Applied Optics
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Lecture 10
Problems on Geometrical Optics

Hello everyone, welcome to my class. Now, this is my tenth class and tenth lecture and today we will solve a few problems which are based on geometrical optics. We are now starting this problem solving lectures because we have covered all the topics of the geometrical optics then after today we will start wave optics. So, let us start, the first problem is from ray tracing and here we will use ray equation and the problem states as follows.

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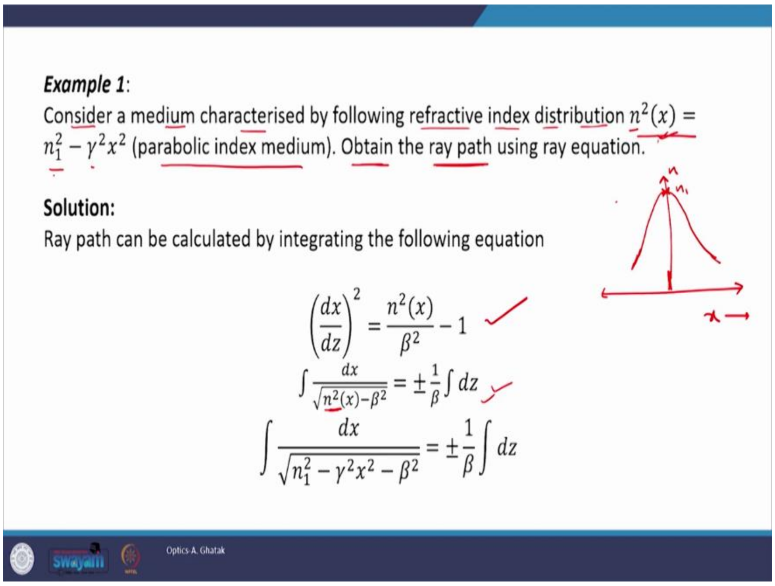

Consider a medium characterized by following refractive index distribution and the square of refractive index which is a function of x is given as $n_1^2 - \gamma^2 x^2$ which of course is a parabolic index medium and what is being asked? the question asked for the ray path, the question asked for ray path using ray equation. Now, you see that this refractive index is parabolic, what is parabolic refractive index? over here on this vertical axis let us plot n here on the horizontal axis let us plot x, x is here and n is being plotted on the vertical axis.

Now, if $x = 0$ then you see that $n^2 = n_1^2$ it means there is some maximum value of the refractive index which is equal to n_1 and if you increase x slowly then the refractive index will decrease, this is how the profile look like therefore this is the parabolic index profile and the question asked for the derivation of ray path using ray equation, then how to do this. We will first start with the ray equation itself this is the ray equation which is given here with this expression.

Now, we want to find the ray path. To find the ray path we will have to of course integrate this ray equation, therefore, integration is performed here it is a first order differential equation therefore we will have to integrate it only once to get the equation and we just did, we took the square root of left and right hand side and then integrated it with respect to, the left hand side is the integration with respect to x while the right hand side is the integration with respect to z and here we substituted for the expression of n^2 from the problem and this is the final expression which has to be integrated.

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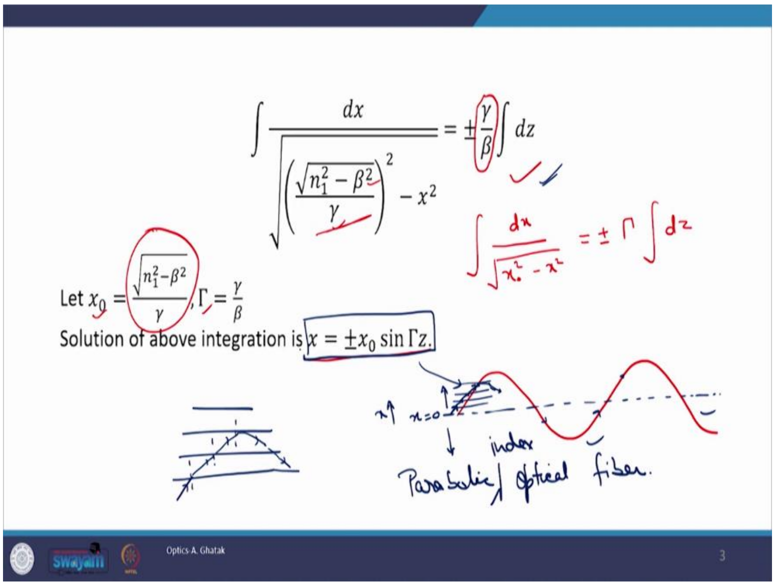

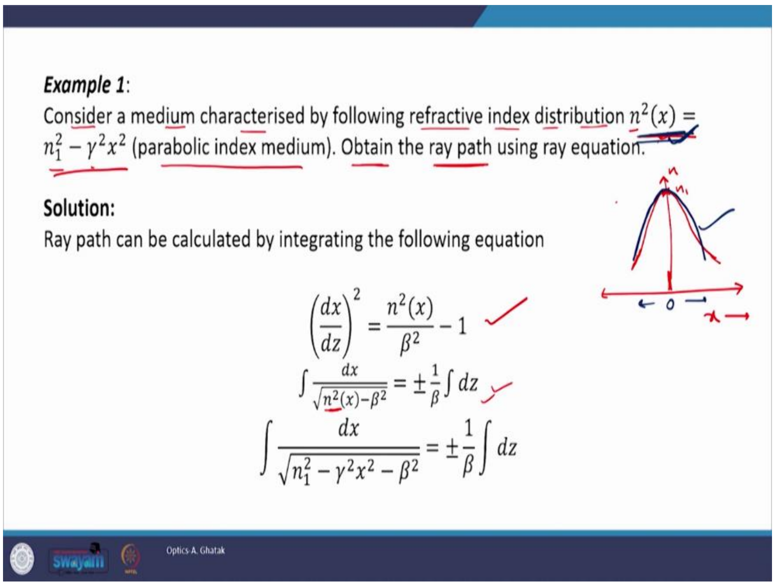

Now after a bit of mathematics we will end up with this expression. Now, the term in the bracket is a constant here n_1^2 is constant, β is independent of x, γ is independent of x. Therefore, we introduce a new parameter which is x_0 and which replaces this term and here this term γ and β is again a constant therefore, we replace $\gamma/\beta = \Gamma$. After these two replacements the equation will look like this $\int dx/\sqrt{x_0^2 - x^2} = \pm \int \Gamma dz$, its easily integrable function and if we integrate it then you will get this relation $x = \pm x_0 sin\, \Gamma z$, Gamma is capital Gamma (Γ) here.

This means that the ray path would be sinusoidal which says that ray path is like this and since the refractive index distribution is, let us choose a different pen to highlight things. Now, we know that suppose there is this axis of symmetry and the ray is going like this as per this expression. Now, the refractive index which is plotted here which is parabolic, this is our parabolic refractive index, this index is maximum at $x = 0$ and if you move away from $x = 0$ the refractive index decreases.

Now the same thing is here, x is equal to zero is this horizontal line, here refractive index is maximum and if you move up or down, this is your x axis, if you move up and up or down, then refractive index should decrease, as per the question. Now, if refractive index is decreasing if we are moving up or down, then we can now divide the medium and small strips and each upper strip is having lower refractive index as compared to the immediately lower strip and what will happen the ray which is going in this direction after getting refracted it will move away from the perpendicular to the interface and each moving away will ultimately give this the ray this path which is given by this red one, because we have this layered index, the ray initially start from this direction, this is the perpendicular it will tilt here, again tilt here and a situation will come where TIR (total internal reflection) will happen and then they will bend back.

And this is how this graded index profile or parabolic index profile gives us a sinusoidal ray path. This is what happens in optical fibers or more precisely graded index optical fiber or parabolic index optical fiber. Due to this type of refractive index profile these optical fibers are able to guide the light along its length, this is an example of light guidance inside a medium but this guidance require a particular distribution of refractive index and once this distribution is given in this problem and therefore, ultimately we have this expression for the ray path.

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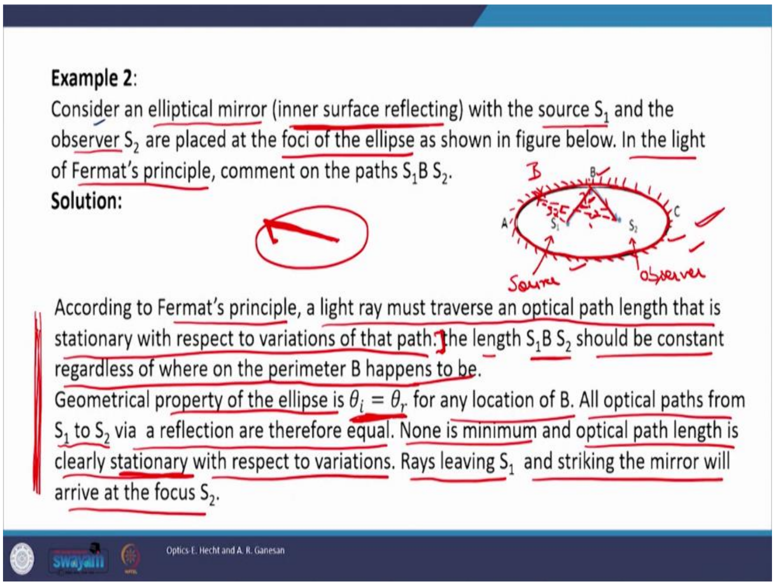

Now, let us go to the next problem, the problem statement is consider an elliptical mirror. Now, the mirror is in the form of an ellipse. This is a mirror and whose inner surface is reflective, the inner surface of this elliptical mirror is reflective. Therefore, any source which is inside this elliptical mirror, if a ray start from the source the light will reflect from this internal mirror and it will be confined within the boundaries of the mirror, it will not go out, the inner surface is reflecting, the point to be noted is that the inner surfaces reflecting.

Now, within this mirror this elliptical mirror there is a source S_1 and an observer S_2 make it a point or it is a source and S_2 is observer and they both are placed on the foci of the ellipse as shown in the figure, as S_1 as on one focus and S_2 is on the second focus. Now, the question asks for exercising Fermat principle and then comment on a particular path, this ray path. In this ray path, the ray is emanating or starting from source S_1 and then it is falling on a point B on the surface of the mirror and after reflecting from point B it goes to observer S_2 and we will have to discuss this scenario in the light of Fermat principal.

Now, let us go to the solution. Now according to Fermat principle, a light ray must traverse an optical path length that is stationary with respect to the variation of that path. If you remember we calculated the variance delta, we took delta of the length the optical path length and if it is not optimum then it should be zero, any variance in the path length should be zero or must be zero. Now, the length S_1BS_2, you see the light is a starting, the ray is a starting from S_1 going to be and then coming back to S_2.

The length S_1BS_2 should be constant regardless of where on the perimeter B happens to be, what this statement says is that the path length S_1BS_2, it should be constant irrespective of the positioning of point B, irrespective of the point of reflection, if we choose another position for B, say B is here, then the this path which is starting from S_1 going to be new B and then again coming back to S_2, this S_1, new B and S_2 this must be equal to S_1BS_2 this is what Fermat said and this is exactly the property of an ellipse we know therefore, the length S_1BS_2 should be constant regardless of where on the perimeter B happens to be, B is the point of reflection.

Now, the geometrical property of the ellipse is that angle of incidence is equal to angle of refraction. If you draw a perpendicular here then this angle must be equal to this, if you draw a perpendicular here then this angle must be equal to the angle of reflection. This is the geometrical property of an ellipse. And therefore, this property would be true or would hold irrespective of the location of B, irrespective of where exactly the point B is on the perimeter, B maybe here, B maybe here, B maybe here, irrespective of the position of the B this property will hold good because of the geometry of the ellipse and therefore, all possible optical path length will be same.

All optical path from S_1 to S_2 via a reflection are therefore equal, they all will be same. We know that ellipse if you start from one foci and then goes to the perimeter and then comes back to the another foci then this path would be the same or if you take two nails and tie a rope between these two nails, and if the length of this rope is longer than the distance between the two nails, then if you put a pen here and then rotate it then it will form a ellipse which means this path length, the constant path length will form an ellipse, its locus would be an ellipse and this is the property of the ellipse and which is being used here in the elliptical mirror.

Now, none is minimum since all the optical paths are the same, there is no minimum neither there is any maximum and optical path length is clearly stationary, we get any stationary path length with respect to the variations. Irrespective of the position of point B, we are getting the same path and which is the exactly is the Fermat principle, the path length should be stationary with respect to the variation in the path, we are varying the path still the path length is coming same again and again. It means that ray leaving S_1 our source and striking the mirror will arrive at focus S_2. Now, this is the discussion in the domain of Fermat principle of this figure.

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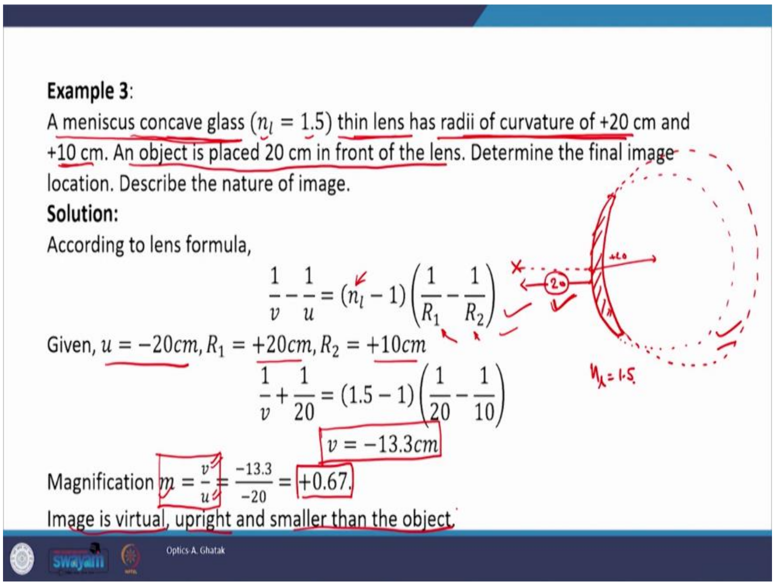

Now, we will move to the next example, which is example number 3. And the statement here is a meniscus concave glass (thin lens) here, meniscus means our lenses in this form and it has radii of curvature of +20 centimeter and +10 centimeter which is of course true because if you form the first sphere from the left most refracting surface, it would be on the right-hand side of the refracting surface therefore, the radius of curvature would be positive and this is +20 centimeter. Similarly, you can form the secondary sphere if you extend the curve and the second sphere is again on the right-hand side of this lens and therefore, it would again be the radius of curvature. This sphere would again be positive.

If now, again I will reread the statement, a meniscus concave glass thin lens has radii of curvature of +20 centimeter and +10 centimeter, it means the two radii, the two spheres are on the right hand side of the lens here as depicted here in this figure. One has a radius of curvature of +20 centimeter and the second refracting surface as the radius of curvature of +10 centimeter and the refractive index of the glass material is 1.5. Now, the question is an object is placed 20 centimeter in front of the lens, this is the axis and the object is placed 20 centimeter in front of the lens. Now, determine the final image location and describe the nature of the image this is what being asked.

Now, we will exercise here the lens formula which $1/v - 1/u = (n - 1)(1/R_1 - 1/R_2)$,, n is replaced by n_l which is the refractive index of the glass. R_1 and R_2 are the radii of curvature which are nothing but +20 centimeter and +10 centimeter. The object is placed at 20 centimeter in the object space therefore u would be -20 centimeter and then after substituting all these into this lens formula, we get the value of v which is the position of the image.

Since v is in minus therefore, the image would be form on the left-hand side of the lens system. Now, once u and v are known, we know we can calculate magnification which is equal to v/u or which is equal to the height of the image upon height of the object, but here are the relevant quantities or v by u therefore, we will use this particular formula m is equal to v/u, will substitute for v and u and this gives the value of magnification as +0.67.

We know that when magnification is positive it means that image is upright, the image is pointing upward, it is directed and the second information which we can draw from the value of magnification is that, that image is smaller than the object because m is less than 1, the magnification m is less than 1 and we know that magnification is the ratio between the height of object and image therefore, image will be smaller than that of the object and these points are written here, images virtual, upright and smaller than the object.

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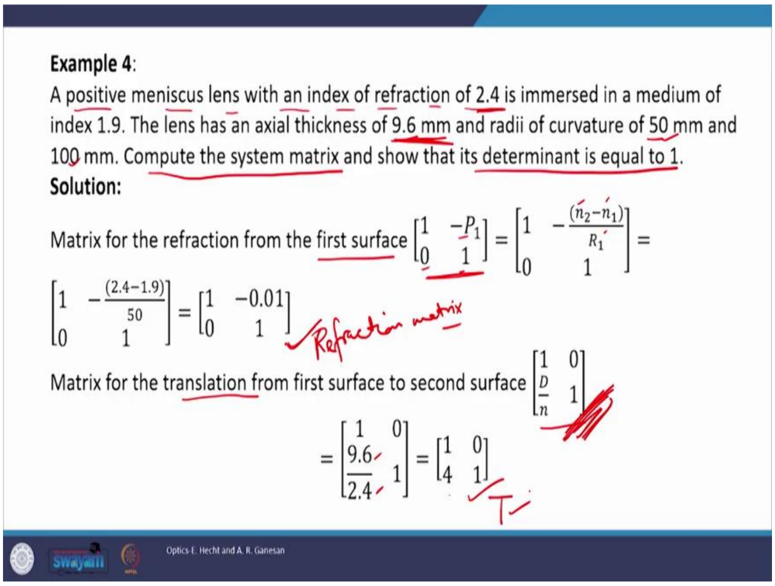

Now, let us move to the next example which is example number 4. The statement is as follows. A positive meniscus lens with an index of refraction of 2.4 is immersed in a medium of index 1.9. The lens has an axial thickness of 9.6 millimeter and radii of curvature of 50 millimeter, 50 millimeter for one surface and 100 millimeter for the other surface here, the radii of curvature 50 millimeter and 100 millimeter. Now it is being asked that compute the system matrix and so that its determinant is equal to 1.

Now, to calculate the system matrix for a lens, we will have to calculate the refraction matrix for first surface and then the translation matrix for the space between the two refracting surface and then again refraction matrix for the second surface because lens has two refracting surface and then in the bulk of the lens in between the two reflecting surface there is a certain thickness in which the light ray travel, it translate.

Therefore, we will have two refracting matrices and one translation matrix and all three matrix we will have to calculate now, and then we will multiply all these matrices, these three matrices to get the system matrix. Then first let us see how to get the matrix, refraction matrix for the first refracting surface, we know refracting, the refraction matrix is given by 1, $-P_1$, 0 and 1. P_1 is the power of the refracting surface which is nothing but $(n_2 - n_1)/R$ we will substitute for n_2, n_1 and R and this will give us this form of the refraction matrix for the left refracting surface.

Now, we will calculate the translation matrix, this is the expression for translation matrix, 1, 0, D/n, and 1. D is the separation or thickness of the lens which is given as 9.6 millimeter, n is the refractive index which is given as 2.4, substitute for D and n and this gives the translation matrix here, translation matrix is here, the refraction matrix is here and this refraction matrix is only for the left interface, the ray refracted from the left interface and then it traveled inside the lens, it traveled through some thickness of the lens which is taken care of by this translation matrix and then again it refracts.

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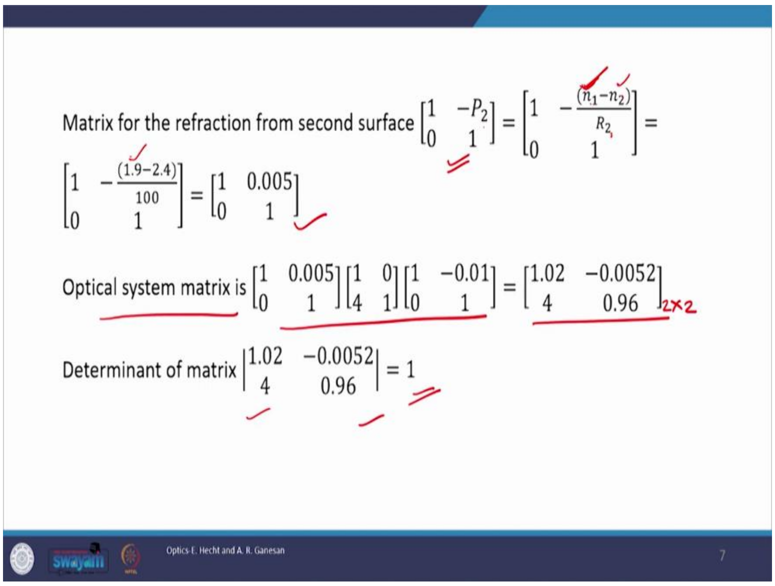

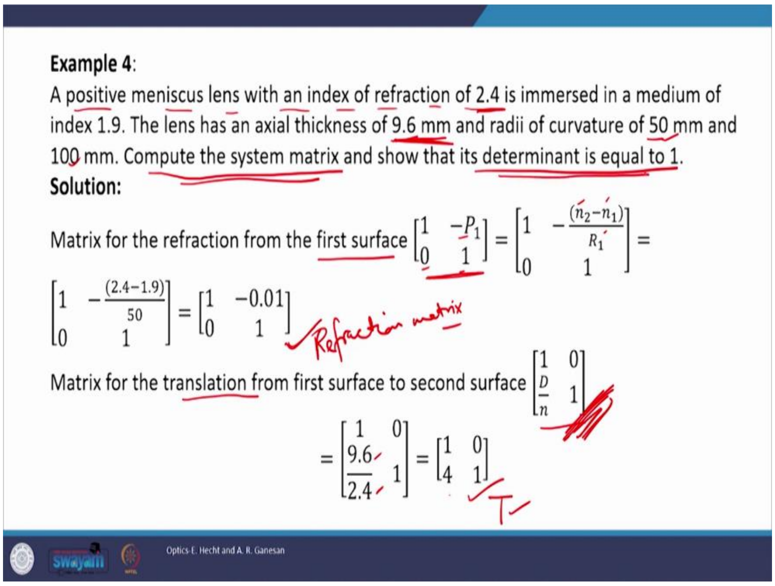

The second refraction is taken care by the second refraction matrix, here P_2 is $(n_1 - n_2)/R_2$, R_2 is the radius of curvature of the second refracting surface, n_1 is the refractive index of air and n_2 is refractive index of the lens material sorry, n_1 is not the refractive index of the air since lens is not kept in air it is kept in a liquid of refractive index 1.9 therefore, we will substitute n_1, we will replace n_1, not with 1 but with 1.9 here, we will always have to keep this in mind and therefore, finally we get the expression of our refraction matrix, the second refraction matrix.

Now, to get the system matrix we will have to multiply all these matrices, the multiplication is performed here and this is the final expression, this is the system matrix. And this is what being asked compute the system matrix and then show that its determinant is equal to 1, if this system matrix is correctly calculated then its determinant must be equal to 1 as we learned before and then here the determinant is calculated and we found that it is equal to 1 and this is all for this problem.

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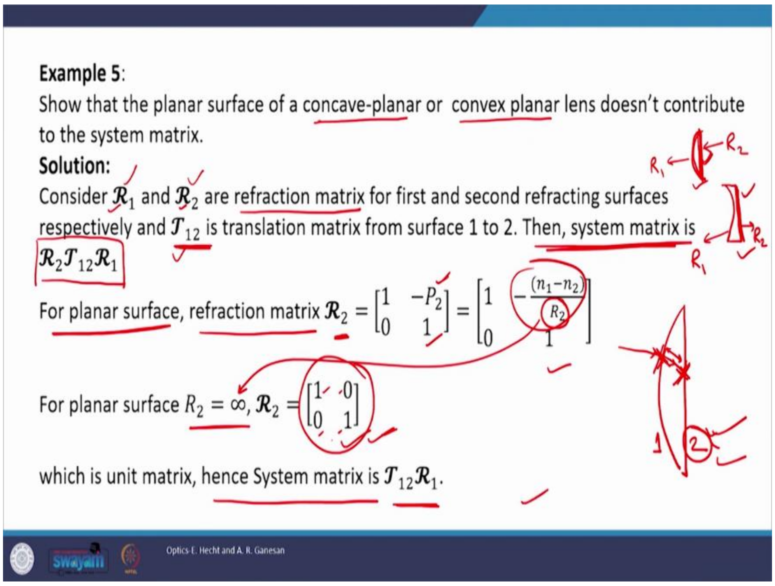

Then we will move to the next problem which is problem number 5, example 5 and the statement is show that the planar surface of a concave planar or convex planar lens does not contribute to the system matrix. I will explain the question for you. The question talks about two lenses, concave planar and convex planar. Now, what is concave planar and convex planar, one surface is like this, other is like this, this is one kind of lens, one surface is like this, other is like this, this is the second kind of lens, this is convex planar and this is concave planar, these are the two type of lenses which the question talks about.

Now, it says that we will have to show, that the planar, this straight edge or this planar surface it will not contribute to the system matrix or alternatively the planar surface can be represented by your matrix which is a unit matrix, the multiplication of this matrix with the system matrix will not alter the system matrix. Therefore, ultimately, we will have to show the matrix for this planar surface is a unit matrix. Now, let us start with the assumption that R_1 and R_2 are the refraction matrices for the first and the second refracting surfaces respectively and if you have a lens and the light is travelling inside it then after refraction it travel through the width of the lens and this is a pure translation.

Therefore, apart from the two refraction which happens at the two refracting surfaces, there is always a translation involved for a lens and therefore, we will also have to incorporate a translation matrix into our calculation. Therefore, to come up with a expression for a system matrix we have to calculate refraction matrix R_1, R_2 and translation matrix T_{12} and this translation matrix is from surface one to surface two. And once all these matrices are known then we can easily calculate the expression for system matrix which is nothing but this expression $R_2T_{12}R_1$ this is what we want to calculate.

Now, let us calculate the refraction matrix for a planar surface this is our concern, this is what the question asked for, the refraction matrix R_2 which is for the planar surface, the second surface here, this will be given by 1, $-P_2$, 0 and P_1. P_2 is the power of the second refracting surface or the power of a planar interface, planar surface and we know the expression of P_2, it is $(n_1 - n_2)/R_2$. Now, what is R_2, R_2 is the radius of curvature of the refracting surface. What is radius of curvature of a plane? Of course, it is infinity, is not?

Therefore, we will replace R_2 with infinity. Once there is a infinity in the denominator the whole quantity here would reduce down to zero. Therefore, the matrix element would be like this 1, 0, 0, 1. And which is nothing but a unit matrix if you multiply this matrix with any other matrix the matrix will remain unchanged.

The R_2 will not affect therefore, the system matrix and this is what was being asked here or we were asked to prove this that the planar surface does not contribute to the system matrix and we prove it here because the planar surface has a matrix which is a unit matrix and therefore, it will not alter the system matrix and the system matrix, the resultant system matrix will be T_{12} into R_1, R_2 is missing, R_2 is nothing but a unit matrix this is all for equation, example 5 or problem 5.

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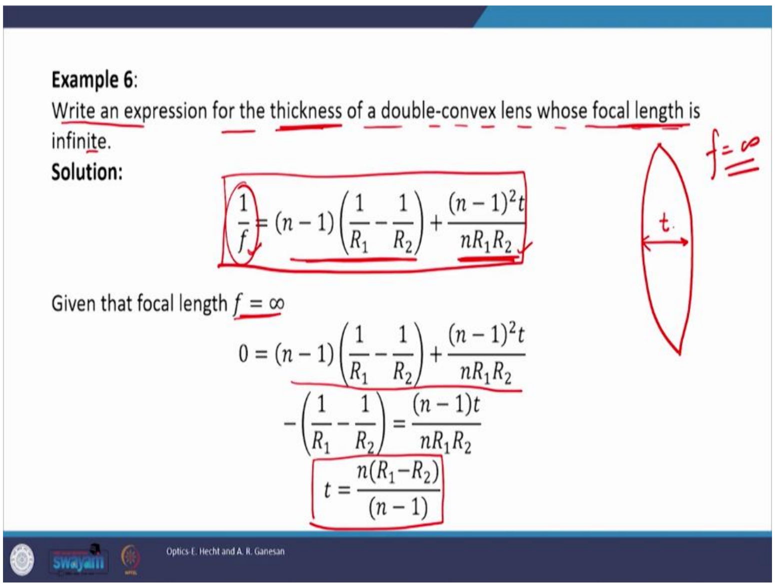

Now, we will move to next problem and which states that write an expression for the thickness of a double convex lens whose focal length is infinite. You have a double convex lens and it is asking for the thickness of a double convex lens, it means it is a thick lens and what is given, given that the focal length of this double convex lens is infinite, f is equal to infinity, this is given and the question is being asked for a thick lens. Therefore, we will apply the formula for focal length of a thick lens and this is the formula focal length for a thick lens. This is the usual formula and there is an additive term which is there due to the finite thickness, for a thicker lens we add this term.

Now, as is given f is equal to infinity and therefore we will replace the left-hand side of this expression by zero because f is in the denominator and then we will write the right hand side as it is. After a bit of simplifications, we can get the expression for the thickness which is nothing but $n(R_1 - R_2)/(n - 1)$, this is the expression for a double convex thick lens whose focal length is infinity, infinite.

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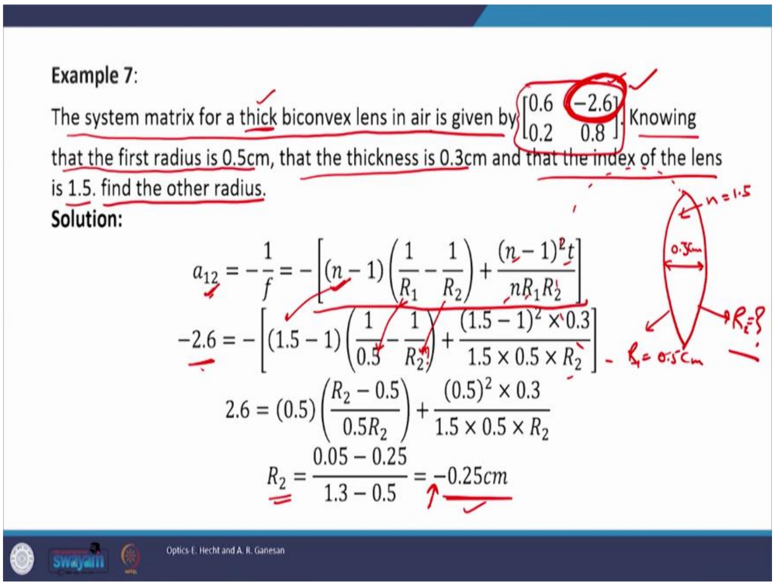

Now, moving to the next problem, the statement here is the system matrix for thick biconvex lens in air is given by, biconvex lens this type of lens and it is thick here, thick biconvex lens in air is given by this matrix here, this matrix give the system matrix of this biconvex lens in air. Now, knowing that the first radius is 0.5 centimeter here R_1 =0.5 centimeter that the thickness is 0.3 centimeter, the thickness of this is 0.3 centimeter and the index of the lens is 1.5 centimeter, the refractive index n is equal to 1.5 sorry, not centimeter. It is the refractive index, it is unitless quantity. Let me correct it, refractive index is 1.5 here, what is being asked, find the other radius this is unknown, the radius of curvature of the right refracting surface is being asked.

The system matrix is given and we know that this term which is sitting on the corner, it has the information of focal length and we will start from here a_{12} which is nothing but this corner term, this is equal to -1/f and we know the formula for a thick lens, which we just used in our previous example, we will just write this formula here. The value of a_{12} is known from the system matrix which is equal to -2.6, just substitute for 2.6 we know the refractive index of the lens which is 1.5, R_1 is known, substitute for R_1, R_2 is not known, you have to find it out. Then substitute for n, t, n and R_1 and R_2 sit as it is.

Now, if you solve it, then it gives the value of $R_2 = -0.25$ which is correct here. If you just make a circle here, this sphere would be on the left-hand side of the refracting surface. Therefore, the minus is correct here, the radius of curvature should be equal to -0.25. These are the examples, which I put to clarify a few concepts which we learn in last few lectures or which we learn while learning the topics of geometrical optics and this is all for this lecture. And I end this class here and see you in the next class. Thank you.