpendicular from the horizontal while α_1 represents the inclination of the incident ray from the horizontal. Similarly, we see that the ray after refraction is inclined at angle α_2 from the horizontal. Using this, we can write $\theta_1 = \varphi_1 + \alpha_1$ while $\theta_2 = \varphi_1 + \alpha_2$.

Now, since, we are in the paraxial design φ_1 is a small and therefore, from the figure we can write $\varphi_1 = x/R$. Where x and R are height of point P and the radius of curvature of the spherical surface respectively. Once these are known, then from equation 6 and 7 what we can write is that $n_1(\varphi_1 + \alpha_1) \approx n_2(\varphi_1 + \alpha_2)$ where what we did is that we substituted for θ_1 and θ_2 from equation 7 into equation 6 and thus we get equation number 8.

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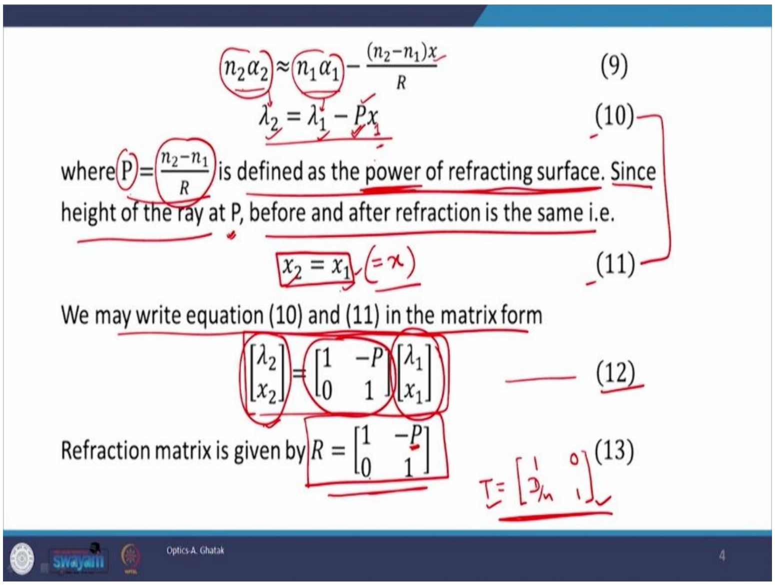

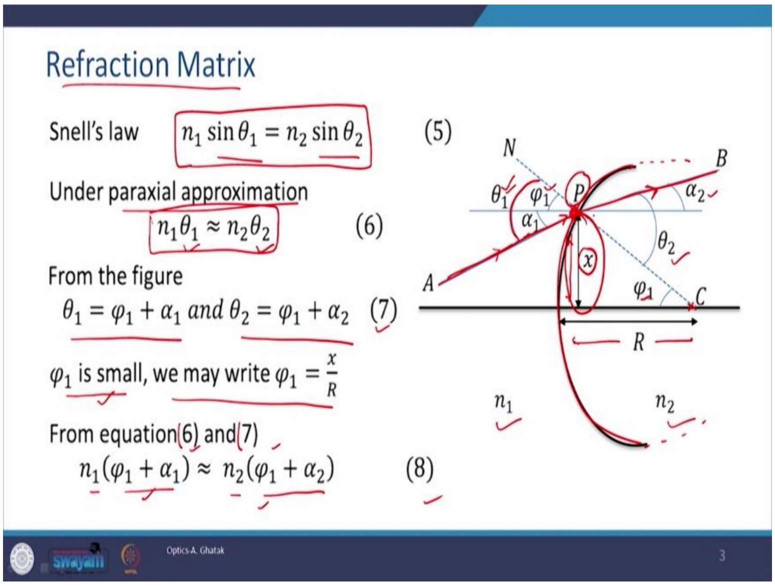

Now, if we expand this, then we will get $n_2\alpha_2 \approx n_1\alpha_1 - \frac{(n_2-n_1)x}{R}$. And we know that $n \times \alpha$ is nothing but direction cosine. Therefore, we can replace $n_2\alpha_2 = \lambda_2$ and $n_1\alpha_1 = \lambda_1$. Where λ_1 and λ_2 are direction cosines after point P. This direction cosines which are λ_1 and λ_2, they respectively represents the direction cosine of the ray before refraction and after reflection, right before refraction and right after refraction.

And we introduce a new parameter P which is expressed as $(n_2 - n_1)/R$ and is defined as the power of the refracting surface. Do not confuse this P with this point P these are 2 different entities, this point is point while this P is a parameter which is defined as $(n_2 - n_1)/R$ and this represent the power of the refracting surface.

Now, since the height of ray at point P before and after refraction is the same. what does it mean is the following: the ray from point A it goes to point P and then as soon as it falls on P it gets refracted and then it goes towards point B, then right before refraction and right after refraction this height x is fixed here. Therefore, what we can write is that x_2 is equal to x_1. What are x_2 and x_1? x_2 and x_1 represents the height of the ray from the axis right after refraction and right before refraction. Since the refraction is happening at point P therefore, x_2 and x_1 would be equal because height of the point P is fixed.

Which is equal to x. Therefore, we may write equation 10 and 11 in the matrix form. We have two equation, equation number 10 and 11. On the left-hand side of 10 and 11 we have λ_2 and x_2 respectively while on the right-hand side we have λ_1 and x_1 and P and x here. We can write, we should replace x with x_1 here for congruence here. Since, x is equal to x_2 and this we may assume that is equal to x because there is only one distance here involved, the height, there is a single height involved height of point P is x, before refraction we are calling it as x_1 and after refraction we are calling it as x_2 and both x_1 and x_2 are the same and they both are equal to x.

Therefore, using equation 10 and 11, we can write this matrix we can represent these two equations in a single matrix form which is given by equation number 12. Where x_1 is equal to x_2 is equal to x. Now, when we write equation 10 and 11 in matrix form, then what we see is that after refraction the coordinates of point P after refraction is related with the coordinates of point P right before refraction and this relation is expressed by equation number 12.

Now, this matrix which is a 2 by 2 matrix is called refraction matrix, always remember this P is the power of refracting surface, whenever it is coming in the form of matrix it is power of refracting surface. Thus, we have a matrix for refraction which is called refraction matrix, we have already derived an expression for translation matrix which is given by $\begin{bmatrix} 1 & 0 \\ D/n & 1 \end{bmatrix}$ here. This we have derived in our previous class and in this expression, D is the distance between two points and a ray travelling between these two points and n is the refractive index of the medium.

Therefore, we have a matrix for translation as well as for refraction. This matrix is called translation matrix while this second matrix is called refraction matrix it represents refraction.

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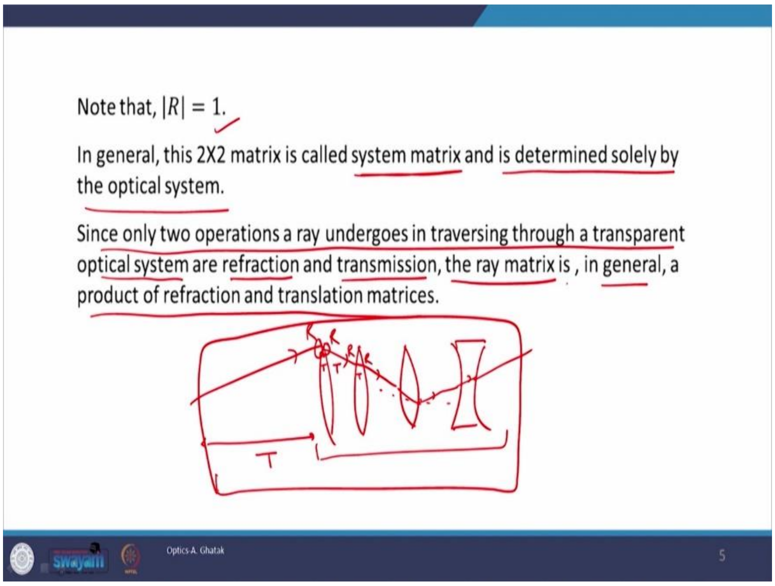

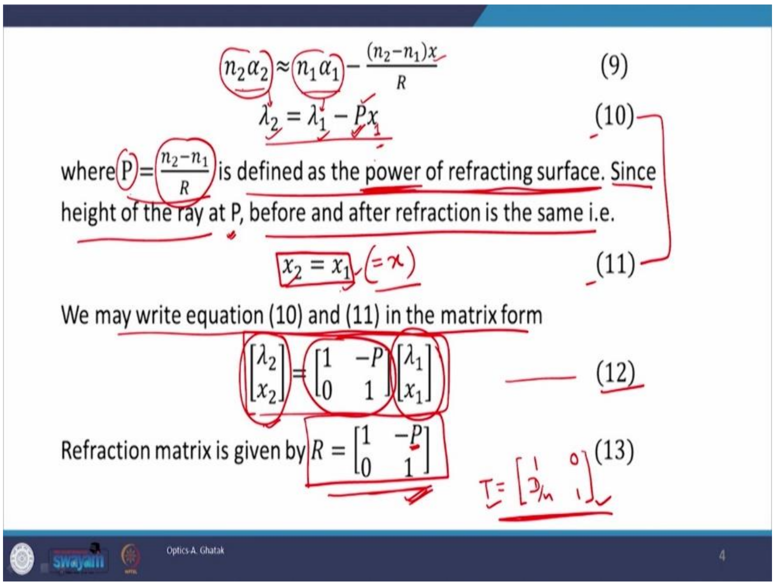

Note here again that if you take determinant of equation number 13, which is your refraction matrix, it is again equal to 1, similar to that the translation matrix, the determinant of refraction matrix is also equal to 1. Now, in general suppose we have a complex system which consists of several lenses, then if you want to calculate the matrix for whole system then you will have to first calculate the matrix for translation and if there is refraction then you have to just put the matrix for refraction and then again if there is some refraction then again put the matrix of reflection and so on and then you multiply all these matrices then you will get a result into matrix which will again be a 2 by 2 matrix and whose determined and again be equal to 1.

And such a matrix is called system matrix. And therefore, in general this 2 by 2 matrix is known as system matrix and it is determined solely by optical system, nothing extra is involved, refraction and translation matrices give rise to system matrix. Now, since only two operations a ray undergoes in travelling through a transparent optical system and what are these two operations, these operations are refraction and transmission. I repeat since only two operations a ray undergoes in traversing through transparent optical systems are refraction and transmission, the ray matrix is in general a product of refraction and translation matrices.

Suppose, we have a number of optical components which are kept in some order and you launch a ray then here it will go through multiple refraction. Now, what you see here is that this is your translation and then here refraction happens and then within the lens again a translation then here again refraction, then again translation, refraction, translation, refraction and so on and so forth. For each refraction and translation, we can write a matrix, for translation, we will write translational matrix translation matrix, for refraction, we will write refraction matrix and multiply them all and the resultant matrix which is called system matrix will have all the information's which all the information's of the phenomena which happen in this region.

All this translation and different refraction information would be embedded in the system matrix which is a 2 by 2 matrix. We will see, we will implement this and we will understand how does it work.

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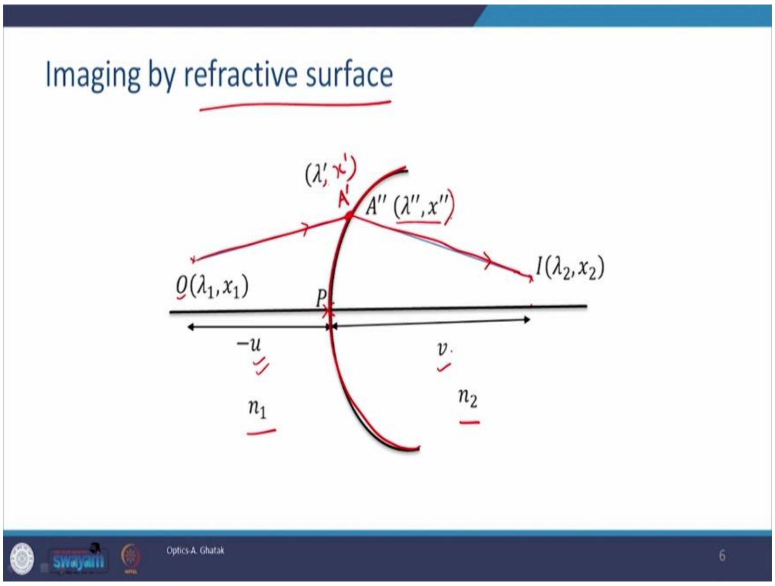

Now, as an example, first consider imaging by a refractive surface. This we have already done; we have calculated, we have seen how refracting surface refract and how it governs the refraction and how to calculate focal length in case of refracting surface or two refracting surfaces. But here in this case, we will implement matrix method for a very simple case of a single refracting surface. Now, this curve represents the refracting surface, this refracting surface divides two media for refractive index n_1 and n_2.

Now, suppose there is a point O from where a ray starts and it falls on a point on a refracting surface and then after refraction it goes towards point I. The coordinates of point O are suppose (λ_1, x_1) and then there is another point whose coordinates are (λ', x') and suppose this point is represented by A' and this A' is on the surface of this curved sphere and it is situated such that on reaching to point A' the ray only undergoes translation and from A' to A" rays undergo refraction and the coordinate of A" is (λ", x").

After emerging at point A" the ray undergoes through a pure translation and through translation it reaches to point I whose coordinates are (λ_2, x_2). I repeat (λ_1, x_1), (λ', x'), (λ", x") and

(λ_2, x_2) are coordinates of O, A', A" and I points respectively. Point O is at a distance minus u from P and point I, the image point is at a distance v from P. P is the point on the axis of this refracting surface the curved refracting surface. Since u is on the left-hand side of P then using our sign convention, we can say that u will be a negative quantity, a negative number while v which is on the right-hand side of P it would be a positive number.

Knowing all this we will write my matrix for this translation, refraction and translation again and we will see how does the resultant system matrix looks like.

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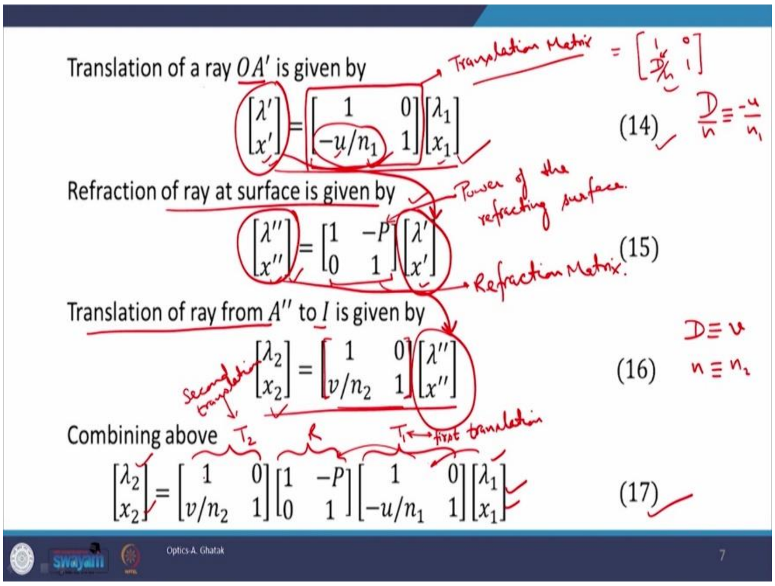

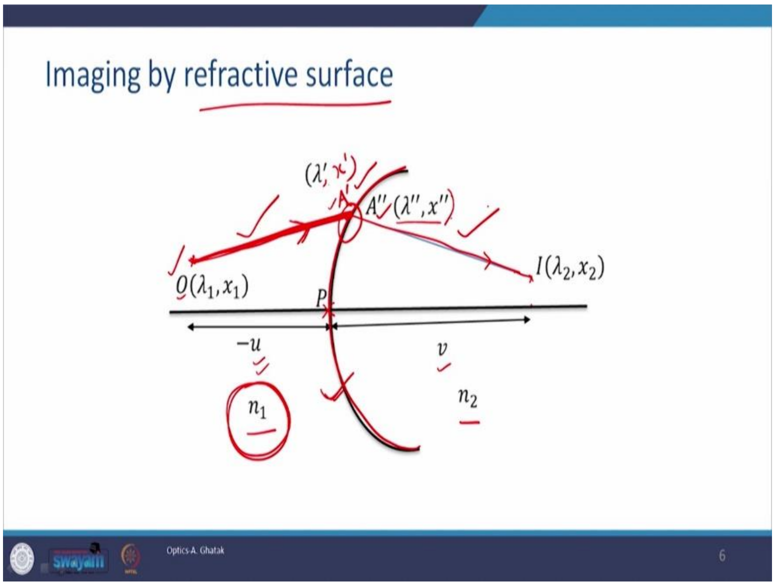

For translation of the ray from point O to A', what are point O and A'?, point O is here and A' is here, for this distance OA' which is a pure translation we will write us matrix which is would be a translation matrix. Now, λ' and x' are the coordinates of point A' and (λ_1, x_1) are the coordinates of point O. Now, these two would be related to this matrix and what is this matrix? This is your translation matrix and we have already derived the equation to connect (λ_1, x_1) and (λ', x') in our previous class.

Therefore, equation 14 represents this connection andthis coordinates of O and A' are related with a translation matrix, we know that the translation matrix has this form here $\begin{bmatrix} 1 & 0 \\ D/n & 1 \end{bmatrix}$ here, D is the separation between horizontal separation between the points within which the ray is travelling. And since the distance here between O and A' is minus u therefore, we will replace D with minus u and therefore, instead of D/n we have -u/n where n_1 is the refractive index after medium as stated before on the left-hand side of this curved surface the refractive index of the medium is n_1.

Therefore, we replaced n in the denominator by n_1. Therefore, D/n is written as -u/n_1 this is element here. Similarly, the coordinates of A" and A' can also be related and since here it is refraction happening, we will use refraction matrix. I repeat the ray starts from O and it goes to A', there is only translation involved, from A' to A" it is refraction and from A" to I it is again only translation. Therefore, for O to A' we will write translation matrix from A' to A" we will write refraction matrix and from A" to I we will again right translation matrix.

Now, for the second case, second case is for refraction we will relate λ" and x" with (λ', x') with a refraction matrix, this is your refraction matrix and here P is your power of the refracting surface, element P is power of the refracting surface and last for the translation from A" to I, we can again write this translation matrix here, the D is replaced by v while n is replaced by n_2 where n_2 is the refractive index of the medium, which is on the right hand side of the refracting surface therefore, we have this type of translation matrix.

Now, we have represented all the translations and refraction in matrix form. What is our ultimate goal? we want to relate (λ", x") with (λ', x'), how to do this? we will substitute (λ', x') from equation number 14 to equation number 15. And this will give the expression for (λ", x") and then the final expression of (λ", x") would be substituted to equation number 16. This will give a relation between (λ_2, x_2) and (λ_1, x_1) and which is given by equation number 17 here.

This relates (λ_1, x_1) which (λ_2, x_2) in matrix form, note the order of the matrices this is your translation matrix, this is your refraction matrix and this is your again translation matrix, this is your first translation, this T is first translation that will let us write it as T_1 and this it would be of T_2 this is for the second translation. What are first and second translation, this is your first translation then refraction here and this is your second translation the same things are represented here in the matrix form.

We have your coordinate of point towards (λ_1, x_1) then it is operated with a translation matrix and then the resultant with reflection matrix and then finally with the again second translation matrix this gives you the resultant expression.

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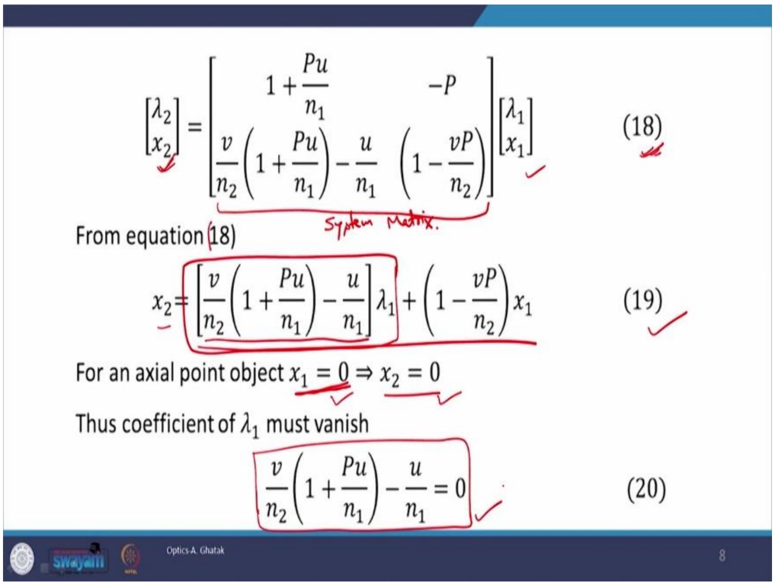

Now, after all these multiplications here, the two translation and one refraction matrices are multiplied and ultimately you get this 2 by 2 matrix, this is your system matrix. Now, system matrix relates to coordinates at point O with the coordinates at point I, the image point. Now, we will expand equation number 18 for the expression of x_2. Now, if we expand 18 for expression of x_2 then we will get this. Now, suppose our object is a point object which is situated on the axis of the system if this is the case then x_1 would be 0 because x_1 is the height of the object from the horizontal axis, from the axis of the system and if it is a point object situated on the axis then this height would be 0.

Therefore, x_1 would be 0 and if our object is a point object which is on the axis of the system and this will lead to a point image and since the image is a point image therefore x_2 would again be equal to 0. Now, once x_1 and x_2 are 0 which is for axial point object and point image then from equation 19, we will get that this term the first term on the right-hand side should be equal to 0 but λ_1 cannot be equal to 0 therefore, the coefficient associated with λ_1 must be equal to 0 and therefore, we will equate this term in the bracket to 0.

Now, if you equate this term to 0 then we will get this relation which is equation number 20. For axial objects, axial point object and axial point image we will get equation number 20.

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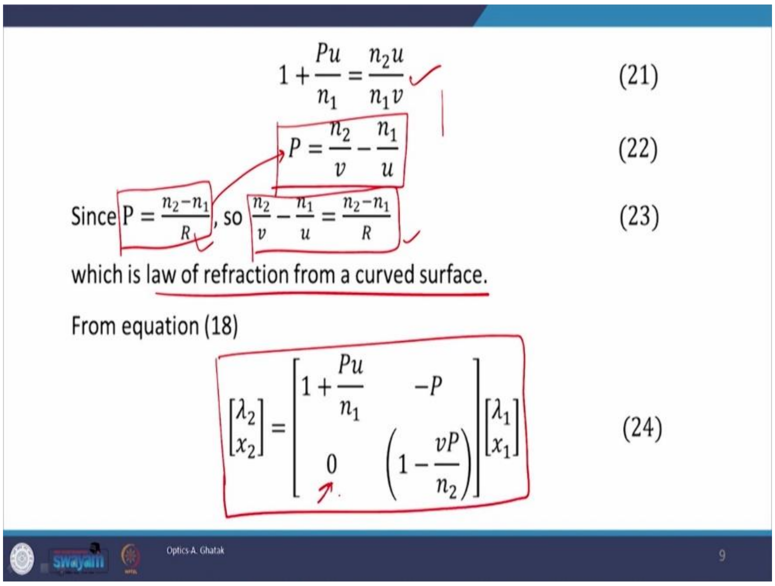

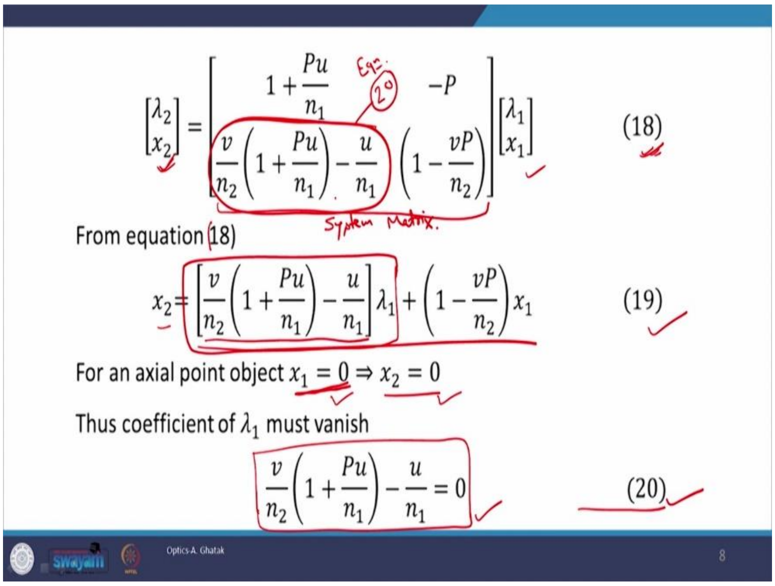

And once, this is given from equation 20. We can after some mathematics we can calculate or we can get an expression of P which is power of our refracting surface. But we know that P is equal to $(n_2 - n_1)/R$, where n_1 and n_2 are refractive indices of the medium 1 and medium 2. The medium which is on the right-hand side of the refracting surface and the medium on the left-hand side of the refracting surface. And n_1 is a medium which is on the left-hand side of the refracting surface and n_2 is the refractive index of the medium which is on the right-hand side of the refracting surface and R is the radius of curvature of this refracting surface or curved surface.

We will substitute this expression of P into expression 22 and this will give equation number 23 here, which is $n_2/v - n_1/u$=$(n_2 - n_1)/R$, which is nothing but law of refraction from a curved surface which we have already derived $n_2/v - n_1/u$=$(n_2 - n_1)/R$ which is a relation which we derived in our previous classes for curved refracting surface which we just implemented Snell's law and from there we derived this relation.

Let us go back again what we did we substituted this term equation, what does equation number 20 represent? Equation number 20 represent this term of the matrix, this is nothing but your equation number 20, we equated this term of the matrix to 0 and this constitute equation number 20. And this gives a valid relation which we derived earlier therefore, this term of this of the matrix would be 0. Once you substitute it to 0 from this equation number 18 we can write this relation, here this term is replaced with 0.

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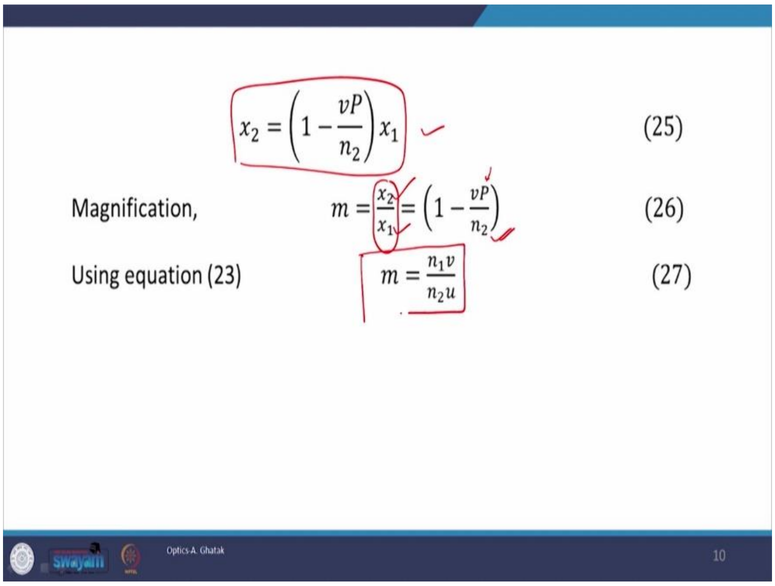

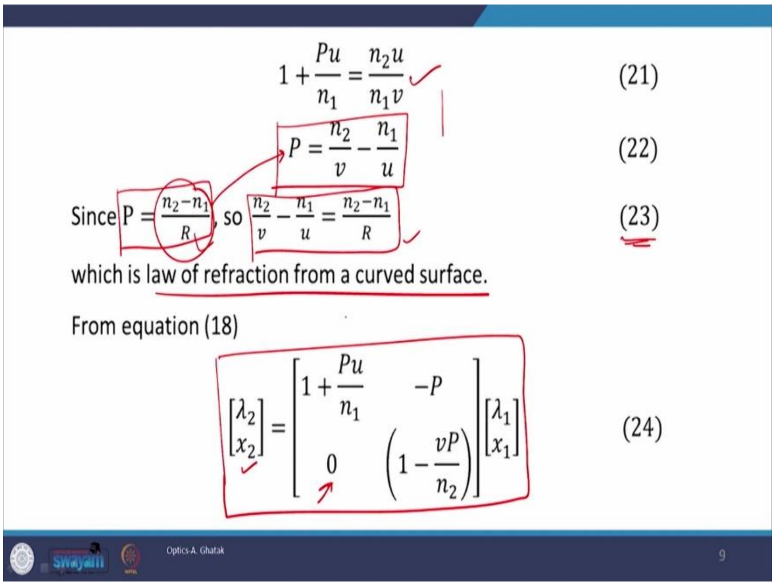

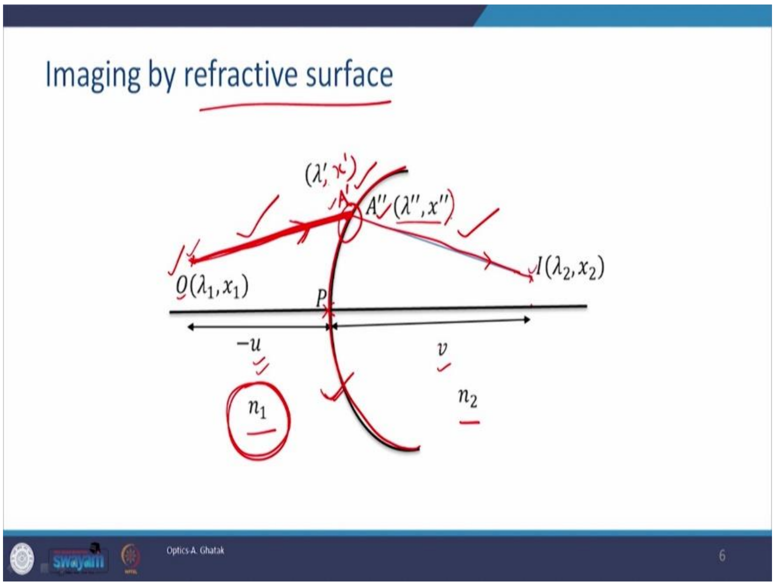

And now again we will expand equation number 24 for expression of x_2 then we will get equation number 25, x_1 and x_2 are height of the object and height of the image. Now, the ratio x_2 to x_1 is nothing but magnification which we have already defined in our earlier classes, we defined magnification as the height of the image upon height of the object and here what is x_2? x_2 is the height of the object, x_1 is the height of the image and from equation 25, we can find this ratio and which is equal to $1-\frac{vP}{n_2}$, where P is again power of the refracting surface.

Now using equation 23 we can simplify equation 26. What is equation 23? This now, we will apply 23 into 26 or we will substitute for the expression of P which is nothing but $(n_2-n_1)/R$ and then we will get this expression for magnification, this is how we can treat refraction from a curved surface using matrix method. We started with point O and then we calculated the magnification and then we calculated also the system matrix. These information should we derived very easily without resorting to the Snell's law.

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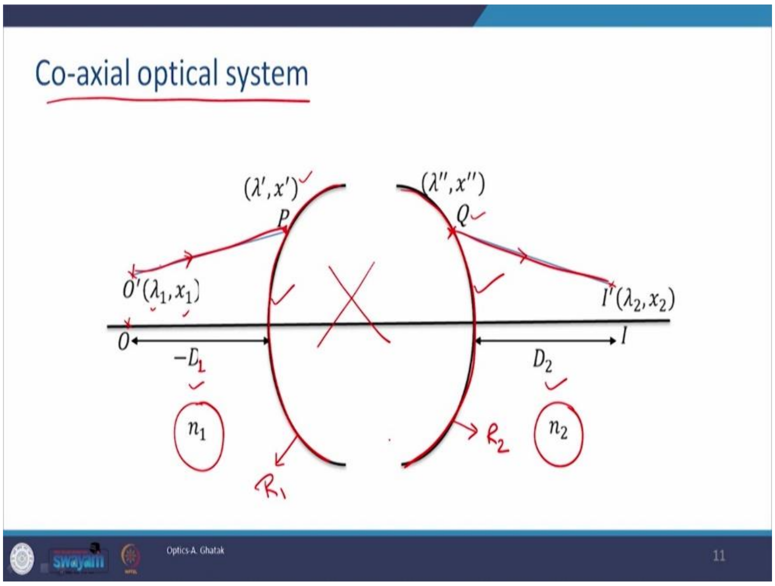

We will again take and another example for and this example is of co-axial optical system. Optical system means instead of having a single refracting surface or some translation, we have suppose a black box with any number of lenses and these lenses of our different kinds, few are concaves, few are concave. Multiple varieties of lenses are kept coaxially in a box and the upper surface of the box is refracting surfaces which are given by these two curves. This left and right curves which binds this optical system. Here, the optical system is within this left and right curve and we do not know what is there inside it, it is like a black box.

Then, now here we will see what help we can get out of the matrix method, what are the given quantities here? There is an object O which is situated here and this object O is at a distance D_1 from the left refracting surface of the optical system. This is the left refracting surface of the optical system, this is the right refracting surface of the optical system and suppose the radius of curvature of the left refracting surface is represented by R_1 and the radius of curvature of the right refracting surfaces represented by R_2. D_1 is the position of the object from the left reflecting surface and D_2 is the position of the image from the right refracting surface.

Now, OO' is the height of the object which is x_1 and the coordinate of O' is (λ_1, x_1) and from O' a ray emanate and it falls on the left refracting surface of the optical system and the coordinate of this point is (λ', x'). We do not know what is happening inside this black box, no information is given, but the ray emanates from point Q here and the coordinate of point Q is $(\lambda'', x'')$and then the point comes at image point I' whose coordinates are (λ_2, x_2).

And the refractive index on the medium on the left-hand side of the optical system is n_1 while the refractive index of the medium on the right-hand side of the optical system is n_2, these information's are given. Now, what to do, we see that from O' the ray is travelling, it is translating to point P then here we will apply your translation. Now, again after emerging from this optical system the ray is again translating from Q to I' with these two translations are known, but within this box what is happening nobody knows and therefore, we cannot form matrix, we cannot form a 2 by 2 matrix for this system.

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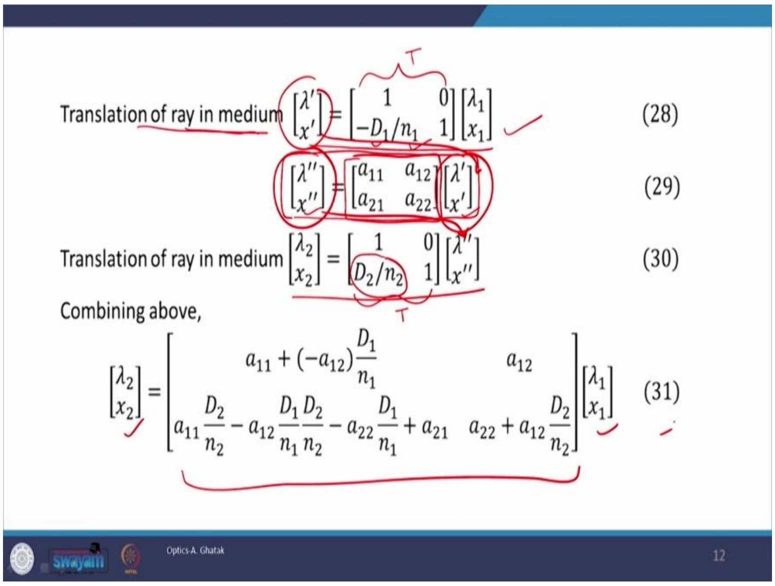

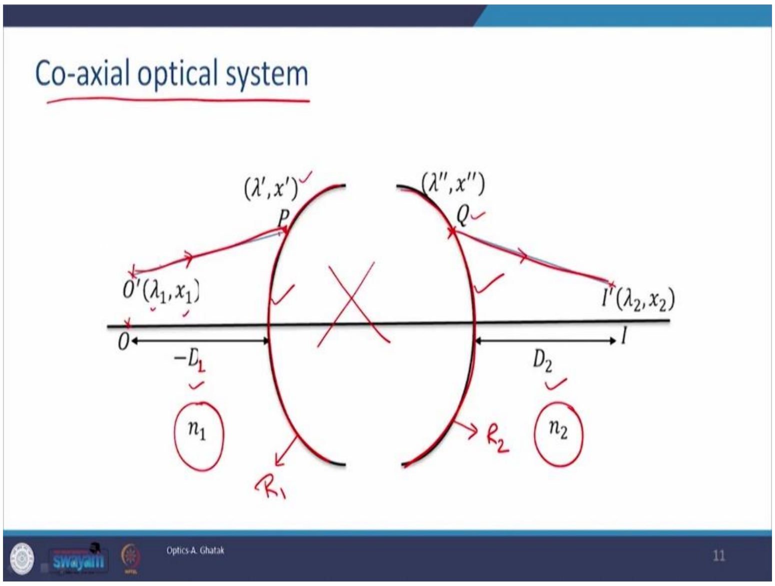

Let us see how to analyze this. For the first translation which is happening in the first medium of refractive index n_1, we can write this equation which is again in matrix form and this is your translation matrix first translation. The refractive index on the left side is n_1 and distance is D_1 therefore, D/n would be replaced by $-D_1/n_1$, minus sign is there because the object is on the left-hand side of the refracting surface. In the black box, in this optical system where no information is given, let us relate the two points what are those two points? Point P and point Q and the coordinates of these two points are related with a matrix again 2 by 2 matrix, but since its elements are not known, we assume these elements of this matrix as a_{11}, a_{12}, a_{21} and a_{22}.

Now, for the third case, which is again translation from Q to I', we can again write this matrix equation where the second translation matrix, this T, can be written very easily where D_2/n_2 is nothing but D_2 is the distance of the image from the right to refracting surface and n_2 is the refractive index of the medium which is on the right side of the optical system. If we know this then we will combine to equation 28, 29, 30 as we did previously. how to combine this? this would be substituted here in equation number 29 and once the expression for (λ'', x'') is known, this would be substituted here in equation 30 this will ultimately relate (λ_2, x_2) with (λ_1, x_1) with this complex matrix, 2 by 2 complex matrix which is given by equation 31. And what are unknowns here? Unknowns are a_{11}, a_{12}, a_{21} and a_{22}.

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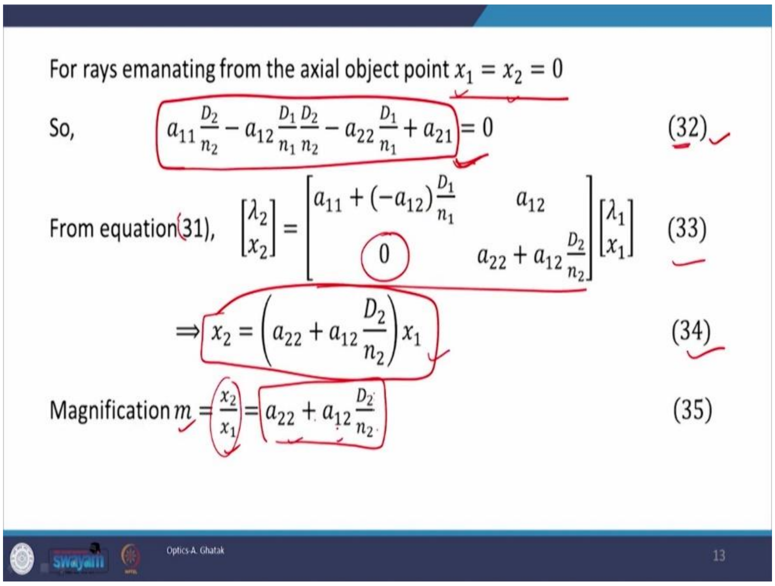

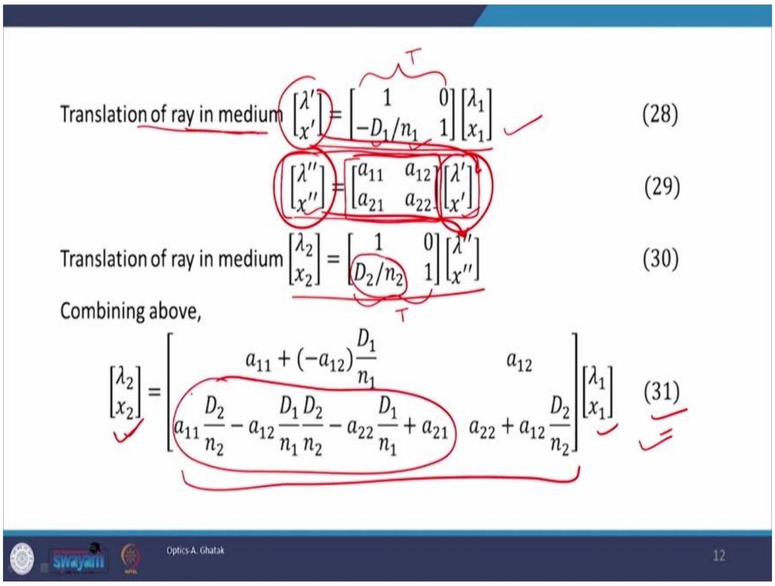

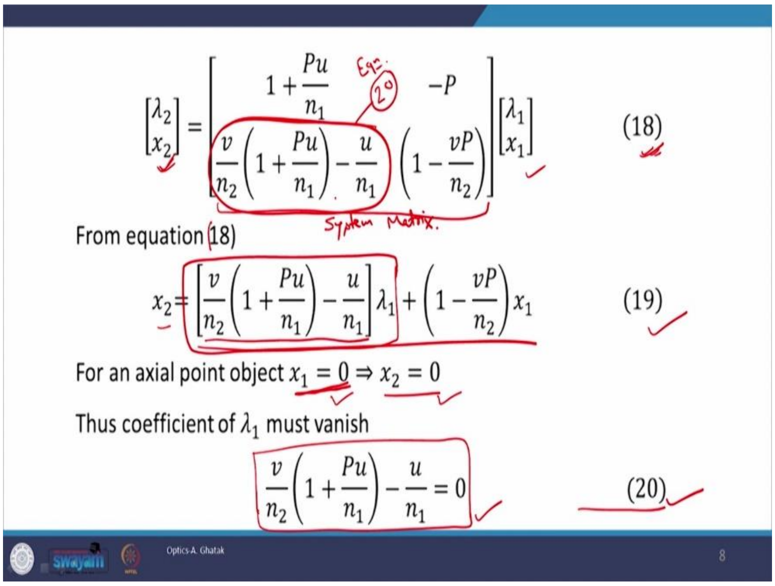

Let us proceed further. As a boundary condition we will again use this suppose, the object is a point object and image is also a point image and they both are axially situated. They are axial point object and axial point image, if this is the case then x_1 and x_2 would be equal to 0, here both x_1 and x_2 will be equal to 0. And if you substitute so, and expand equation number 31 for x_2, the expression of x_2 as we did previously then the coefficient of λ_1, this will be the coefficient of λ_1 and this must be equal to 0.

We are just repeating this step which we did previously here, we are having a equation number 18 then we expand at 18 to get an expression for x_2 and this term the coefficient of λ_1 must be equal to 0, the same thing is being repeated here. This is the coefficient of λ_1 which we now have equated to 0 which is our equation number 32. Then the equation 31 which is our master equation here, this term would be equal to 0, since it is equated to 0 we will get equation number

32. Here, this term is 0 now, we will again expand equation 32 to get an expression of x_2 and this will lead to equation number 34.

Now, let us again calculate as we did previously, let us again try to find the expression for magnification which is equal to x_2/x_1, the height of image to the height of object, this ratio is magnification. Now, from equation 34 if you calculate x_2/x_1 then you get this expression which is $a_{22} + a_{12} \times D_2/n_2$, this is the expression for magnification. Once we know what magnification is now here which is $a_{22} + a_{12} \times D_2/n_2$.

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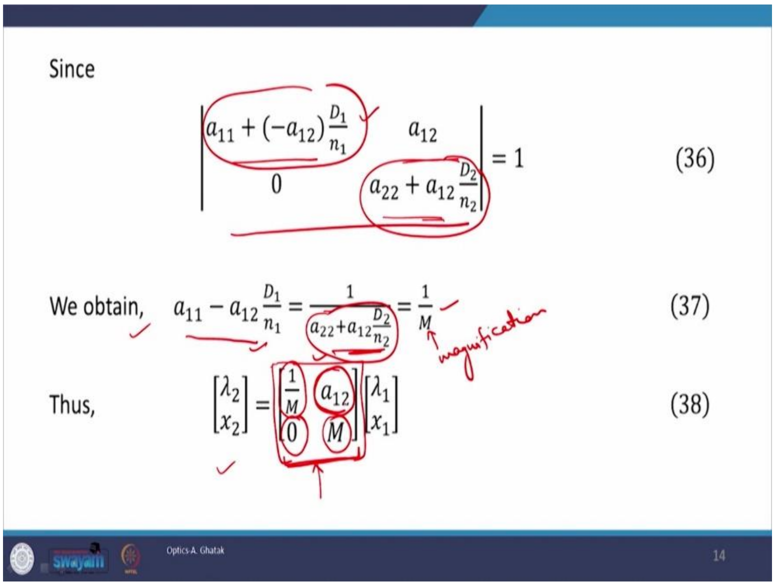

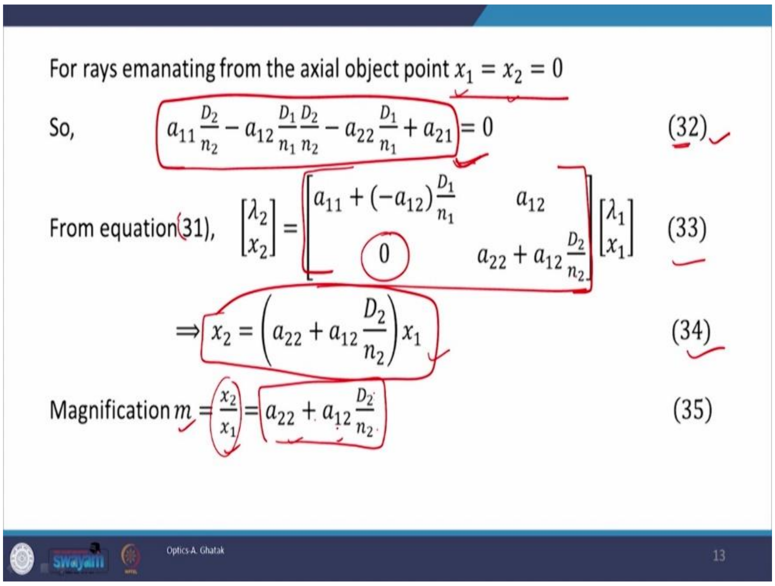

And there is one more condition on this matrix, this is our matrix and on this matrix which is a system matrix the second condition is that it its determinant must be equal to unity. Now, let us exercise the second condition which is the determinant of the system matrix must be equal to unity then what we get is the following we obtain this expression is equal to 1 by this expression. What it is written here? But this term is equal to magnification therefore, we can write:$1/(a_{22} + a_{12} \times D_2/n_2) = 1/M$, M is nothing but your magnification.

It means the first term of the after matrix is equal to magnification and inverse of the second term, sorry, first term is equal to inverse of magnification and the fourth term is equal to magnification and therefore, the resultant form of the system matrix would be something like this, this is the final form of the resultant matrix.

Now, conclusively we started with a system matrix where there were so many unknown terms which were a_{11}, a_{12}, a_{21} and a_{22}. Now, after analyzing the system using matrix method, we found a simplified matrix whose first term is inverse of magnification, last term is magnification, this term is equal to 0. Only unknown is this term a_{12}. a_{12} is only unknown we are left with. Now, we will see like in we will study a few more topic and see what a_{12} represents.

But for now, we know that if a very complex optical system is given, then we can form its system matrix. When can we form its system matrix? Before if we know what is the position of object, what is the height of the object and if we know what is the position of the image and what is the height of the image then with this information, we can find three element of the 2 by 2 system matrix, only one unknown is left, this is the beauty of the matrix method and for this unknown, we will do further analysis and we will see what happens. And this is all for today. We will talk more about the implementation of the matrix method in the next class.