Applied Optics
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Lecture: 32
Fraunhofer Diffraction

Hello everyone, welcome back to my class. In the last class we started module 7, wherein we talked about diffraction. We observed the similarities between interference and diffraction, and then we talked about the different classes of diffraction, wherein we saw that diffraction is divided in two parts the first is called Fraunhofer diffraction and the second is called Fresnel diffraction.

Now today, we start Fraunhofer diffraction and just to revise Fraunhofer diffraction is the class of diffraction, wherein both the source and the screen is effectively at infinity from the diffracting element or from the aperture plane.

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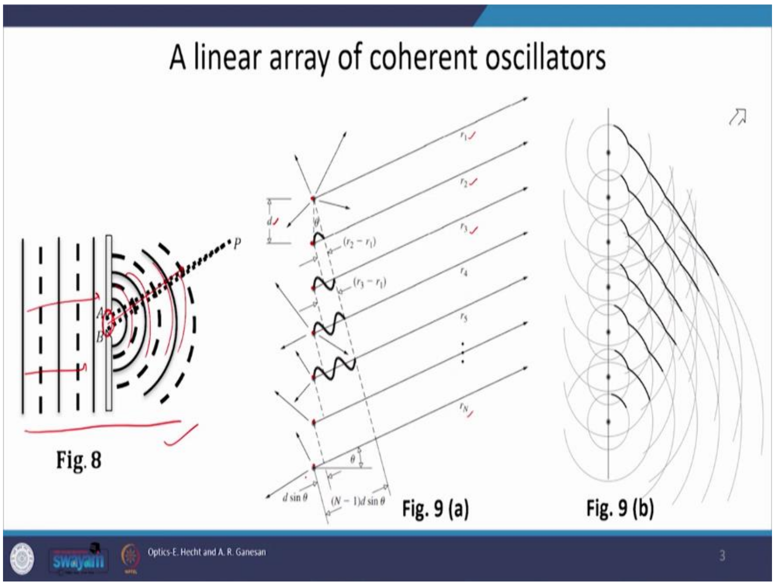

Now this is the usual case which we observe in with the aperture, we launch a plane wave here and then due to the aperture we see the formation of a spherical wave front and we see that light bends. This bending usually name as diffraction. Now in Fraunhofer domain, we will generalize this case, we know that in the Huygen's principle that each point on the wave front works as a secondary source.

Now if the light is being launched on the aperture, then each point on the aperture will behave as a source of secondary wavelet. Now to generalize this further here we assume that we have a large number of point sources which are given here and all these point sources are separated by a distance d.

And the point sources are very closely spaced and the observation point is very far from these point sources. And therefore, we may assume that these point sources, they are in phase and the observation point is, point is at a distance r_1 from first source, r_2 from second source, r_3 from third source and r_N from N^{th} source.

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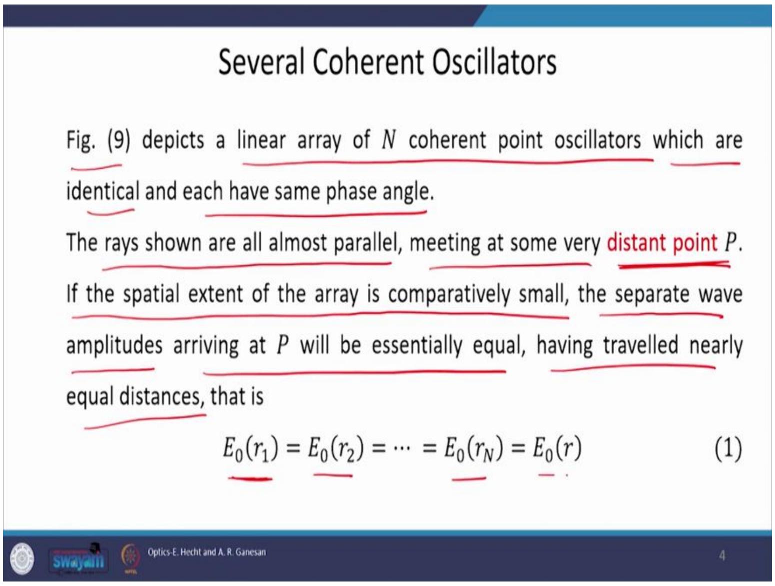

Now this is what is written here in the figure 9, in the last figure, it depicts a linear array of N coherent point oscillator which are identical and each have same phase angle. The ray shown are almost parallel meeting at some very distant point P, make it a point observation point P is very far from these point sources or point oscillators. If the spatial extent of array is comparatively small, thus the separate wave amplitudes arriving at P will be essentially equal, having travelled nearly equal distances.

Now, suppose in this plane, we have the point sources whose spatial extent like all these points are just lying within this distance and the observation point is so far from this arrangement that this distance is negligible as compared to the distance from this point sources to the point of observation P. In that particular case we may assume that each individual source emits a ray which is reaching at point P and the amplitude from each source which is reaching at P is same.

And we also assume that all these point sources they are in the same phase. In this circumstances, $E_0(r_1)$ which is the amplitude, which is received at point P from first point oscillator is equal to that of second point oscillator is equal to that of N^{th} point oscillator and we assume that all these are equal to $E_0(r)$.

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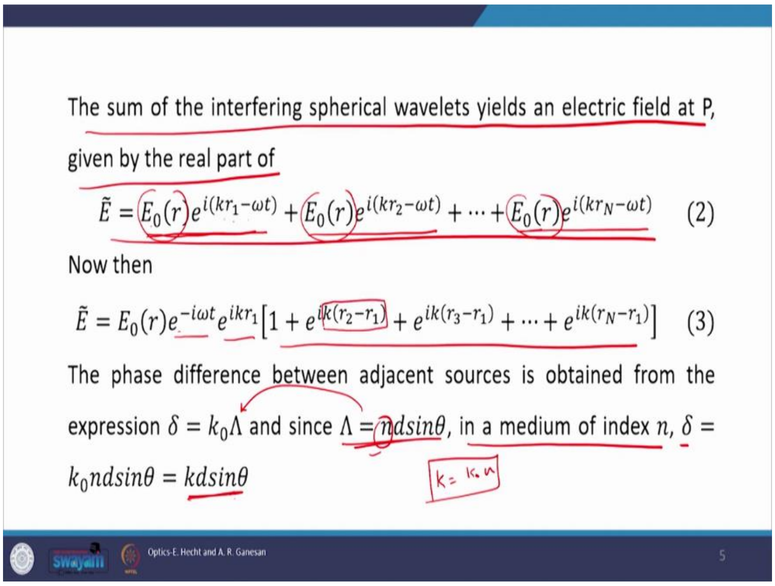

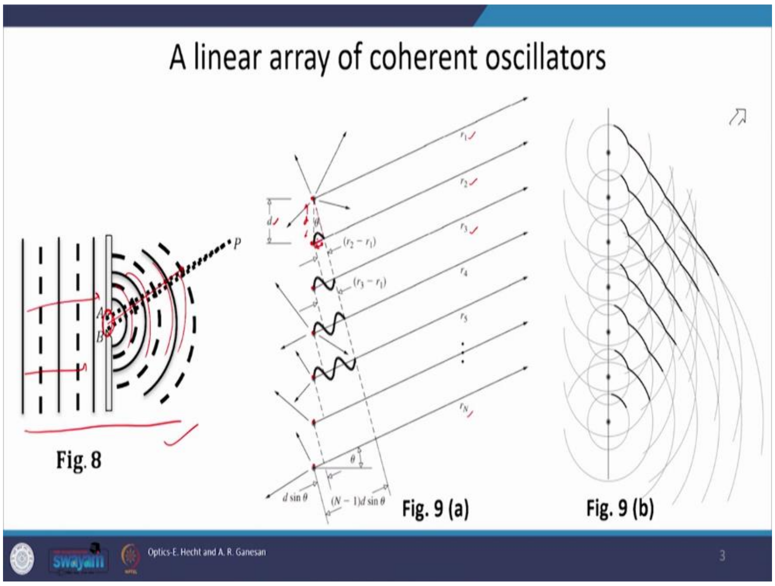

Now the sum of interfering spherical wavelets yield an electric field at P given by real part of this expression. What does this expression represent? This expression in complex representation represents the field received at point P from all these N number of point oscillators. The first oscillator emits $E_0(r)e^{i(kr_1-\omega t)}$, similarly, the second emit this and the last one emit this.

Since we already assumed that the amplitudes are the same therefore these quantities are same here. Now let us take it for $e^{-i(\omega t)}$ and $e^{i(kr_1)}$ out of the bracket, then we are left with this series. Now let us calculate the phase difference between adjacent sources. Now we know that the adjacent sources are separated by a distance d. And how to calculate the phase difference? Phase difference is k_0 into path difference. And the path difference from the figure, here we know that the separation is d here.

And then the path difference would be this. Where this is the perpendicular which we draw from first ray path to the secondary path. And this would be equal to dsinθ, where θ is this angle. With this, we can write that optical path difference between the adjacent rays is $ndsin\theta$, we will substitute it here to get phase difference, and we also assume that this point oscillator are kept in a medium of refractive index n. Therefore, n is appearing here, therefore the ultimately the expression for phase difference is $kdsin\theta$ where $k = k_0 n$. Now once this phase difference is known we can write it here, $k(r_1 - r_2)$ is the phase difference.

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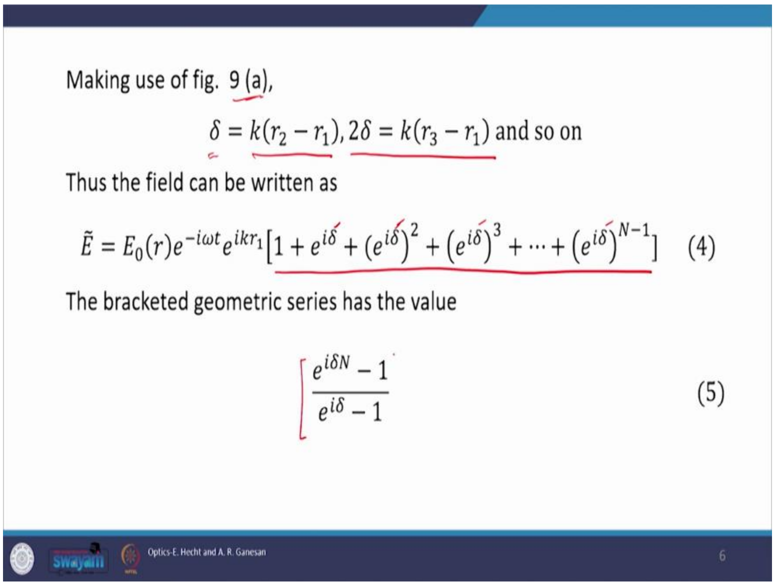

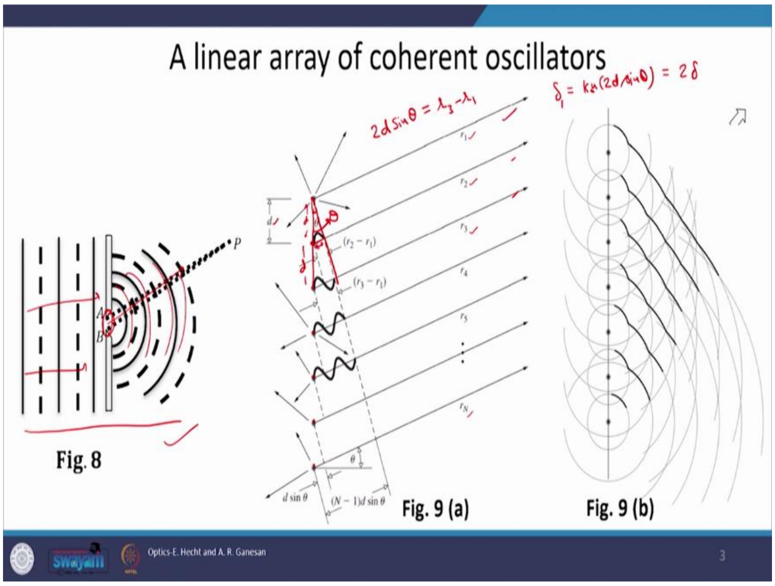

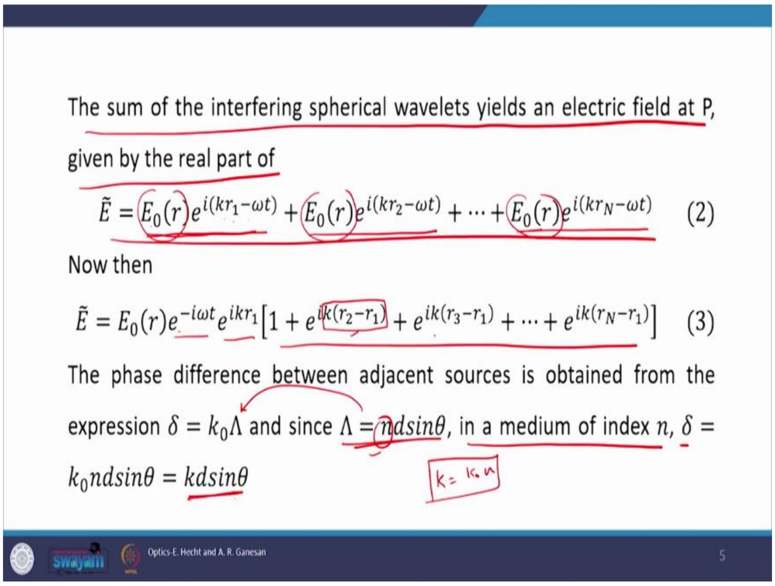

Therefore, from equation, from figure 9 (a) phase difference is $k(r_2 - r_1)$. But what would be the phase difference between the first ray and the third ray? Let us go back into the figure, this is the first ray and this is the third ray. To calculate the phase difference, we draw perpendicular here, we join these two lines and angle here is again θ and now its total distance is 2 d. Then the path difference would be 2d sinθ, which is $r_3 - r_1$. Therefore, the corresponding phase difference would be $k_0 2dsin\theta$. Let us call it δ_1, which would be equal to twice δ.

δ is nothing but it is k_0 d sinθ but if you also take the refractive index of the medium then it would be quite better because refractive index also plays a role in defining the path difference. And this is what is done here. The $2\delta = k(r_3 - r_1)$, this is the phase difference between first and three. Similarly, between first and four you will get $k(r_4 - r_1)$. And therefore, you can replace the phases in occurring in this series.

The first will be δ, second would be 2δ , third would be 3δ, and last would be (N-1) δ. Now you see that this series in the bracket, it is nothing but GP and we know the formula to add them up. If you add them up you get this term, it is for $e^{i(\delta N\text{-}1)}/e^{i(\delta-1)}$.

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The bracketed term can be rearranged into the form

$$\frac{e^{\frac{iN\delta}{2}}\left[e^{\frac{iN\delta}{2}} - e^{\frac{-iN\delta}{2}}\right]}{e^{\frac{i\delta}{2}}\left[e^{\frac{i\delta}{2}} - e^{\frac{-i\delta}{2}}\right]} \quad (6)$$

which can equivalently be written as $e^{\frac{i(N-1)\delta}{2}}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right)$

The field then becomes

$$\tilde{E} = E_0(r)e^{-i\omega t}e^{i[kr_1+(N-1)\delta/2]}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right) \quad (7)$$

Notice that if we define R as the distance from the centre of the line of oscillators to the point P, that is

$$R = \frac{1}{2}(N-1)d\sin\theta + r_1 \quad (8)$$

Then the field becomes

$$\tilde{E} = E_0(r)e^{i(kR-\omega t)}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right) \quad (9)$$

Finally, then, the flux density distribution within the diffraction pattern due to N coherent, identical, distant point sources in a linear array is proportional to $\tilde{E}\tilde{E}^*/2$ for complex E and is given by

Making use of fig. 9 (a),

$$\delta = k(r_2 - r_1),\ 2\delta = k(r_3 - r_1) \text{ and so on}$$

Thus the field can be written as

$$\tilde{E} = E_0(r)e^{-i\omega t}e^{ikr_1}\left[1 + e^{i\delta} + (e^{i\delta})^2 + (e^{i\delta})^3 + \cdots + (e^{i\delta})^{N-1}\right] \quad (4)$$

The bracketed geometric series has the value

$$\left[\frac{e^{i\delta N} - 1}{e^{i\delta} - 1}\right] \quad (5)$$

Let us simplify this a bit more then we can take $e^{iN\delta/2}$ out of the bracket from the numerator and from the denominator, we take it power $i\delta/2$ out of the bracket, with this the bracketed term get rearranged and we get equation number 6. This expression is familiar to us and then we know that this is nothing but $(e^{i\theta} - e^{-i\theta})/2i$.

And then what is this? This is your $(cos\theta + isin\theta - cos\theta + isin\theta)/2i$ and this is equal to $sin\theta$. Similarly, with the denominator then we will exercise this formula here. And this will result, this will result this term. This whole term will now reduce to this term because this bracket term in the numerator will reduce down to $sinN\delta/2$. Similarly, bracket bracketed term in the denominator will reduce down to $sin\delta/2$. And this is nothing but $e^{i(N-1)\delta/2}$.

This we got after solving the bracketed term in equation number 4. Apart from this bracketed term we have this extra multiplicative term. Let us take them also into account and then the final expression for the field at a point of observation now looks like this, which is given by equation number 7.

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Notice that if we define R as the distance from the centre of the line of oscillators to the point P, that is

$$R = \frac{1}{2}(N-1)d\sin\theta + r_1 \qquad (8)$$

Then the field becomes

$$\tilde{E} = E_0(r)e^{i(kR-\omega t)}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right) \qquad (9)$$

Finally, then, the flux density distribution within the diffraction pattern due to N coherent, identical, distant point sources in a linear array is proportional to $\tilde{E}\tilde{E}^*/2$ for complex E and is given by

A linear array of coherent oscillators

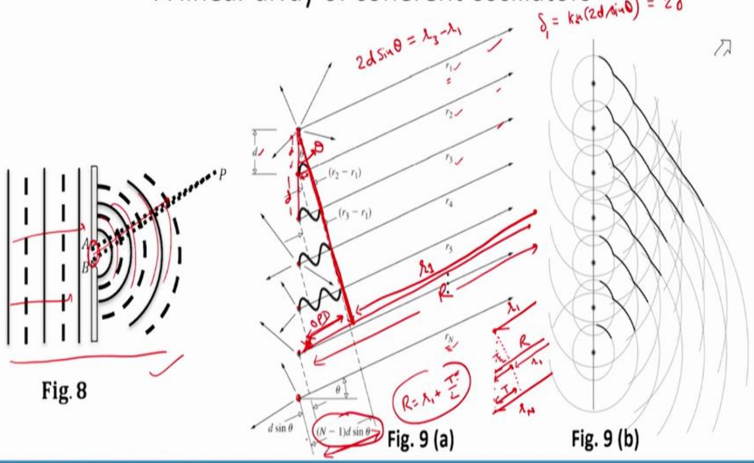

Fig. 8 Fig. 9 (a) Fig. 9 (b)

The bracketed term can be rearranged into the form

$$\frac{e^{\frac{iN\delta}{2}}\left[e^{\frac{iN\delta}{2}} - e^{\frac{-iN\delta}{2}}\right]}{e^{\frac{i\delta}{2}}\left[e^{\frac{i\delta}{2}} - e^{\frac{-i\delta}{2}}\right]} \qquad (6)$$

which can equivalently be written as $e^{\frac{i(N-1)\delta}{2}}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right)$

The field then becomes

$$\tilde{E} = E_0(r)e^{-i\omega t}e^{i[kr_1+(N-1)\delta/2]}\left(\frac{\sin N\delta/2}{\sin\delta/2}\right) \qquad (7)$$

Now, if we define R as the distance from the center of the oscillator to point P then what would be the expression of R. Let us go back into the first figure. Now here, let us pick some point which is center. Let us say that it is the center here and from the center the point P is at a distance R.

Then if we have to draw to calculate the path length difference, we will have to draw a perpendicular from here on R and as soon as a draw perpendicular from the first oscillator on R then this distance will be equal to r_1 and this is will be your extra path, this is optical path length difference. Hope this is clear.

Now since this point source is at the center of this oscillator array therefore this optical path length difference would be half of the optical path length difference between r_1 and r_n. This is our array and r is at the center this is your r, this is your r_1, this is your r_{N} and we drew this perpendicular from here to here therefore this distance we say as r_1 and this optical path length difference this would be half of this, if this is total then it would be total by 2.

Therefore, R would be r_1+T/2. What is T? T is nothing but $(N-1)dsin\theta$. Therefore, the expression of R would be this. This is $(N-1)dsin\theta$, which is optical path length difference between first and the last N^{th} oscillator and we take half of this because r is the distance from the center of the line oscillator to the point P and r_1 is the distance of the P from first point source.

Since we drew the perpendicular, then before the perpendicular we have OPD, after the perpendicular we have r. Therefore, the addition of these two term will give us R. Now this term, the right hand side of R, is appearing here in equation number 7. Now let us replace these things with R and the field expression get modifies and we get this as equation number 9. Here you see that R is appearing in the exponent.

Once the field is known we need to calculate the flux density or irradiance, how to calculate this. The flux density distribution within the diffraction pattern due to N coherent identical distant point sources in a linear array would be proportional to $\tilde{E}\tilde{E}^*/2$, this we know already.

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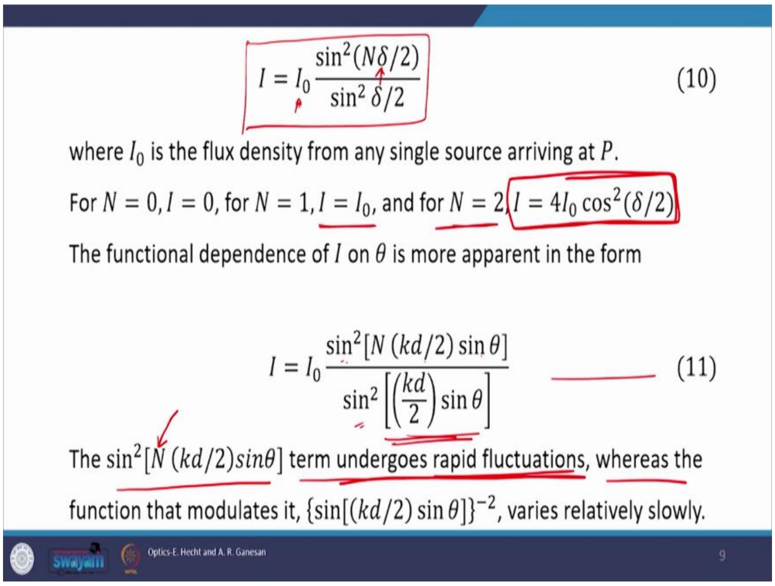

And to after the calculation, we get this expression for irradiance. Where I_0 is the flux density from any single source. We know that all these oscillators are identical then I_0 is the irradiance or flux density of any of these point oscillator. Now if you substitute N is equal to 0 in equation number 10 then you get irradiance value or flux density value 0.

And which is very much obvious, if you have no oscillator, no source you will not get any irradiance at the point of observation P. But if the number of point oscillator is 1 then $I = I_0$, which is again very much obvious because I_0 is the irradiance due to one source.

But for N=2, you get $4I_0 cos^2\delta/2$. And if you remember, this expression we have witnessed in Young's double slit experiment. Now the functional dependence of I on θ is more apparent if you expand δ, δ is your phase difference which is kdsinθ. Now let us replace δ with kd sinθ in equation 10, then we get equation number 11.

In the numerator we have N (kd/2) sinθ while in the denominator we have (kd/2) sinθ and we are taking sign of these quantities. The $\sin^2[N(kd/2)sin\theta]$ term undergoes rapid fluctuations. Why? Because N is sitting here N is large number of oscillator, number of oscillator is N which is very large therefore the oscillations which the numerator will produce will be large, the fluctuation is rapid here.

Whereas in the denominator, we have a function which is relatively slowly varying and therefore this rapid fluctuations of numerator is modulating the slowly varying function which is sitting in the denominator.

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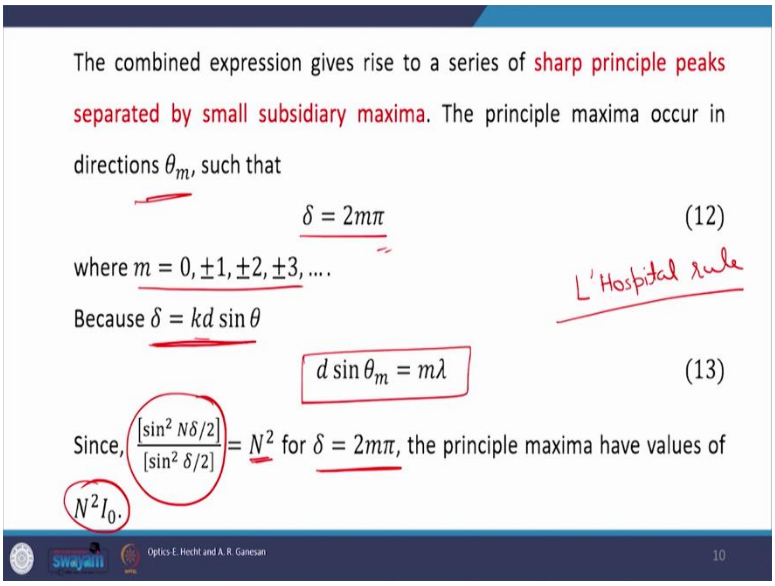

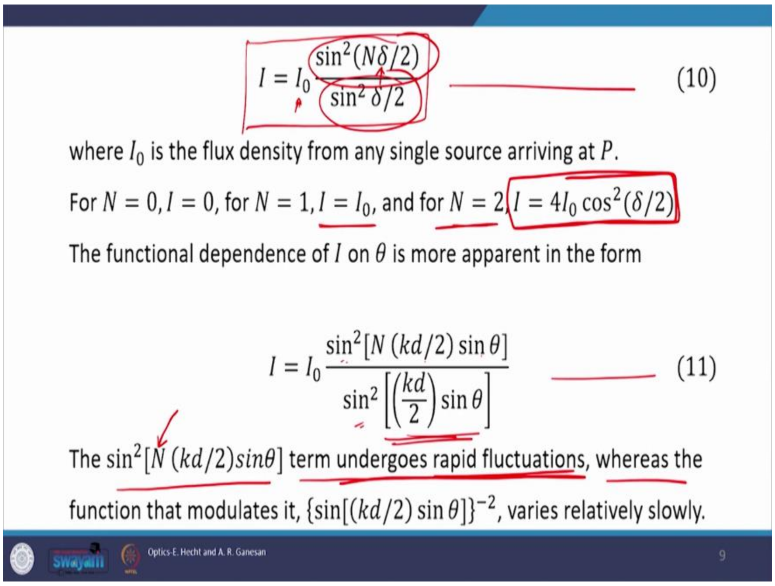

And therefore, what you will get ultimately if you plot this equation number 11, then the combined expression gives rise to a series of sharp principal peaks separated by small subsidiary maxima. You will have several sharp peaks which would be separated by small maxima. And what would be the position of maxima when will we get maxima, the conventional trick is when the phase difference δ is integral multiple of 2π we will get principle maxima.

If the direction θ_m, m is such that $\delta = 2m\pi$ where m is an integer we get maxima. But δ is equal to kd sinθ in terms of d sinθ this equation reduced to this expression d $\sin\theta_m = m\lambda$, the path difference must be integral multiple of wavelength. Now going back into equation 10, we see that in the numerator we have $sin^2 N\delta/2$ and in the denominator we have $sin^2\delta/2$.

How to evaluate this? Because when $\delta = 0$ both in the numerator and denominator we have 0 by 0 form. Here we exercise L'Hospital rule, we will use this rule. In this rule, we will differentiate both numerator and denominator with respect to the variable and after doing this we get N^2 the value of this ratio is equal to N^2 for δ is equal to 2 mπ.

And from here we can easily calculate the irradiance, the resultant irradiance which would be equal to $N^2 I_0$, the principal maxima value would be equal to $N^2 I_0$, as you increase the number of point sources, as you increase the number of point oscillator, the resultant irradiance at the screen at any point will increase rapidly. How rapidly? $N^2 I_0$.

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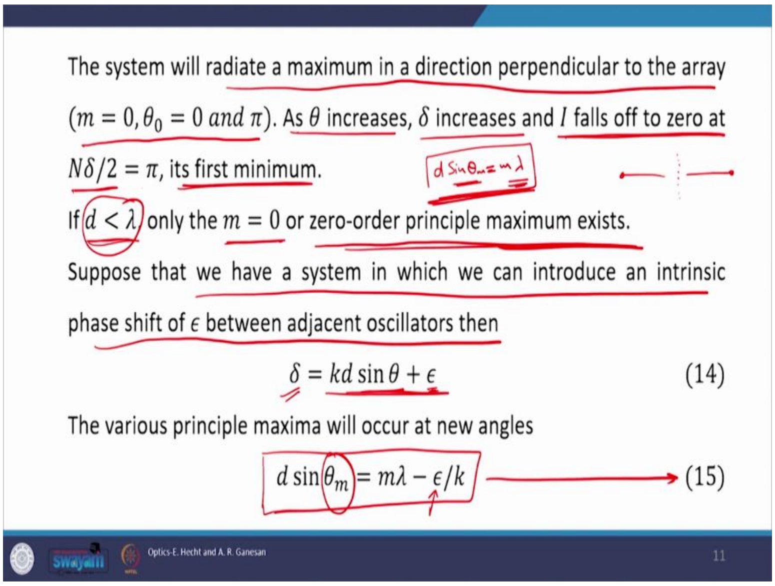

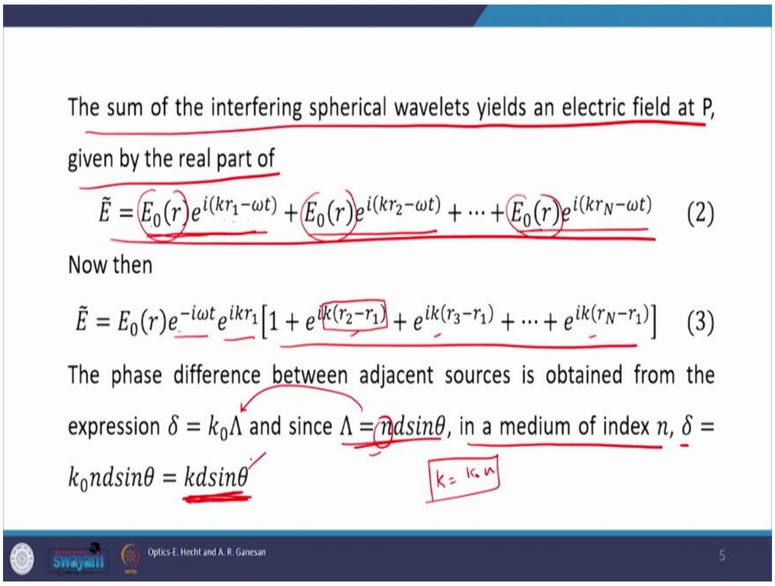

Now the system will radiate a maximum in a direction perpendicular to the array, m is equal to 0, θ is equal to 0 and π, which is of course very much clear, these are the point oscillator, maxima will be here either here or here because the system is emitting in both the direction. Now as θ increases δ increases and I falls off to 0. You see phase is kd sinθ, as you increase θ, δ will increase and as δ increases what will happen I will reduce down the irradiance will falls up and it will become 0, when $N\delta/2 = \pi$. And this is the position of first minimum, this is the condition for first minima.

Now just look into this expression, $dsin\theta_m = m\lambda$. In the right hand side, we have fixed quantity. Now if d is much much smaller than λ, just for the correctness of this equality sinθ now must be greater than 1, which is not possible. Therefore, for d smaller than λ, the only solution which exist is m is equal to 0. It, therefore in this case zero order principle maximum exist, when d is less than λ.

Only zero order principle maximum will exist when d is smaller than λ. Now suppose that we have a system in which we can introduce an intrinsic phase shift of ϵ between adjacent oscillators. We started with a set of oscillators, set of linear oscillators which were oscillating in same phase. And which were separated by a distance d.

Now what we are saying is that the adjacent oscillators are now shifted in phase by ϵ, therefore the phase difference between the waves emitting from the adjacent source will have contribution from the path length difference and will have contribution from the initial phase difference. And this ϵ represents the initial phase difference between the adjacent oscillators. Therefore, we will sum up these two contribution to come up with an expression of the resultant phase difference.

Therefore, the condition of maxima which is $dsin\theta_m = m\lambda$ will get modified and it will modifies to equation number 15, $dsin\theta_m = m\lambda - \epsilon/k$. Therefore, the angular direction of the maxima now becomes a function of ϵ, the initial phase. It means with the addition of some initial phase or deliberate addition of initial phase, we can tilt whole fringe pattern. We can tilt or we can manipulate the position of central maxima or principle maxima.

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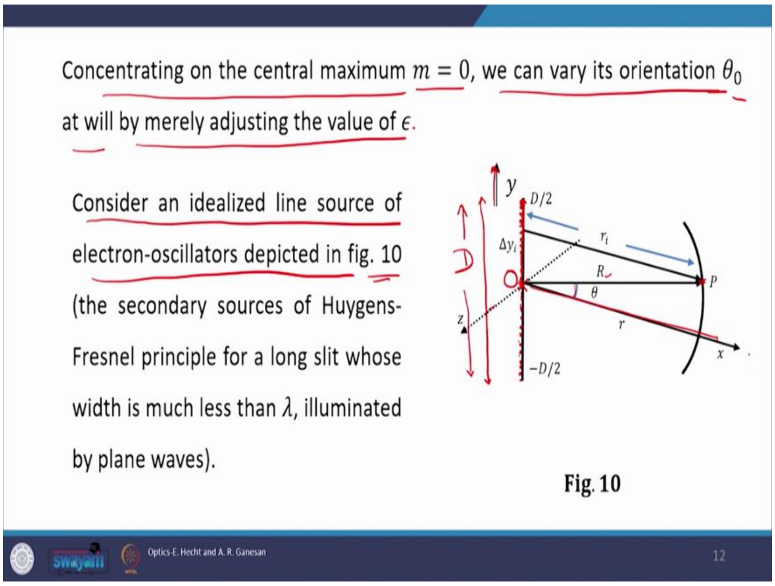

Now concentrating on the central maximum which is m=0, we can vary its orientation θ_0 at will by merely adjusting the value of ϵ. Now let us consider another case or a bit idealized case. Now till now, we were considering point oscillators which were separated by certain distance D but what if this D is extremely small, if D is extremely small then this array of line oscillator can be termed as line source.

Now, consider an idealized line source of electron oscillators as shown in figure 10. Now here these are the point oscillators, which are very closely spaced, almost touching each other, the D the separation is infinitesimally small now, therefore we can safely say line charge.

And the overall length of this line charge is D, this is D and they are placed at the center here. And since O is origin, the line charge is placed at origin in the vertical up direction, the length of the line charge is D/2 in vertical down direction, the length of this line charge element is also D/2. Axis is pointing in this direction, y axis is pointing vertically up and z is coming out of the plane of the paper. Now also assume a point of observation P here which is at a distance R away from origin O and when you join P with O then this line makes an angle θ with x axis.

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Now each point emits a spherical wave front here, each point on the line source now will emit a spherical wave front. And how to write the expression of the field for this is spherical wave front? By this relation $(\epsilon_0/r)(sin\omega t - kr)$. ϵ_0 is amplitude, r is the distance, we know that as you go away from the point charge from the source point, the field will decay down. Therefore, r is sitting in the denominator and the usual phase function sign. And ϵ_0 is the amplitude, the source strength.

Now the present situation is distinct from the previous figure, from the previous analysis where we were having this point oscillators. Now what is the difference? The difference is that here the sources are very weak, their number N is tremendously large, huge number of sources are here and the separation between them is vanishingly small.

Since we are calling them as a line source therefore the separation is vanishingly small. Now if the D is entire length of the array then segment Δy_i will contain $\Delta y_i N/D$ sources. Now suppose this is the your line element which is emitting and then out of in this line element we choose a segment of length y_i. Now the total number of point sources on this line segment is N and total length is D, therefore number of point sources per unit length would be N/D, which is number of point sources per unit length.

Once the number of point sources per unit length is known then in length element $\Delta y_i$the number of point sources would be $\Delta y_i N/D$.

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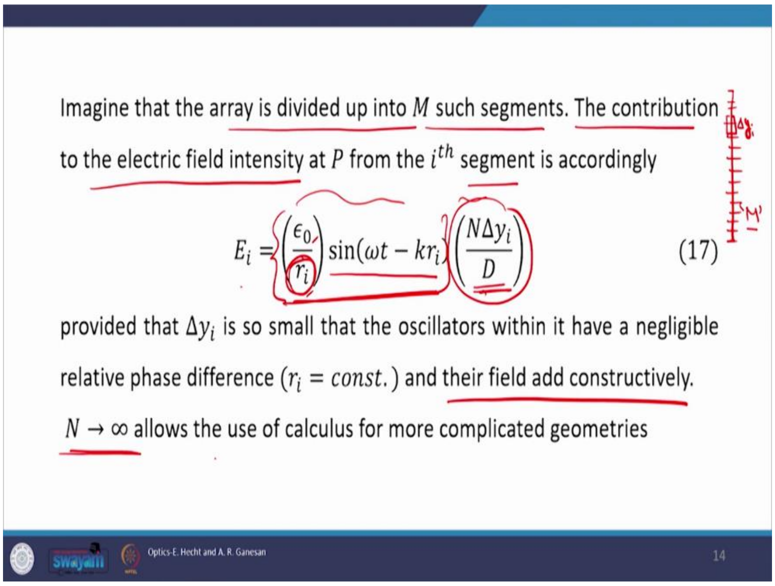

Now imagine that array is divided into M such segments. This was the segment which we were considering and in this segment we took Δy_i. Now assume that there are M number of such segment here the total number of segment is M and total number of point sources is N.

Now the contribution to the electric field intensity at P from the i^{th} segment is $E_i = (\epsilon_0/r_i)sin(\omega t - kr_i)(N\Delta y_i/D)$, $N\Delta y_i/D$ represents the number of point sources in length element Δy_i or in segment Δy_i. And this is contribution from single point source and therefore we have to multiply the contribution from the one source into the number of sources in length element y_i.

But while doing so we have assume that in length element y_i or in segment y_i the segment is so small that the points sources are oscillator, which is lying in this segment have negligible relative phase difference. This r_i is constant for all oscillators lying within Δy_i, r_i is independent of y_i, if we fix, if we take a particular segment then within that segment r_i is constant.

And we also assume they are in same phase therefore we also assume that their field add constructively. Now since the separation is vanishingly small and N is very huge, the number of oscillators is huge. Therefore, we can use calculus for more complicated geometry. We will see how to use it.

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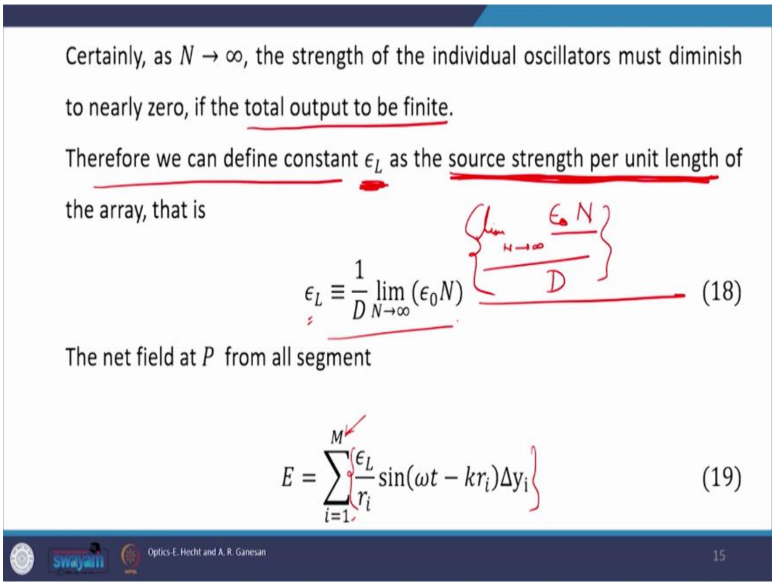

Now, we are observing electric field strength at a point of observation P. And this electric field strength will be resultant of the electric field contribution from N number of point source, which is there within a line source which is of length D. But if we assume that N is approaching towards infinity then each point source lying on this line source, the strength of all individual point source must diminish nearly to 0. Why is it so?

Because if they are contributing hugely and N is infinity then at the point of observation, we will receive infinite field. Because the point sources are infinite and they have some finite value of strength. Therefore, as N approaches to infinity, the strength of individual source must approach to 0. Because we want the total output to be finite, total output must be finite. Therefore, we can define a constant ϵ_L as source strength per unit length.

This will resolve the problem of infinite field at observation point P. What, how to define source strength per unit length? It can be defined like this, we have total number of sources N and strength of each source is ϵ_0 that therefore total strength would be $\epsilon_0 N$.

Since L is in increasing to infinity, under this limit if we divide it with the D then it gives strength of the source per unit length and which is given by equation number 18. This is how ϵ_L is defined. Once ϵ_L is defined, then we can express the total field at point of observation P from all the segment and we know that we have M number of segment. And Δy_i is one of such segments, we will sum over i now.

Now this is the expression for one of the segment and we now sum over i, i varies from 1 to M, M is the number of segment. And now since, the length of the segment is very small, then the number of segment would also be very huge.

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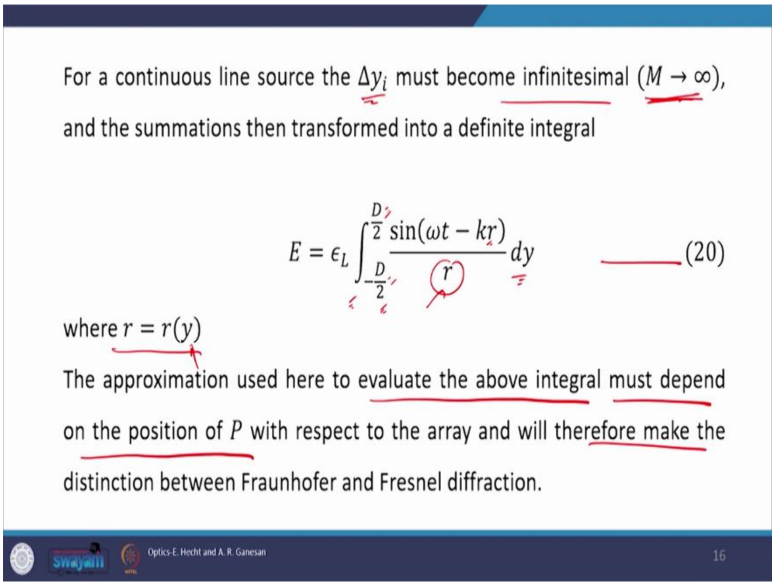

And therefore, we can replace the summation with integration. And this is what exactly is done in equation number 20, the summation is replaced by this integration. And since the line element is of length D there of course the integration limit will vary from -D/2 to +D/2 because line element is placed at the center and half of its length is above in positive y direction and half of its length is in negative y direction. Therefore, the limit would be from -D/2 to +D/2 number of the length element is increasing till to an infinity, it is infinitesimal.

Therefore, Δy_i goes to dy and where r which is sitting in the denominator and here its would be a function of y. Now here the approximation used to evaluate the above integral must depend upon the position of P because r is here, which is a function of y. And therefore, this expression 20, it makes the distinction between Fraunhofer and Fresnel diffraction because to solve 20, we will have to make some approximation. We will be either in the Fraunhofer domain or Fresnel domain which depends upon r or which depend upon the finiteness of the source. How big the length of the line source is, I stop here, thank you for listening me.